ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, f.1

ISOPTICS OF OPEN ROSETTES II

BY

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Abstract. In this paper we give some new properties of isoptics of open rosettes. We introduce quasiperiodic open rosettes, consider isoptics of pairs of quasiperiodic open rosettes and give some of their properties.

Mathematics Subject Classification 2000: 53A04.

Key words: rosette, open, convex, isoptic, parametrization, periodic, quasiperiodic, curvature, width, disjoint, pair.

1. Introduction.

In this section we recall some facts from [1].

Definition 1. *A plane, open curve C with positive curvature is said to be an open rosette.*

Let C be an open rosette with a support function $p(t)$, $t \in \mathbb{R}$. Parametrization of C is given by $z(t) = p(t)e^{it} + p'(t)ie^{it}$. We fix a point $z(t_0)$ and denote the tangent to C at $z(t_0)$ by l_0. Next, we choose t_1, $t_0 < t_1$ closest to t_0 in the sense of parametrization and we denote the tangent line at this point by l_1. We choose t_1 in such a way that an angle between tangents l_0 and l_1 equals to $\pi - \alpha$. Then we define following points

$$t_2 = t_1 + 2\pi = t_0 + \alpha + 2\pi$$

$$t_3 = t_2 + 2\pi = t_0 + \alpha + 4\pi$$

$$\vdots$$

$$t_k = t_{k-1} + 2\pi = t_0 + \alpha + 2(k-1)\pi$$

$$\vdots$$

We denote by $l_1, l_2, \ldots, l_k, \ldots$ tangents to C at these points.

Definition 2. *The cut locus of common points of the tangents l_0 and l_k forms α-isoptic of k-th order of open rosette. We denote this curve by $C_{\alpha,k}$ and call it α, k-isoptic.*

Theorem 1. *Let C be an open rosette with $p(t)$ as its support function and let $C_{\alpha,k}$ be its isoptic. We have the following equation for parametrization of $C_{\alpha,k}$*

$$z_{\alpha,k}(t) = p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{p(t+2(k-1)\pi+\alpha)}{\sin\alpha}\right) ie^{it}$$

for $t \in \mathbb{R}, k \in \mathbb{N}$.

Remark 1. Curvature of an open rosette parametrized by the support function is given by

$$k(t) = \frac{1}{p(t)+p''(t)}, \quad t \in \mathbb{R}$$

2. Curvature of an isoptic of an open rosette

Theorem 2. *Let C be an open rosette parametrized by $z(t) = p(t)e^{it} + p'(t)ie^{it}$, $t \in \mathbb{R}$ and $C_{\alpha,k}$ be its isoptic. Curvature of $C_{\alpha,k}$ is given by*

$$\kappa_{\alpha,k}(t) = \frac{2|q(t)|^2 - [q(t), q'(t)]}{|q(t)|^3} \sin\alpha$$

where

$$q(t) = z(t) - z(t+2(k-1)\pi+\alpha).$$

Proof. Differentiaiting equation from theorem 1 with respect to t we get

$$\begin{aligned}
z'_{\alpha,k}(t) &= \left(p'(t) + p(t)\cot\alpha - \frac{p(t+2(k-1)\pi+\alpha)}{\sin\alpha}\right) e^{it} \\
&\quad + \left(p(t) - p'(t)\cot\alpha + \frac{p'(t+2(k-1)+\alpha)}{\sin\alpha}\right) ie^{it} \\
z''_{\alpha,k}(t) &= \left(p(t) + p''(t) + 2p'(t)\cot\alpha - \frac{2p(t+2(k-1)\pi+\alpha)}{\sin\alpha}\right) e^{it} \\
&\quad + \Bigg(p(t)\cot\alpha + 2p'(t) - p''(t)\cot\alpha \\
&\quad - \frac{p(t+2(k-1)+\alpha)}{\sin\alpha} + \frac{p''(t+2(k-1)+\alpha)}{\sin\alpha}\Bigg) ie^{it}.
\end{aligned}$$

By using following notation

$$\begin{aligned}
b(t) &= p(t+2(k-1)\pi+\alpha)\sin\alpha + p'(t+2(k-1)\pi+\alpha)\cos\alpha - p'(t)\\
B(t) &= p(t) - p(t+2(k-1)\pi+\alpha)\cos\alpha + p'(t+2(k-1)\pi+\alpha)\sin\alpha\\
\lambda(t) &= b(t) - B(t)\cot\alpha\\
\rho(t) &= B(t) + b(t)\cot\alpha
\end{aligned}$$

we can write $z'_{\alpha,k}(t)$ and $z''_{\alpha,k}(t)$ as

$$\begin{aligned}
z'_{\alpha,k}(t) &= -\lambda(t)e^{it} + \rho(t)ie^{it}\\
z''_{\alpha,k}(t) &= -(\lambda'(t)+\rho(t))e^{it} + (\rho'(t)-\lambda(t))ie^{it}.
\end{aligned}$$

Curvature $\kappa_{\alpha,k}(t)$ of the isoptic $C_{\alpha,k}$ can be computed from the formula

$$\kappa_{\alpha,k}(t) = \frac{\left[z'_{\alpha,k}(t), z''_{\alpha,k}(t)\right]}{|z'_{\alpha,k}(t)|^3},$$

where $[u,v] = [a+bi, c+di] = ad-bc$. Since

$$\left[z'_{\alpha,k}(t), z''_{\alpha,k}(t)\right] = \frac{b^2(t)+B^2(t)+b'(t)B(t)-b(t)B'(t)}{\sin^2\alpha}$$

and

$$|z'_{\alpha,k}(t)| = \frac{\sqrt{b^2(t)+B^2(t)}}{\sin\alpha}$$

then

$$\kappa_{\alpha,k}(t) = \frac{b^2(t)+B^2(t)+b'(t)B(t)-b(t)B'(t)}{(b^2(t)+B^2(t))^{\frac{3}{2}}}\sin\alpha.$$

We also have

$$\begin{aligned}
q(t) &= z(t) - z(t+2(k-1)\pi+\alpha) = B(t)e^{it} - b(t)ie^{it},\\
q'(t) &= (b(t)+B'(t))e^{it} + (B(t)-b'(t))ie^{it}.
\end{aligned}$$

So

$$\kappa_{\alpha,k}(t) = \frac{2|q(t)|^2 - [q(t), q'(t)]}{|q(t)|^3}\sin\alpha. \qquad \square$$

Theorem 3. *Let C be an open rosette parametrized by $z(t) = p(t)e^{it} + p'(t)ie^{it}$ where $p(t), t \in \mathbb{R}$ is its support function. Let $q(t) = z(t) - z(t+2(k-1)\pi+\alpha)$. Isoptic $C_{\alpha,k}$ is an open rosette if and only if $[q(t), q'(t)] \le 2|q(t)|^2$.*

Proof. Knowing curvature of $C_{\alpha,k}$ from theorem 2 the proof is immediate. $\square$

3. Quasiperiodic open rosettes and its isoptics

Definition 3. *Let C be an open rosette and $p(t),\ t \in \mathbb{R}$ be its support function. We call the rosette C a quasiperiodic open rosette if the following condition (the condition of quasiperiodicity) holds $\exists_{j\neq 0}\forall_t\exists_{L(t)\neq 0}\ p(t+2\pi j) = p(t) + L(t)$ where $L(t)$ is periodic function of period $2\pi j$.*

The equation of tangent to C at a point $z(t) = p(t)e^{it} + p'(t)ie^{it}$ is given by the formula $x\cos t + y\sin t - p(t) = 0$. So the condition of quasiperiodicity means that the distance between parallel tangents at points $z(t)$ and $z(t + 2\pi j)$ are a periodic function and also that tangents at points $z(t), z(t + 2\pi j), z(t + 4\pi j), \ldots$ are equidistant (see fig. 1). If curve is given by a parametrization then we compute the distance between appropriate tangents.

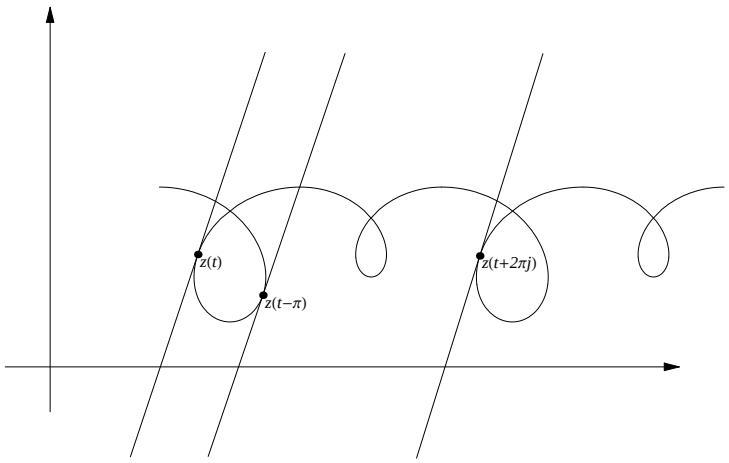

Figure 1: Quasiperiodic open rosette

Definition 4. *Let C be an open rosette and $p(t)$ be its support function. We call $w(t) = |p(t + \pi) - p(t)|$ the width of an open rosette.*

Theorem 4. *The width of quasiperiodic open rosette is periodic function.*

Proof. Let $p(t)$ be support function of open rosette and $w(t)$ be its

width. Using the condition of quasiperiodicity we obtain

$$w(t+2\pi j) = |p(t+2\pi j+\pi) - p(t+2\pi j)| = \\ = |p(t+\pi) + L(t) - p(t) - L(t)| = |p(t+\pi) - p(t)| = w(t). \quad \square$$

Theorem 5. *Let C be a quasiperiodic open rosette and $C_{\alpha,k}$ be its isoptic. The isoptic $C_{\alpha,k}$ is quasiperiodic open rosette if and only if $[q(t), q'(t)] \leq 2|q(t)|^2$, where $q(t) = z(t) - z(t+2(k-1)\pi+\alpha)$.*

Proof. Triangles with vertices at points $z(t), z_{\alpha,k}(t), z(t+2(k-1)\pi+\alpha)$ and $z(t+2\pi j), z_{\alpha,k}(t+2\pi j), z(t+2\pi j+2(k-1)\pi+\alpha)$ are congruent (for notation see fig. 2). It follows from parallelism of tangents at points $z(t), z(t+2\pi j)$ and $z(t+2(k-1)\pi+\alpha), z(t+2\pi j+2(k-1)\pi+\alpha)$ and periodicity of $q(t)$. From sine theorem we have that angles between tangent at $z(t)$ and $z'_{\alpha,k}(t)$ and between tangent at $z(t+2\pi j)$ and $z'_{\alpha,k}(t+2\pi j)$ are equal so $z'_{\alpha,k}(t)$ and $z'_{\alpha,k}(t+2\pi j)$ are parallel. Distance between tangents at $z(t)$ and $z(t+2\pi j)$ is a periodic function then so is distance between $z'_{\alpha,k}(t)$ and $z'_{\alpha,k}(t+2\pi j)$. This means that condition of quasiperiodicity is fulfilled. $\square$

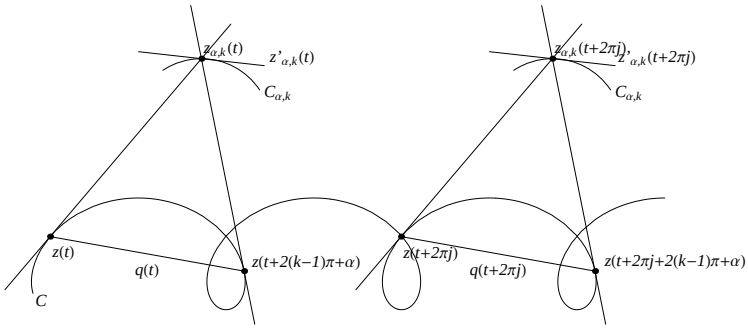

Figure 2: Quasiperiodic open rosette and its isoptic for $j = 2$ and $k = 1$.

Remark 2. We write the condition of quasiperiodicity in terms of the support function. Intresting problem is to find conditions for the curvature of quasiperiodic open rosette. Periodicity of curvature is not sufficient however in the following examples the curvature is periodic so it can be a necessary condition.

4. Examples of quasiperiodic open rosettes

Example 1. Let $p(t) = 4 + t\sin t$ be a support function of a curve (fig. 3). It is an open rosette since its curvature is given by

$$\kappa(t) = \frac{1}{4 + 2\sin t} > 0.$$

If we choose $j \neq 0$ then $p(t + 2\pi j) = p(t) + 2\pi j \sin t$ and $L(t) = 2\pi j \sin t$ is a periodic function.

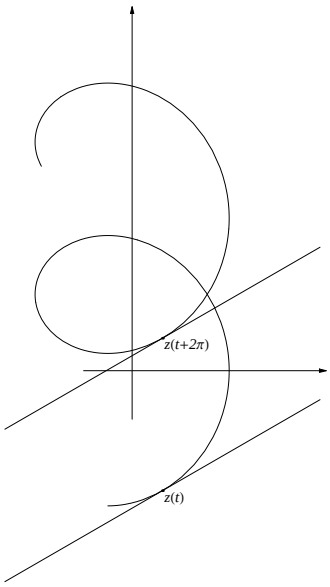

Figure 3: Curve with support function $p(t) = 4 + t\sin t$.

Example 2. Consider an open rosette with the support function given by $p(t) = t\cos t + \cos^2 t + 6$ (fig. 4). Its curvature equals

$$\kappa(t) = \frac{1}{6 - \cos^2 t - 2\sin t + 2\sin^2 t} > 0$$

And we obtain $L(t) = 2\pi j \cos t$ which is a periodic function for any $j \neq 0$.

Example 3. Curve from fig. 5 consists of the same segments (from $z(t)$ to $z(t + 2\pi j)$). If we choose $j = 4$ then there exists appropriate $L(t)$.

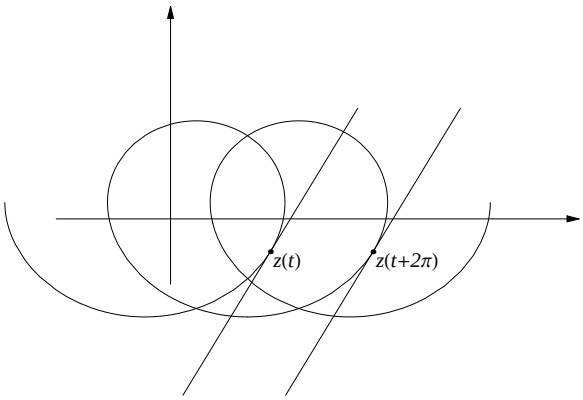

Figure 4: Curve with support function $p(t) = t\cos t + \cos^2 t + 6$.

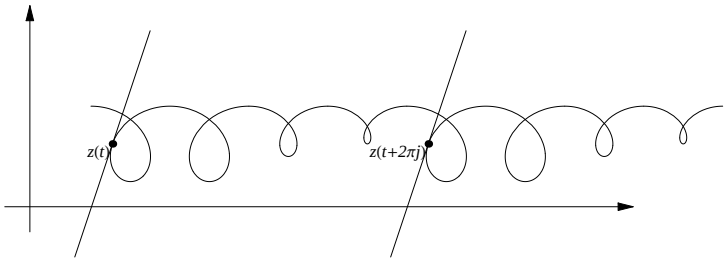

Figure 5: Quasiperiodic open rosette with $j = 4$.

5. Isoptics of pairs of quasiperiodic open rosettes. Let C_1 and C_2 be disjoint quasiperiodic open rosettes parametrized by $z_1(t) = p_1(t)e^{it} + p_1'(t)ie^{it}$, $z_2(t) = p_2(t)e^{it} + p_2'(t)ie^{it}$, where $p_1(t)$, $p_2(t)$, $t \in \mathbb{R}$ are their support functions. Fix a point $z_1(t)$ and define points $z_2(t_k) = z_2(t + 2(k-1)\pi + \alpha)$, where $k \in \mathbb{N}$, $\alpha \in (0, \pi)$. Let l_0 denote tangent line to C_1 at point $z_1(t)$ and l_k denote tangent line to C_2 at point $z_2(t_k)$. Note that tangent lines l_0 and l_k form an angle α. Let $z_{\alpha,k}(t)$ denote the intersection point of tangent lines l_0 and l_k (fig. 6).

Definition 5. *The curve $C_{\alpha,k} : z = z_{\alpha,k}(t)$, where α is fixed is said to be the α-isoptic of k-th order of pair of quasiperiodic open rosettes C_1 and C_2.*

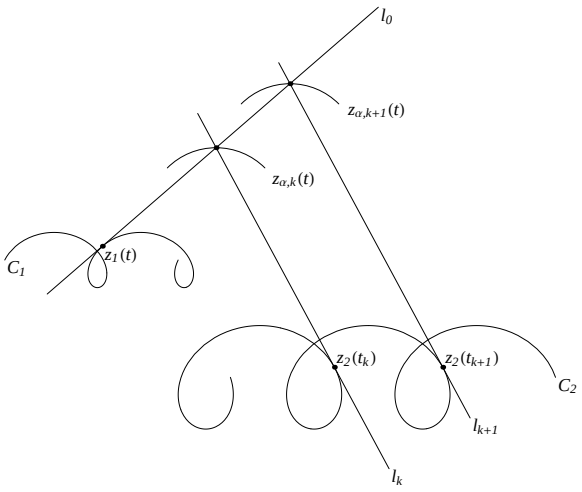

Figure 6: Isoptic of pair of quasiperiodic open rosettes.

Theorem 6. *Under the above notation the parametrization of isoptic* $C_{\alpha,k}$ *is given by*

$$z_{\alpha,k}(t) = p_1(t)e^{it} + \left(-p_1(t)\cot\alpha + \frac{p_2(t+2(k-1)\pi+\alpha)}{\sin\alpha}\right) ie^{it}.$$

Proof. Let $q(t) = z_1(t) - z_2(t+2(k-1)\pi+\alpha)$ and

$$\begin{aligned} b(t) &= [q(t), e^{it}] = \\ &= -p_1'(t) + p_2(t+2(k-1)\pi+\alpha)\sin\alpha + p_2'(t+2(k-1)\pi+\alpha)\cos\alpha \\ B(t) &= [q(t), ie^{it}] = \\ &= p_1(t) - p_2(t+2(k-1)\pi+\alpha)\cos\alpha + p_2'(t+2(k-1)\pi+\alpha)\sin\alpha \end{aligned}$$

The equation of the α, k-isoptic can be written as

$$\begin{aligned} z_{\alpha,k}(t) &= z_1(t) + \lambda(t)ie^{it} = \\ &= z_2(t+2(k-1)\pi+\alpha) + \mu(t)ie^{i(t+2(k-1)\pi+\alpha)} \end{aligned}$$

so we obtain

$$\begin{aligned} q(t) &= z_1(t) - z_2(t+2(k-1)\pi+\alpha) = \\ &= -\lambda(t)ie^{it} + \mu(t)ie^{i(t+2(k-1)\pi+\alpha)} \end{aligned}$$

We multiply the above equation

$$[q(t), e^{it}] = [-\lambda(t)ie^{it} + \mu(t)ie^{i(t+2(k-1)\pi+\alpha)}, e^{it}]$$
$$[q(t), ie^{it}] = [-\lambda(t)ie^{it} + \mu(t)ie^{i(t+2(k-1)\pi+\alpha)}, ie^{it}]$$

to get

$$b(t) = \lambda(t) - \mu(t)\cos\alpha$$
$$B(t) = -\mu(t)\sin\alpha$$

and finally

$$\lambda(t) = b(t) - B(t)\cot\alpha$$
$$\mu(t) = \frac{-B(t)}{\sin\alpha}$$

Then we obtain the equation

$$z_{\alpha,k}(t) = p_1(t)e^{it} + \left(-p_1(t)\cot\alpha + \frac{p_2(t+2(k-1)\pi+\alpha)}{\sin\alpha}\right)ie^{it}. \qquad \square$$

Theorem 7. *Let C_1, C_2 be disjoint quasiperiodic open rosettes parametrized by $z_1(t) = p_1(t)e^{it} + p_1'(t)ie^{it}$ and $z_2(t) = p_2(t)e^{it} + p_2'(t)ie^{it}$ where $p_1(t)$, $p_2(t)$, $t \in \mathbb{R}$ are their support functions. Let $C_{\alpha,k}$ be isoptic of C_1 and C_2. Curvature of $C_{\alpha,k}$ is given by*

$$\kappa_{\alpha,k}(t) = \frac{2|q(t)|^2 - [q(t), q'(t)]}{|q(t)|^3}\sin\alpha$$

where

$$q(t) = z_1(t) - z_2(t + 2(k-1)\pi + \alpha).$$

Proof. From theorem 6 we have

$$z_{\alpha,k}(t) = p_1(t)e^{it} + \left(-p_1(t)\cot\alpha + \frac{p_2(t+2(k-1)\pi+\alpha)}{\sin\alpha}\right)ie^{it}$$

and

$$b(t) = -p_1'(t) + p_2(t + 2(k-1)\pi + \alpha)\sin\alpha + p_2'(t + 2(k-1)\pi + \alpha)\cos\alpha$$
$$B(t) = p_1(t) - p_2(t + 2(k-1)\pi + \alpha)\cos\alpha + p_2'(t + 2(k-1)\pi + \alpha)\sin\alpha$$

To finish the proof we use the same notation and computation as in theorem 2. $\square$

Theorem 8. *Let C_1, C_2 be disjoint quasiperiodic open rosettes parametrized by $z_1(t) = p_1(t)e^{it} + p_1'(t)ie^{it}$ and $z_2(t) = p_2(t)e^{it} + p_2'(t)ie^{it}$ where $p_1(t)$, $p_2(t)$, $t \in \mathbb{R}$ are their support functions. Isoptic $C_{\alpha,k}$ of rosettes C_1, C_2 is an open rosette if and only if $[q(t), q'(t)] \leq 2|q(t)|^2$, where $q(t) = z_1(t) - z_2(t + 2(k-1)\pi + \alpha)$.*

Proof. The proof is simple consequence of theorem 7. □

Remark 3. Note that isoptic of pair of disjoint quasiperiodic open rosette need not to be a quasiperiodic open rosette (fig. 7).

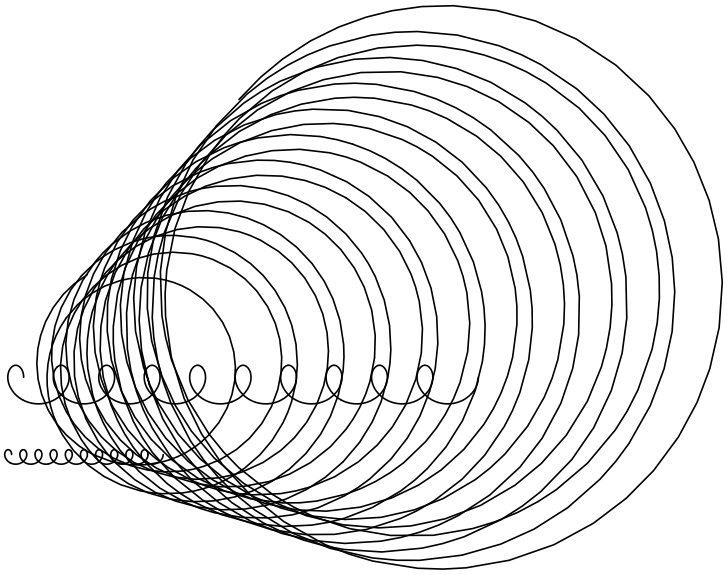

Figure 7: Isoptic of pairs of quasiperiodic open rosettes which is not quasiperiodic.

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Received: 16.X.2006

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