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A NOTE ON ČEBYŠEV TYPE INEQUALITIES

BY

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Abstract. In the present note we establish two new inequalities similar to Čebyšev inequality involving twice differentiable functions.

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1. Introduction. For two absolutely continuous functions $f, g : [a, b] \to R$, consider the functional

$$\begin{aligned} T(f,g) &= \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\ &\quad -\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \end{aligned}$$

where the involved integrals exist. The following classical inequality is well known in the literature as the Čebyšev [1] inequality (see also [8, p.207]) :

$$|T(f,g)| \le \frac{1}{12}(b-a)^2 \left\|f'\right\|_\infty \left\|g'\right\|_\infty, \tag{1.1}$$

provided $f', g' \in L_\infty[a, b]$.

Since its appearence in 1882, the inequality (1.1) has been studied extensively and numerous variants, generalizations and extensions appeared in the literature, see [4, 5, 7, 8] and the references given therein. The main objective of this note is to establish new bounds on the right hand side in (1.1) involving first and second derivatives of functions f, g . The analysis used in the proofs is based on the integral identities proved in [2] and [3] and our results provide new estimates on these types of inequalities.

2. Statement of results. In what follows R and $'$ denotes the set of real numbers and the derivative of a function. We use the following notations to simplify the details of presentation. For a suitable function $h : [a,b] \to R$ we set

$$E\left(h', h''\right)(x) = \left(x - \frac{a+b}{2}\right) h'(x) - \frac{1}{b-a}\int_a^b k(x,t)\, h''(t)\, dt,$$

$$\begin{aligned} H\left(h, h''\right)(x) &= \frac{h(b) - h(a)}{b-a}\left(x - \frac{a+b}{2}\right) \\ &\quad + \frac{1}{(b-a)^2}\int_a^b \int_a^b p(x,t)\, p(t,s)\, h''(s)\, dsdt, \end{aligned}$$

and define $\|h\|_\infty = \sup_{t\in[a,b]} |h(t)| < \infty$ and let

$$L(x) = \frac{1}{b-a}\int_a^b |k(x,t)|\, dt,$$

$$M(x) = \frac{1}{(b-a)^2}\int_a^b \int_a^b |p(x,t)\, p(t,s)|\, dsdt,$$

where $k(x,t), p(x,t) : [a,b]^2 \to R$ are given by

$$k(x,t) = \begin{cases} \frac{(t-a)^2}{2} & \text{if } t \in [a,x] \\ \frac{(t-b)^2}{2} & \text{if } t \in (x,b], \end{cases}$$

$$p(x,t) = \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b]. \end{cases}$$

By simple computation we have (see [2, p. 34] and [3, p.70])

$$L(x) = \frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2,$$

and

$$M(x) = \frac{1}{2}\left\{\left[\frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right]^2 + \frac{1}{12}\right\}(b-a)^2.$$

Our main results are given in the following theorems.

Theorem 1. *Let $f, g : [a, b] \to R$ be continuous functions on $[a, b]$ and twice differentiable on (a, b) , whose first and second derivatives $f', f'', g', g'' : (a, b) \to R$ are bounded on (a, b). Then*

$$|T(f, g)| \leq \frac{1}{b-a} \int_a^b A(x)\, dx, \tag{2.1}$$

where

$$\begin{aligned} A(x) &= \left(x - \frac{a+b}{2}\right)^2 \|f'\|_\infty \|g'\|_\infty \\ &\quad + \left|x - \frac{a+b}{2}\right| \left\{\|f'\|_\infty \|g''\|_\infty + \|g'\|_\infty \|f''\|_\infty\right\} L(x) \\ &\quad + \|f''\|_\infty \|g''\|_\infty L^2(x), \end{aligned} \tag{2.2}$$

for $x \in [a, b]$

Theorem 2. *Let $f, g : [a, b] \to R$ be continuous functions on $[a, b]$ and twice differentiable on (a, b) whose second derivatives $f'', g'' : (a, b) \to R$ are bounded on (a, b) . Then*

$$|T(f, g)| \leq \frac{1}{b-a} \int_a^b B(x)\, dx, \tag{2.3}$$

where

$$\begin{aligned} B(x) &= |FG| \left(x - \frac{a+b}{2}\right)^2 \\ &\quad + \left|x - \frac{a+b}{2}\right| \left\{|F| \|g''\|_\infty + |G| \|f''\|_\infty\right\} M(x) \\ &\quad + \|f''\|_\infty \|g''\|_\infty M^2(x), \end{aligned} \tag{2.4}$$

in which

$$F = \frac{f(b) - f(a)}{b-a}, \quad G = \frac{g(b) - g(a)}{b-a}. \tag{2.5}$$

3. Proofs of Theorems 1 and 2. From the hypotheses of Theorem 1, we have the following identities (see [2, p.35] , and also [6]):

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=E\left(f',f''\right)(x), \tag{3.1}$$

$$g(x)-\frac{1}{b-a}\int_a^b g(t)\,dt=E\left(g',g''\right)(x), \tag{3.2}$$

for $x\in[a,b]$. Multiplying the left sides and right sides of (3.1) and (3.2), dividing the resulting identity by $(b-a)$ and integrating over $[a,b]$ we get

$$T(f,g)=\frac{1}{b-a}\int_a^b E\left(f',f''\right)(x)\,E\left(g',g''\right)(x)\,dx. \tag{3.3}$$

It is easy to observe that

$$\begin{aligned}
&\left|E\left(f',f''\right)(x)\,E\left(g',g''\right)(x)\right|=\left|\left(x-\frac{a+b}{2}\right)^2 f'(x)\,g'(x)\right.\\
&-\left(x-\frac{a+b}{2}\right)\left\{f'(x)\left(\frac{1}{b-a}\int_a^b k(x,t)\,g''(t)\,dt\right)\right.\\
&\left.+g'(x)\left(\frac{1}{b-a}\int_a^b k(x,t)\,f''(t)\,dt\right)\right\}\\
&\left.+\left(\frac{1}{b-a}\int_a^b k(x,t)\,f''(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b k(x,t)\,g''(t)\,dt\right)\right|\\
&\leq\left(x-\frac{a+b}{2}\right)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty\\
&+\left|x-\frac{a+b}{2}\right|\left\{\left\|f'\right\|_\infty\left\|g''\right\|_\infty+\left\|g'\right\|_\infty\left\|f''\right\|_\infty\right\}L(x)\\
&+\left\|f''\right\|_\infty\left\|g''\right\|_\infty L^2(x)=A(x).
\end{aligned} \tag{3.4}$$

From (3.3), using the properties of modulus and (3.4), the required inequality in (2.1) follows.

From the hypotheses of Theorem 2, we have the following identities (see [3, p.71]):

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=H\left(f,f''\right)(x), \tag{3.5}$$

$$g(x) - \frac{1}{b-a}\int_a^b g(t)\,dt = H\left(g, g''\right)(x), \tag{3.6}$$

for $x \in [a,b]$. Multiplying the left sides and right sides of (3.5) and (3.6), dividing the resulting identity by $(b-a)$ and integrating over $[a,b]$ we get

$$T(f,g) = \frac{1}{b-a}\int_a^b H\left(f, f''\right)(x) H\left(g, g''\right)(x)\,dx. \tag{3.7}$$

A simple computation shows that

$$\begin{aligned}
\left|H\left(f, f''\right)(x) H\left(g, g''\right)(x)\right| &= \left|FG\left(x - \frac{a+b}{2}\right)^2\right. \\
&+ \left(x - \frac{a+b}{2}\right)\left\{F\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)\,p(t,s)\,g''(s)\,dsdt\right. \\
&\left. +G\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)\,p(t,s)\,f''(s)\,dsdt\right\} \\
&+ \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)\,p(t,s)\,f''(s)\,dsdt\right) \\
&\left. \times \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)\,p(t,s)\,g''(s)\,dsdt\right)\right| \\
\leq |FG|\left(x - \frac{a+b}{2}\right)^2 &+ \left|x - \frac{a+b}{2}\right|\left\{|F|\left\|g''\right\|_\infty + |G|\left\|f''\right\|_\infty\right\} M(x) \\
&+ \left\|f''\right\|_\infty \left\|g''\right\|_\infty M^2(x) = B(x).
\end{aligned} \tag{3.8}$$

From (3.7), using the properties of modulus and (3.8) we get the desired inequality in (2.3).

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