ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
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WEAK COFIBRATIONS FOR C^*-ALGEBRAS

BY

ALINA TOFAN

Abstract. In this Note we define the notion of weak cofibration in the homotopy theory of C^*- algebras. This notion is correlated with the notion of cofibration for C^*-algebras introduced by SCHOCHET in [8] (also studied in [6] and [7]). We establish some 'topological' characterizations and other properties of weak cofibrations of C^*- algebras. Particularly we derive a result which corresponds to a Dold's type theorem in Top.

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Key words: C^*-algebra, $*$-homomorphism, homotopy for $*$-homomorphisms, cofibration and weak cofibration of C^*- algebras, Dold's type theorem.

1. Introduction. In the category of topological spaces the notion of *weak cofibration* was introduced by the name of *h-cofibration* by TOM DIECK, KAMPS and PUPPE in their Homotopietheorie ([2], Ch.I §2) in connection with the study of cofiber homotopy equivalences. This notion was inspired by the notion of *weak fibration* before introduced by DOLD [3] and WEINZWEIG [9] in connection with the study of fiber homotopy equivalences. A map $u : A \to X$ is a weak cofibration if there exists a cofibration $v : A \to Y$ such that the topological spaces X and Y have the same homotopy type as spaces under A. Weak cofibrations satisfy a weak form of the homotopy extension property (*WHEP*). This property can be formulated in two alternative ways which are equivalent (see [2], Ch. I, and [4], Ch. 6). Either way we are given a space E, a map $f : X \to E$ and a homotopy $G : A \times I \to E$ such that $G_0 = f \circ u$. In the first formulation the property is that there exists a homotopy $F : X \times I \to E$ such that $F_0 \simeq f$ under A and such that $F \circ (u \times id_I) = G$. In the second formulation we further require that G is semistationary, in the sense that $G(a, .)$ is stationary on $[0, \frac{1}{2}]$ for

each point $a \in A$, and then the property is that there exists a homotopy $F : X \times I \to E$ such that $F_0 = f$ and such that $F \circ (u \times id_I) = G$.

Since to a problem of cofiber homotopy equivalence in the category Top corresponds a problem of homotopy equivalence *'over'* in the category of C^*-algebras and moreover all three characterizations of a topological weak cofibration can be translated in the homotopic language of C^*-algebras, the consideration of the notion of weak cofibration for C^*-algebras appears to be very natural. This is more so as the notion of cofibration for C^*-algebras was already studied in [8] (see also [6], [7]).

We prove the equivalence of the three 'topological' characterizations (theorems 3.3 and 3.4) and we establish also some other properties. Particularly we derive a result which corresponds to a 'Dold's type theorem' in Top (Theorem 5.3).

2. The notion of homotopy for $*$-homomorphisms. By a $*$-homomorphism we shall mean a homomorphism of C^*-algebras. We recall from [1] the definition of the homotopy relation between $*$-homomorphisms.

Given a C^*-algebra A and a (locally) compact space Y, denote by AY the C^*-algebra of (vanishing at infinity) continuous functions on Y into A. If $Y = I := [0, 1]$ then, for every $t \in I$, denote by $\rho_t : AI \to A$ the $*$-homomorphism defined by $\rho_t(\alpha) = \alpha(t), \forall \alpha \in AI$. If $\phi : A \to B$ is a $*$-homomorphism and Y is a (locally)compact space, then ϕ induces a $*$-homomorphism $\phi Y : AY \to BY$ given by $\phi Y(u) = \phi \circ u, \forall u \in AY$.

Definition 2.1. (i) *The $*$-homomorphisms $\eta, \xi : A \to B$ are homotopic if there exists a $*$-homomorphism $\Phi : A \to BI$ satisfying $\rho_0 \circ \Phi = \eta$ and $\rho_1 \circ \Phi = \xi$. We shall use the notation $\eta \overset{h}{\sim} \xi$ and Φ will be called a $*$-homotopy.*

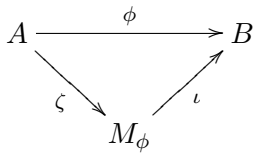
(ii) *The C^*- algebras A and B have the same homotopy type if there exists $\eta : A \to B$ and $\xi : B \to A$ verifying $\eta \circ \xi \overset{h}{\sim} id_B$ and $\xi \circ \eta \overset{h}{\sim} id_A$.*

Definition 2.2. (i) *Let $\phi : A \to B$ and $\phi' : A' \to B$ be $*$-homomorphisms. A $*$-homomorphism $\eta : A \to A'$ is called over B if it is a $*$-homomorphism in the ordinary sense and $\phi' \circ \eta = \phi$.*

(ii) *Let $\eta, \xi : A \to A'$ be $*$-homomorphisms over B. A $*$-homotopy over B of η into ξ is a $*$-homotopy $\Phi : A \to A'I$, with $\Phi : \eta \overset{h}{\sim} \xi$, which is a $*$-homomorphism over B at each stage of deformation, i.e., $\phi' \circ \rho_t \circ \Phi = \phi, \forall t \in I$. We shall use the notation $\Phi : \eta \overset{h}{\sim}_B \xi$.*

Definition 2.3. [8] (see also [6], [7]). *A $*$-homomorphism $\phi : A \to B$ is a cofibration (or a $*$-cofibration) if for a C^*-algebra D, a $*$-homomorphism $\psi : D \to A$ and a $*$-homotopy $\Phi : D \to BI$ of $\phi \circ \psi$, i.e., $\rho_0 \circ \Phi = \phi \circ \psi$, there exists a $*$-homotopy $\Psi : D \to AI$ of ψ, i.e., $\rho_0 \circ \Psi = \psi$, such that $\phi I \circ \Psi = \Phi$.*

Proposition 2.4. [6] (see also [8]). *For any $*$-homomorphism $\phi : A \to B$ there exists a commutative diagram*



$\phi = \iota \circ \zeta$, with ζ a strong deformation $*$-homomorphism and ι a $*$-cofibration.

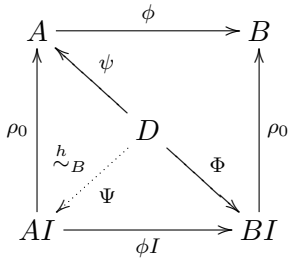
3. Weak cofibrations

Definition 3.1. *A $*$-homomorphism $\phi : A \to B$ is a weak cofibration of C^*-algebras (or a $*$-weak cofibration) if there exists a $*$-cofibration $\phi' : A' \to B$ such that A and A' (ϕ and ϕ') have the same $*$-homotopy type over B.*

Example 3.2. A $*$-homomorphism $\phi : A \to B$ is a $*$-weak cofibration if A has the same $*$-homotopy type over B with a directed sum $p_B : A' \oplus B \to B$ (cf. [7], Ex. 2.2).

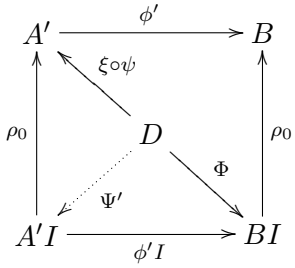
Theorem 3.3. *A $*$-homomorphism $\phi : A \to B$ is a $*$-weak cofibration if and only if it has a weak form of $*$- homotopy extension property ($*$-WHEP) in the following sense: for a C^*-algebra D, a $*$-homomorphism $\psi : D \to A$ and a $*$-homotopy $\Phi : D \to BI$ for $\phi \circ \psi$, i.e., $\rho_0 \circ \Phi = \phi \circ \psi$, there exists a $*$-homotopy $\Psi : D \to AI$ such that $\rho_0 \circ \Psi \overset{h}{\sim}_B \psi$ and $\phi I \circ \Psi = \Phi$.*

This property is schematically illustred by the following diagram



Proof. Suppose that $\phi : A \to B$ is a $*$-weak cofibration and $\psi : D \to A$, $\Phi : D \to BI$ are $*$-homomorphisms with $\rho_0 \circ \Phi = \phi \circ \psi$.

Let $\phi' : A' \to B$ be a $*$-cofibration $*$-homotopically equivalent over B with ϕ by the $*$-homomorphisms $\eta : A \to A'$ and $\xi : A' \to A$. Then we have $\phi' \circ (\xi \circ \psi) = (\phi' \circ \xi) \circ \psi = \phi \circ \psi = \rho_0 \circ \Phi$. Then by $*$-HEP for ϕ', there exists a $*$-homotopy $\Psi' : D \to A'I$, with $\rho_0 \circ \Psi' = \xi \circ \psi$ and $\phi' I \circ \Psi' = \Phi$.



We define $\Psi = \xi I \circ \Psi' : D \to AI$. This $*$-homotopy satisfies $\phi I \circ \Psi = \phi I \circ (\xi I \circ \Psi') = (\phi \circ \xi) I \circ \Psi' = \phi' I \circ \Psi' = \Phi$ and $\rho_0 \circ \Psi = \rho_0 \circ (\xi I \circ \Psi') = \xi \circ (\rho_0 \circ \Psi') = \xi \circ (\eta \circ \psi) = (\xi \circ \eta) \circ \psi \overset{h}{\sim} \psi$. Moreover, if $\Lambda : A \to AI$ is a $*$-homotopy over B between $\xi \circ \eta$ and id_A, then $\Lambda \circ \psi : D \to AI$ satisfies the relations:$\rho_0 \circ (\Lambda \circ \psi) = (\rho_0 \circ \Lambda) \circ \psi = (\xi \circ \eta) \circ \psi$, $\rho_1 \circ (\Lambda \circ \psi) = (\rho_1 \circ \Lambda) \circ \psi = id_A \circ \psi = \psi$ and $\phi \circ \rho_t \circ (\Lambda \circ \psi) = (\phi \circ \rho_t \circ \Lambda) \circ \psi = \phi \circ \psi$. Thus we obtain that $\Lambda \circ \psi$ is a $*$-homotopy, $\Lambda \circ \psi : \rho_0 \circ \Psi \overset{h}{\sim} \psi$ over B. Hence ϕ has $*$-WHEP.

Conversely, now suppose that ϕ has $*$-WHEP.

Let $M_\phi = \{(a, \alpha) \in A \oplus BI / \alpha(1) = \phi(a)\}$ be the mapping cylinder of ϕ. We know (from Prop. 2.4)that $\phi = \iota \circ \zeta$, where $\iota : M_\phi \to B$ is defined by $\iota((a, \alpha)) = \alpha(0)$ and $\zeta : A \to M_\phi$ is given by $\zeta(a) = (a, \alpha_{\phi(a)})$, with $\alpha_{\phi(a)}$ the constant path defined by $\alpha_{\phi(a)}(t) = \phi(a), \forall t \in I$. Moreover ι is a $*$-cofibration. We can prove that the C^*-algebras A and M_ϕ have the same $*$-homotopy type over B.

The $*$-homomorphism $\eta : M_\phi \to A$ defined by $\eta((a, \alpha)) = a$ satisfies the conditions $\eta \circ \zeta = id_A$ and $\zeta \circ \eta \overset{h}{\sim} id_{M_\phi}$, but η is not over B. For the $*$-homotopy $\Phi : M_\phi \to BI$, defined by $\Phi((a, \alpha)) = \widehat{\alpha}$, we have $\rho_0 \circ \Phi = \phi \circ \eta$. By $*$-WHEP, there exists $\Gamma : M_\phi \to AI$, such that $\phi I \circ \Gamma = \Phi$ and $\rho_0 \circ \Gamma \overset{h}{\sim}_B \eta$. Let $\Lambda : M_\phi \to AI$ be a corresponding $*$-homotopy, i.e., $\rho_0 \circ \Lambda = \rho_0 \circ \Gamma$, $\rho_1 \circ \Lambda = \eta$ and $\phi \circ (\rho_t \circ \Lambda) = \phi \circ \eta, \forall t \in I$.

The $*$-homomorphism $\xi = \rho_1 \circ \Gamma : M_\phi \to A$ is over B since $(\phi \circ \xi)((a, \alpha)) = \phi(\Gamma((a, \alpha))(1)) = (\phi I \circ \Gamma)((a, \alpha))(1) = \Phi((a, \alpha))(1) = \widehat{\alpha}(1) =$

$\alpha(0) = \iota((a,\alpha))$. Obviously that ζ is also over B. We shall prove that ζ and ξ define a $*$-homotopy equivalence over B between ϕ and ι.

At first, we see that a $*$-homomorphism $\Lambda' : M_\phi \to AI$ is well defined by

$$\Lambda'((a,\alpha))(t) = \begin{cases} \Lambda((a,\alpha))(1-2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \Gamma((a,\alpha))(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

and we consider the composition $\Lambda' \circ \zeta : A \to AI$. For this $*$-homotopy we have $\rho_0 \circ (\Lambda' \circ \zeta) = (\rho_0 \circ \Lambda') \circ \zeta = (\rho_0 \circ \Lambda) \circ \zeta = \eta \circ \zeta = id_A$ and $\rho_1 \circ (\Lambda' \circ \zeta) = (\rho_1 \circ \Lambda') \circ \zeta = (\rho_1 \circ \Gamma) \circ \zeta = \xi \circ \zeta$. Moreover, the $*$-homotopy, $\Lambda' \circ \zeta$ is over B, i.e., $\phi \circ \rho_t \circ (\Lambda' \circ \zeta) = \phi$, since $(\phi \circ \rho_{1-2t} \circ \Lambda \circ \zeta)(a) = (\phi \circ \rho_{1-2t} \circ \Lambda)((a, \alpha_{\phi(a)})) = (\phi \circ \eta)((a, \alpha_{\phi(a)})) = \phi(a)$ and $(\phi \circ \rho_{2t-1} \circ \Gamma \circ \zeta)(a) = (\phi \circ \rho_{2t-1} \circ \Gamma)((a, \alpha_{\phi(a)})) = (\phi I \circ \Gamma)((a, \alpha_{\phi(a)}))(2t-1) = \Phi((a, \alpha_{\phi(a)}))(2t-1) = \widehat{\alpha}_{\phi(a)}(2t-1) = \phi(a)$. Thus we have obtained that $\Lambda' \circ \zeta : id_A \overset{h}{\sim}_B \xi \circ \zeta$. The second composition is given by $(\zeta \circ \xi)((a,\alpha)) = \zeta(\Gamma((a,\alpha))(1)) = (\Gamma((a,\alpha))(1), \alpha_{\alpha(0)})$. Consider the $*$-homotopy defined by $\Psi : M_\phi \to M_\phi I$ defined by

$$\Psi((a,\alpha))(t) = \begin{cases} (\Gamma((a,\alpha))(1-2t), \alpha_{2t}), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (\Lambda((a,\alpha))(2t-1), \alpha), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

where $\alpha_t : I \to B$ is given by $\alpha_t(\tau) = \alpha(t\tau)$. Ψ is well defined in $t = \frac{1}{2}$ since $\rho_0 \circ \Gamma = \rho_0 \circ \Lambda$ and $\alpha_1 = \alpha$.

Moreover, $\phi(\Gamma((a,\alpha))(1-2t) = (\phi \circ \Gamma((a,\alpha)))(1-2t) = (\phi I \circ \Gamma)((a,\alpha))(1-2t) = \Phi((a,\alpha))(1-2t) = \widehat{\alpha}(1-2t) = \alpha(2t) = \alpha_{2t}(1)$ and $\phi(\Lambda((a,\alpha))(2t-1) = \phi(\rho_{2t-1} \circ \Lambda)((a,\alpha))) = \phi(\eta((a,\alpha))) = \phi(a) = \alpha(1)$. Thus $\Psi((a,\alpha))(t) \in M_\phi, \forall t \in I$ and therefore Ψ is well defined. At last, by a simple computation it follows that $\rho_0 \circ \Psi = \zeta \circ \xi$, $\rho_1 \circ \Psi = id_{M_\phi}$ and Ψ is a $*$-homotopy over B. This ends the proof. □

Theorem 3.4. *Let $\phi : A \to B$ be a $*$-homomorphism and $0 < \varepsilon < 1$. Then the following statements are equivalent:*

(i) *ϕ is a $*$-weak cofibration .*

(ii) *For an arbitrary C^*-algebra D, an arbitrary $*$-homomorphism $\psi : D \to A$ and a 'semistationary' $*$-homotopy $\Phi : D \to AI$, satisfying $\rho_t \circ \Phi = \phi \circ \psi$ for any $t \in [0, \varepsilon]$, there exists a $*$-homotopy $\Psi : D \to AI$ with the properties : $\rho_0 \circ \Psi = \psi$ and $\phi I \circ \Psi = \Phi$.*

Proof. For (i)⇒ (ii), let $\phi : A \to B$ be a $*$-weak cofibration. Let ψ and Φ be $*$-homomorphisms as in the statement (ii).

For the $*$-homotopy $\Phi' : D \to BI$, defined by $\Phi'(d)(t) = \Phi(d)((1-\varepsilon)t+\varepsilon)$, we have $\rho_0 \circ \Phi' = \rho_\varepsilon \circ \Phi = \phi \circ \psi$. By $*$-WHEP for ϕ, there exists a $*$-homotopy $\Gamma : D \to AI$ such that $\rho_0 \circ \Gamma \overset{h}{\sim}_B \psi$, by a $*$-homotopy $\Lambda : D \to AI$, and $\phi I \circ \Gamma = \Phi'$. Denote $J = [0, \varepsilon]$ and $J' = [\varepsilon, 1]$. The $*$-homomorphism $\Psi' : D \to AJ'$, defined by $\Psi'(d)(t) = \Gamma(d)(\frac{t-\varepsilon}{1-\varepsilon})$, satisfies $\phi I \circ \Psi' = \Phi$, since $(\phi I \circ \Psi')(d)(t) = (\phi I)(\Gamma(d)(\frac{t-\varepsilon}{1-\varepsilon})) = (\phi I \circ \Gamma)(d)(\frac{t-\varepsilon}{1-\varepsilon}) = \Phi'(d)(\frac{t-\varepsilon}{1-\varepsilon}) = \Phi(d)(t)$. Similarly, the $*$-homomorphism $\Psi'' : D \to AJ$, defined by $\Psi''(d)(t) = \Lambda(d)(\frac{\varepsilon-t}{\varepsilon})$, satisfies $\phi I \circ \Psi'' = \Phi$. Moreover, $\rho_\varepsilon \circ \Psi' = \rho_\varepsilon \circ \Psi''$. Thus, there exists a $*$-homotopy $\Psi : D \to AI$ well defined by $\Psi(d)|_{[0,\varepsilon]} = \Psi''(d)$ and $\Psi(d)|_{[\varepsilon,1]} = \Psi'(d)$, $\forall d \in D$. This satisfies $\phi I \circ \Psi = \Phi$ and $\rho_0 \circ \Psi = \rho_0 \circ \Psi'' = \rho_1 \circ \Lambda = \psi$.

For (ii)$\Rightarrow$(i) it is sufficient to consider $\varepsilon = \frac{1}{2}$. We shall prove that ϕ has $*$-WHEP. If $\psi : D \to A$ and $\Phi : D \to BI$ are $*$-homomorphisms such that $\rho_0 \circ \Phi = \phi \circ \psi$, then we can apply (ii) for ψ and the semistationary $*$-homotopy $\Phi : D \to BI$, defined by $\Phi'(d)(t) = \Phi(d)(max(2t-1, 0))$. Then there exists $\Psi' : D \to AI$ satisfying the conditions $\rho_0 \circ \Psi' = \psi$ and $\phi I \circ \Psi' = \Phi'$. Define a $*$-homotopy $\Psi : D \to AI$, by $\Psi(d)(t) = \Psi'(d)(\frac{1+t}{2})$. Then $\phi I \circ \Psi = \Phi$, since $(\phi I \circ \Psi)(d)(t) = \phi(\Psi'(d)(\frac{1+t}{2})) = (\phi \circ \Psi')(d)(\frac{1+t}{2}) = \Phi'(d)(\frac{1+t}{2}) = \Phi(d)(t)$. It remains to see that $\rho_0 \circ \Psi \overset{h}{\sim}_B \psi$. For this consider the $*$-homotopy $\Lambda : D \to AI$, defined by $\Lambda(d)(t) = \Psi'(d)(\frac{1-t}{2})$, which verifies $\rho_0 \circ \Lambda = \rho_{\frac{1}{2}} \circ \Psi' = \rho_0 \circ \Psi$ and $\rho_1 \circ \Lambda = \rho_0 \circ \Psi' = \psi$. Moreover, the $*$-homotopy $\Lambda : \rho_0 \circ \Psi \overset{h}{\sim} \psi$ is over B, since $(\phi \circ \rho_t \circ \Lambda)(d) = \phi(\Psi'(d)(\frac{1-t}{2})) = \Phi'(d)(\frac{1-t}{2}) = \Phi(d)(max(-t, 0)) = \Phi(d)(0) = (\phi \circ \psi)(d)$, $\forall d \in D$. Thus we have proved that ϕ has $*$-WHEP and by Th.3.3 the proof of this theorem is complete. □

4. Properties

Proposition 4.1. *If $\phi : A \to B$ is a $*$-homomorphism of C^*-algebras, $\phi' : A' \to B$ is a $*$-weak cofibration, $\eta : A' \to A$ is a $*$-homomorphism over B and η admits a $*$-homotopy right inverse over B, then ϕ is a $*$- weak cofibration.*

Proof. Let $\eta' : A \to A'$ be a $*$-homotopy right inverse over B of η. Then $\phi \circ \eta = \phi'$, $\phi' \circ \eta' = \phi$ and $\eta \circ \eta' \overset{h}{\sim}_B id_A$. If D is a C^*-algebra and $\psi : D \to A$, $\Phi : D \to BI$ are $*$-homomorphisms such that $\rho_0 \circ \Phi = \phi \circ \psi$, then $\phi' \circ (\eta' \circ \psi) = (\phi' \circ \eta') \circ \psi = \phi \circ \psi = \rho_0 \circ \Phi$. The $*$-weak cofibration

ϕ' has $*$-WHEP (Th. 3.3) which implies that there exists $\Psi' : D \to A'I$ satisfying $\phi' I \circ \Psi' = \Phi$ and $\rho_0 \circ \Psi' \overset{h}{\sim}_B \eta' \circ \psi$. Define $\Psi = \eta I \circ \Psi'$. We have $\phi I \circ \Psi = (\phi I \circ \eta I) \circ \Psi' = \phi' I \circ \Psi' = \Phi$ and $\rho_0 \circ \Psi = \rho_0 \circ (\eta I \circ \Psi') = \eta \circ (\rho_0 \circ \Psi') \overset{h}{\sim}_B \eta \circ \eta' \circ \psi \overset{h}{\sim}_B \psi$. By Theorem 3.3 it follows that ϕ is indeed a $*$-weak cofibration. □

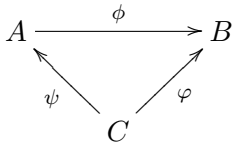
Proposition 4.2. *Let $\phi : A \to B$ be a $*$-weak cofibration. Let $\phi' : A' \to B$ and $\psi : A' \to A$ be $*$-homomorphisms such that $\phi \circ \psi \overset{h}{\sim} \phi'$. Then there exists a $*$-homomorphism over B, $\psi' : A' \to A$, such that $\psi' \overset{h}{\sim} \psi$.*

Proof. Let $\Phi : A' \to BI$ be a $*$-homotopy with $\rho_0 \circ \Phi = \phi \circ \psi$ and $\rho_1 \circ \Phi = \phi'$. By $*$-WHEP, there exists $\Psi : A' \to AI$ with the following properties: $\phi I \circ \Psi = \Phi$ and $\rho_0 \circ \Psi \overset{h}{\sim}_B \psi$. Define $\psi' = \rho_1 \circ \Psi$. Then we have $\psi' \overset{h}{\sim} \rho_0 \circ \Psi \overset{h}{\sim} \psi$. The $*$-homomorphism ψ' is over B, since $\phi \circ \psi' = \phi \circ (\rho_1 \circ \Psi) = \rho_1 \circ \phi I \circ \Psi = \rho_1 \circ \Phi = \phi'$. □

Corollary 4.3. *Let $\phi : A \to B$ be a $*$-weak cofibration. If ϕ admits a right inverse up to $*$-homotopy, then ϕ admits a right inverse.*

Proof. The assertion is just a special case of Prop.4.2, for ϕ, id_B and ψ. Thus there exists ψ' a $*$-homomorphism over B, hence $\phi \circ \psi' = id_B$. □

Proposition 4.4. *Let a $*$-homotopy commutative diagram be given:*



Suppose that ϕ is a $*$-weak cofibration. Then A is a 'canonical retract' of the relative mapping cylinder $M_\varphi(\phi)$.

Particularly, if ϕ is a $*$-weak cofibration and it admits a $*$-homotopy right inverse, then A is a canonical retract of the cylinder AI.

Proof. For the relative mapping cylinder, $M_\varphi(\phi) = \{(c, \gamma) \in C \oplus AI / \phi(\gamma(1)) = \varphi(c)\}$, there exists the projection $\pi_C : M_\varphi(\phi) \to C$, $\pi_C((c, \gamma)) = c$. We need to define a $*$-homomorphism $\theta : C \to M_\varphi(\phi)$ such that $\pi_C \circ \theta = id_C$.

Let $\Phi : C \to BI$ be a $*$-homotopy for the equivalence $\phi \circ \psi \overset{h}{\sim} \varphi$. Since $\rho_0 \circ \Phi = \phi \circ \psi$ and ϕ has $*$-WHEP, there exists $\Psi : C \to AI$, a $*$-homotopy,

with $\phi I \circ \Psi = \Phi$ and $\rho_0 \circ \Psi \overset{h}{\sim}_B \psi$. We can define a $*$-homomorphism $\theta : C \to M_\varphi(\phi)$, by $\theta(c) = (c, \Psi(c))$. We have $\gamma = \Psi(c)$ and $\phi(\gamma(1)) = \rho_1((\phi I \circ \Psi)(c)) = \rho_1(\Phi(c)) = \varphi(c)$. Thus θ is well defined and $\pi_C \circ \theta = id_C$.

For the particular case, we take $C = B$ and $\varphi = id_B$. Then $M_\varphi(\phi)$ can be identified with AI. □

Proposition 4.5. i) *If $\phi : A \to B$ is a $*$-weak cofibration and $\psi : B \to C$ is a $*$-cofibration, then $\psi \circ \phi$ is a $*$-weak cofibration.*

ii) *If $\phi : A \to B$ is a $*$-weak cofibration and $\eta : A' \to A$ is a $*$-homotopy equivalence over B, then $\phi \circ \eta$ is a $*$-weak cofibration.*

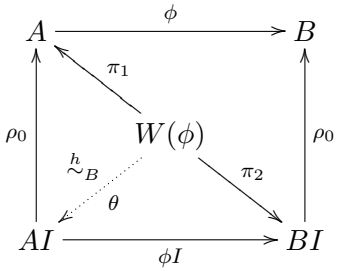
Proof. i) The composition of two $*$-cofibrations is also a $*$-cofibration, (see [7]). If $\phi' : A_1 \to B$ is a $*$-cofibration , then $\psi \circ \phi'$ is a $*$-cofibration . Moreover, if A_1 and A have the same $*$-homotopy type over B, they have also the same homotopy type over C.

ii) If $\phi' : A_1 \to B$ is a $*$-cofibration of the same $*$-homotopy type over B as ϕ then $\phi \circ \eta$ have the same homotopy type over B as ϕ. □

Proposition 4.6. *Let $\phi : A \to B$ be a $*$-homomorphism. Consider the pull-back of B along the pair $(\phi, \rho_0 : BI \to B)$, i.e., $W(\phi) = A(B; \phi, \rho_0) = \{(a, \beta) \in A \oplus BI / \phi(a) = \beta(0)\}$, ([1], p. 21), and the $*$-homomorphism $k : AI \to W(\phi)$, defined by $k(\alpha) = (\alpha(0), \phi \circ \alpha)$.*

If ϕ is a $*$-weak cofibration, k admits a $*$-homotopy right inverse over B.

Proof. The $*$-homomorphisms $\pi_1 : W(\phi) \to A$, $\pi_1((a, \beta)) = a$, and $\pi_2 : W(\phi) \to BI$, $\pi_2((a, \beta)) = \beta$, satisfy the relation $\rho_0 \circ \pi_2 = \phi \circ \pi_1$. By $*$-WHEP, there exists a $*$-homotopy $\theta : W(\phi) \to AI$, such that $\rho_0 \circ \theta \overset{h}{\sim}_B \pi_1$ and $\phi I \circ \theta = \pi_2$.



Let $\Phi : W(\phi) \to AI$ be a $*$-homotopy such that $\rho_0 \circ \Phi = \rho_0 \circ \theta$, $\rho_1 \circ \Phi = \pi_1$ and $\phi \circ (\rho_t \circ \Phi) = \phi \circ \pi_1$. We have $(k \circ \theta)((a, \beta)) = ((\rho_0 \circ \theta)((a, \beta))$,

$(\phi I \circ \theta)((a,\beta))) = ((\rho_0 \circ \theta)((a,\beta)), \pi_2((a,\beta)) = ((\rho_0 \circ \theta)((a,\beta)),\beta)$. We shall prove that $k \circ \theta \overset{h}{\sim} id_{W(\phi)}$.

The $*$-homotopy $\Gamma : W(\phi) \to W(\phi)I$, with $\Gamma((a,\beta))(t) = (\Phi((a,\beta))(t),\beta)$, is well defined since $\phi(\Phi((a,\beta))(t)) = (\phi\circ(\rho_t\circ\Phi))((a,\beta)) = (\phi\circ\pi_1)((a,\beta)) = \phi(a) = \beta(0)$. Moreover, $(\rho_0 \circ \Gamma)((a,\beta)) = ((\rho_0 \circ \Phi)((a,\beta)),\beta) = (k \circ \theta)((a,\beta))$. Thus, Γ is a $*$-homotopy over B for the equivalence $k\circ\theta\overset{h}{\sim}_B id_{W(\phi)}$.□

Proposition 4.7. *If one of the $*$-homomorphisms $\phi : A \to B$ and $\psi : C \to B$ is a $*$-weak cofibration, then the double mapping cylinder $M_{(\phi,\psi)}$ and the pull-back $A(B;\phi,\psi)$ have the same $*$-homotopy type.*

Proof. The double mapping cylinder is $M_{(\phi,\psi)} := \{(a,c,\beta) \in A \oplus C \oplus BI/\phi(a) = \beta(0), \psi(c) = \beta(1)\}$, [6], and the pull-back is $A(B;\phi,\psi) = \{(a,c) \in A \oplus C/\phi(a) = \psi(c)\}$, ([1], p. 21). Suppose that ϕ is a $*$-weak cofibration. Consider the $*$-homomorphisms $k : AI \to W(\phi)$, $\theta : W(\phi) \to AI$ and $\Phi : W(\phi) \to AI$ defined in the proof of Prop. 4.6, with the properties: $(\phi \circ \theta)((a,\beta)) = \beta$, $\Phi((a,\beta))(0) = \theta((a,\beta))(0)$, $\Phi((a,\beta))(1) = a$ and $\phi(\Phi((a,\beta))(t) = \phi(a)$. We shall prove that the $*$-homomorphism

$\eta : A(B;\phi,\psi) \to M_{(\phi,\psi)}$, $\eta((a,c)) = (a,c,\beta_{\phi(a)})$, with $\beta_{\phi(a)}(t) = \phi(a), \forall t \in I$, and $\xi : M_{(\phi,\psi)} \to A(B;\phi,\psi)$, $\xi((a,c,\beta)) = (\theta((a,\beta))(1),c)$ establish a $*$-homotopy equivalence.

We have $(\xi \circ \eta)((a,c)) = \xi((a,c,\beta_{\phi(a)})) = (\theta((a,\beta_{\phi(a)}))(1),c)$ and we define $\Lambda : A(B;\phi,\psi) \to A(B;\phi,\psi)I$, a $*$-homotopy for $id_{A(B;\phi,\psi)} \overset{h}{\sim} \xi \circ \eta$, by

$$\Lambda((a,c))(t) = \begin{cases} (\Phi((a,\beta_{\phi(a)}))(1-2t),c), & \text{if } 0 \le t \le \frac{1}{2}, \\ (\theta((a,\beta_{\phi(a)}))(2t-1),c), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

At first, Λ is well defined, since $\phi(\Phi((a,\beta_{\phi(a)}))(1-2t)) = \phi(a) = \psi(c)$, $\phi(\theta((a,\beta_{\phi(a)}))(2t-1) = (\phi \circ \theta((a,\beta_{\phi(a)})))(2t-1) = \beta_{\phi(a)}(2t-1) = \phi(a) = \psi(c)$, and for $t = \frac{1}{2}$ the two expressions are equal: $\Phi((a,\beta_{\phi(a)}))(0) = \theta((a,\beta_{\phi(a)}))(0)$. We can easily verify that $\rho_0 \circ \Lambda = id_{A(B;\phi,\psi)}$ and $\rho_1 \circ \Lambda = \xi \circ \eta$.

The second composition is

$$(\eta \circ \xi)((a,c,\beta)) = \eta((\theta((a,\beta))(1),c)) = (\theta((a,\beta))(1),c,\beta_{\psi(c)})$$

and a $*$-homotopy for $id_{M_{(\phi,\psi)}} \overset{h}{\sim} \eta \circ \xi$ is $\Gamma : M_{(\phi,\psi)} \to M_{(\phi,\psi)}I$, with

$$\Gamma((a,c,\beta))(t) = \begin{cases} (\Phi((a,\beta))(1-2t),c,\beta)), & \text{if } 0 \le t \le \frac{1}{2}, \\ (\theta((a,\beta))(2t-1),c,\beta_{2t-1}), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

where $\beta_t : I \to B$ is defined by $\beta_t(\tau)) = \beta(1-(1-t)(1-\tau))$.

If $t = \frac{1}{2}$,we have $\Phi((a,\beta))(0) = \theta((a,\beta))(0)$ and $\beta_0 = \beta$.

For $0 \leq t \leq \frac{1}{2}$, $\phi(\Phi((a,\beta))(1-2t) = \beta(0)$ and $\beta(1) = \psi(c)$. For $\frac{1}{2} \leq t \leq 1$, $\phi(\theta((a,\beta))(2t-1)) = \beta(2t-1) = \beta_{2t-1}(0)$ and $\beta_{2t-1}(1) = \beta(1) = \psi(c)$. Thus Γ is well defined. Moreover, $\rho_0 \circ \Gamma = id_{M_{(\phi,\psi)}}$ and $(\rho_1 \circ \Gamma)((a,c,\beta)) = (\theta((a,\beta))(1),c,\beta) = (\theta((a,\beta))(1),c,\beta_{\psi(c)}) = (\eta \circ \xi)((a,c,\beta))$, since $\beta_1(\tau) = \beta(1) = \psi(c) = \beta_{\psi(c)}(\tau)$. Hence $\rho_1 \circ \Gamma = \eta \circ \xi$. □

Corollary 4.8. *If $\phi, \phi' : A \to A'$ are $*$-weak cofibrations and $\psi, \psi' : C \to B$ are $*$-homomorphisms satisfying $\phi \overset{h}{\sim} \phi'$, $\psi \overset{h}{\sim} \psi'$, then the pull-backs $A(B;\phi,\psi)$ and $A(B;\phi',\psi')$ have the same $*$-homotopy type.*

Proof. If $\Lambda : C \to BI$ is a $*$-homotopy with $\rho_0 \circ \Lambda = \psi$ and $\rho_1 \circ \Lambda = \psi'$, then the $*$-homomorphisms $\eta : M_{(\phi,\psi)} \to M_{(\phi,\psi')}$, $\eta((a,c,\beta)) = (a,c,\beta * \Lambda(c))$, and $\xi : M_{(\phi,\psi')} \to M_{(\phi,\psi)}$, $\xi((a,c,\beta)) = (a,c,\beta * \widehat{\Lambda(c)})$, define a $*$-homotopy equivalence. (Here $*$ denotes the usual composition of paths and $\widehat{\Lambda(c)}$ is the inverse path of $\Lambda(c)$ in B). Similarly, from $\phi \overset{h}{\sim} \phi'$ it follows $M_{(\phi,\psi')} \overset{h}{\sim} M_{(\phi'\psi')}$ and we conclude that $M_{(\phi,\psi)} \overset{h}{\sim} M_{(\phi'\psi')}$.

By Prop. 4.7, for ϕ a $*$-weak cofibration, we have $M_{(\phi,\psi')} \overset{h}{\sim} A(B;\phi,\psi)$ and for ϕ' a $*$-weak cofibration, $M_{(\phi',\psi')} \overset{h}{\sim} A(B;\phi',\psi')$. By these $*$-homotopy equivalences we obtain that $A(B;\phi,\psi)$ and $A(B;\phi',\psi')$ have the same $*$-homotopy type. □

Proposition 4.9. *If $\phi : A \to B$ is a $*$-weak cofibration and $\eta : A' \to B$ is a $*$-homomorphism, then $\pi_2 : A(B;\phi,\eta) \to A'$, $\pi_2((a,a')) = a'$, is a $*$-weak cofibration.*

Proof. We shall denote by $C := A(B;\phi,\eta) = \{(a,a') \in A \oplus A'/\phi(a) = \eta(a')\}$ the pull-back and π_1, π_2 the projections, i.e., $\pi_1((a,a')) = a$, $\pi_2((a,a')) = a'$.

By Th. 3.3, it is enough to prove that π_2 has $*$-WHEP.

Let D be a C^*-algebra and $\psi : D \to C$, $\Phi \to A'I$ be such that $\rho_0 \circ \Phi = \pi_2 \circ \psi$. It follows that $\rho_0 \circ (\eta I \circ \Phi) = \phi \circ (\pi_1 \circ \psi)$.

The $*$-weak cofibration ϕ has $*$-WHEP, thus there exists $\Gamma : D \to AI$ such that $\phi I \circ \Gamma = \eta I \circ \phi$ and $\rho_0 \circ \Gamma \overset{h}{\sim} \pi_1 \circ \psi$, by a $*$-homotopy $\Lambda : D \to AI$ satisfying $\rho_0 \circ \Lambda = \rho_0 \circ \Gamma$, $\rho_1 \circ \Lambda = \pi_1 \circ \psi$ and $\phi \circ (\rho_t \circ \Lambda) = \phi \circ (\pi_1 \circ \psi), \forall t \in I$. A$*$-homotopy $\Psi : D \to CI$ is well defined by $\Psi(d)(t) = (\Gamma(d)(t), \Phi(d)(t))$,

since $\phi(\Gamma(d)(t)) = (\phi \circ \Gamma(d))(t) = (\phi I \circ \Gamma)(d)(t) = (\eta I \circ \Phi)(d)(t) = (\eta \circ \Phi(d))(t) = \eta(\Phi(d))(t)$. Thus the condition $\pi_2 I \circ \Psi = \Phi$ is fulfilled.

We define a $*$-homotopy $\Lambda' : D \to CI$, by $\Lambda'(d)(t) = (\Lambda(d)(t), (\pi_2 \circ \psi)(d))$. If $\psi(d) = (a, a')$, then $\phi(\Lambda(d)(t)) = (\phi \circ \rho_t \circ \Lambda)(d) = (\phi \circ \pi_1 \circ \psi)(d) = \phi(a)$ and $(\eta \circ (\pi_2 \circ \psi))(d) = \eta(a') = \phi(a)$. Thus $\Lambda'(d)(t) \in C = A(B; \phi, \eta)$ and Λ' is well defined. Then we have $(\rho_0 \circ \Lambda')(d) = ((\rho_0 \circ \Lambda)(d), (\pi_2 \circ \psi)(d)) = ((\rho_0 \circ \Gamma)(d), (\rho_0 \circ \Phi)(d)) = (\rho_0 \circ \Psi)(d)$ and $(\rho_1 \circ \Lambda')(d) = ((\rho_1 \circ \Lambda)(d), (\pi_2 \circ \psi)(d)) = ((\pi_1 \circ \psi)(d), (\pi_2 \circ \psi)(d)) = \psi(d)$. Thus we have $\Lambda' : \rho_0 \circ \Psi \overset{h}{\sim} \psi$, and the $*$-homotopy Λ' is over A', since $\pi_2 \circ (\rho_t \circ \Lambda') = \pi_2 \circ \psi, \forall t \in I$. We can conclude that π_2 has $*$-WHEP and therefore it is a $*$-weak cofibration. □

Proposition 4.10. *If $\phi : A \to B$ is a $*$-weak cofibration, then $\phi I : AI \to BI$ is also a $*$-weak cofibration.*

Proof. Let $\phi' : A' \to B$ be a $*$-cofibration $*$-homotopically equivalent with ϕ over B. Denote by $\eta : A \to A'$ and $\xi : A' \to A$ two $*$-homomorphisms over B establishing a $*$-homotopy equivalence over B. Consider the $*$-homomorphism $\phi' : A'I \to BI$, which by [7], Cor. 3.6, is a $*$-cofibration. We can prove that this is $*$-homotopically equivalent with ϕI over BI. The $*$-homomorphisms which establish this equivalence are $\eta I : AI \to A'I$ and $\xi I : A'I \to AI$. These are $*$-homomorphisms over BI: if $\alpha \in AI$, $(\phi' I \circ \eta I)(\alpha) = \phi' I(\eta \circ \alpha) = \phi' \circ \eta \circ \alpha = \phi \circ \alpha = \phi I(\alpha)$, i.e., $\phi' I \circ \eta I = \phi I$, and analogously, $\phi I \circ \xi I = \phi' I$. Moreover, $\xi I \circ \eta I = (\xi \circ \eta) I$. Then, by hypothesis, there exists a $*$-homotopy $\Lambda : A \to AI$ satisfying the conditions:

(i) $\rho_0 \circ \Lambda = \xi \circ \eta$.

(ii) $\rho_1 \circ \Lambda = id_A$.

(iii) $\phi \circ \rho_t \circ \Lambda = \phi, \forall t \in I$.

Consider $\Lambda I : AI \to (AI)I$ and $\widetilde{\Lambda} : AI \to (AI)I$ defined by $\widetilde{\Lambda}(\alpha)(t)(\tau) = \Lambda I(\alpha)(\tau)(t), \forall \alpha \in AI, \forall t, \tau \in I$. These are $*$-homomorphisms. For $\alpha \in AI$ and $t \in I$, we have $(\rho_t \circ \widetilde{\Lambda})(\alpha) = \rho_t \circ \Lambda \circ \alpha$. It follows:

(i)' $(\rho_0 \circ \widetilde{\Lambda})(\alpha) = \rho_0 \circ \Lambda \circ \alpha = \xi \circ \eta \circ \alpha = ((\xi \circ \eta) I)(\alpha)$, i.e., $\rho_0 \circ \widetilde{\Lambda} = \xi I \circ \eta I$.

(ii)'$(\rho_1 \circ \widetilde{\Lambda})(\alpha) = \rho_1 \circ \Lambda \circ \alpha = id_A \circ \alpha = \alpha$, i.e., $\rho_1 \circ \widetilde{\Lambda} = id_{AI}$.

(iii)' $(\phi I \circ \rho_t \circ \widetilde{\Lambda})(\alpha) = \phi I(\rho_t \circ \Lambda \circ \alpha) = \phi \circ \rho_t \circ \Lambda \circ \alpha = \phi \circ \alpha = \phi I(\alpha)$, i.e., $\phi I \circ \rho_t \circ \widetilde{\Lambda} = \phi I$.

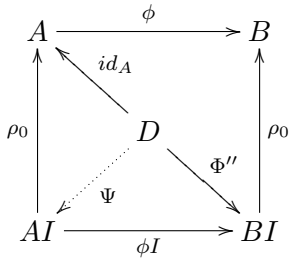
Therefore we have proved that $\widetilde{\Lambda}$ is $*$-homotopy over BI between $\xi I \circ \eta I$ and id_{AI}. Analogously, using a $*$-homotopy over B between $\eta \circ \xi$ and $id_{A'}$, we obtain a $*$-homotopy over BI between $\eta I \circ \xi I$ and $id_{A'I}$. □

5. A Dold type theorem via $*$-WHEP. This section is devoted to prove a Dold type theorem for $*$-homomorphisms. At first we give two lemmas necessary for the proof.

Lemma 5.1. *If $\phi : A \to B$ is a $*$-weak cofibration and $\eta : A \to A$ is a $*$-homomorphism over B such that $\eta \overset{h}{\sim} id_A$, then η admits a $*$-homotopy right inverse over B.*

Proof. If $\Phi : A \to AI$ is a $*$-homotopy such that $\rho_0 \circ \Phi = \eta$ and $\rho_1 \circ \Phi = id_A$, then we define a semistationary $*$-homotopy $\Phi' : A \to AI$, by $\Phi'(a)(t) = \Phi(a)(max(2t-1,0), \forall a \in A, \forall t \in I$. Then $\Phi'(a)(t) = \eta(a)$ for any $t \in [0, \frac{1}{2}]$ and $\rho_1 \circ \Phi' = id_A$.

The $*$-homomorphism $\Phi'' = \phi I \circ \Phi' : A \to BI$ is a $*$-homoptopy for an equivalence $\phi \overset{h}{\sim} \phi$. For any $t \in [0, \frac{1}{2}]$ and $\forall a \in A$, we have $(\rho_t \circ \Phi'')(a) = (\phi I \circ \Phi')(a)(t) = (\phi \circ \Phi'(a))(t) = \phi(\eta(a)) = \phi(a) = (\phi \circ id_A)(a)$. By Th.3.3, there exists $\Psi : A \to AI$ with the properties $\rho_0 \circ \Psi = id_A$ and $\phi I \circ \Psi = \Phi''$.



Then $\xi = \rho_1 \circ \Psi$ is a $*$-homomorphism over B, since $\phi \circ \xi = \phi(\rho_1 \circ \Psi) = \rho_1 \circ (\phi I \circ \Psi) = \rho_1 \circ \Phi'' = \phi$.

The $*$-homotopy $\Gamma : A \to AI$, given by

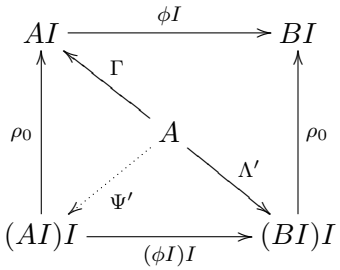
$$\Gamma(a)(\tau) = \begin{cases} \eta(\Psi(a)(1-2\tau)), & \text{if } 0 \leq \tau \leq \frac{1}{2}, \\ \Phi'(a)(2\tau - 1), & \text{if } \frac{1}{2} \leq \tau \leq 1. \end{cases}$$

is well defined since for $\tau = \frac{1}{2}$ we have $\eta \circ (\rho_0 \circ \Psi) = \eta = \rho_0 \circ \Psi'$.

Moreover, $\Gamma : \eta \circ \xi \overset{h}{\sim} id_A$. But this $*$-homotopy is not over B. Then we consider $\Lambda : A \to (BI)I$, defined by:

$$\Lambda(a)(t)(\tau) = \begin{cases} \phi(\Psi(a)(1-2\tau(1-t))), & \text{if } 0 \leq \tau \leq \frac{1}{2}, \\ \phi(\Phi'(a)(1-2(1-\tau)(1-t))), & \text{if } \frac{1}{2} \leq \tau \leq 1. \end{cases}$$

For $\tau = \frac{1}{2}$ we have $\phi(\Psi(a)(t)) = (\phi I \circ \Psi)(a)(t) = \Phi''(a)(t) = (\phi I \circ \Phi')(a)(t) = \phi(\Phi'(a)(t))$. Thus, Λ is well defined. It follows by computation that $\Lambda(a)(0)(\tau) = \phi(\Gamma(a)(\tau))$ and $\Lambda(a)(t)(0) = \phi(a)$. Then the $*$-homotopy $\Lambda' : A \to (BI)I$, $\Lambda'(a)(t) = \Lambda(a)(max(2t-1), 0))$ satisfies $\rho_t \circ \Lambda' = \phi \circ \Gamma$ for any $t \in [0, \frac{1}{2}]$. Now we use the result of Prop. 4.10 from which ϕI is a $*$-weak cofibration. Then by Th. 3.4 there exists $\Psi' : A \to (AI)I$ such that $\rho_0 \circ \Psi' = \Gamma$ and $(\phi I)I \circ \Psi' = \Lambda'$.



Then: $\eta \circ \xi = \eta \circ (\rho_1 \circ \Psi) = \rho_0 \circ \Gamma = \rho_0 \circ (\rho_0 \circ \Psi') \overset{h}{\sim} \rho_0 \circ (\rho_1 \circ \Psi') \overset{h}{\sim} \rho_1 \circ (\rho_1 \circ \Psi') \overset{h}{\sim} \rho_1 \circ (\rho_0 \circ \Psi') = \rho_1 \circ \Gamma = id_A$. All $*$-homotopies are over B, since $(\phi \circ \rho_0 \circ (\rho_t \circ \Psi'))(a) = \phi((\Psi'(a)(t)(0) = \Lambda'(a)(t)(0) = \phi(a)$, $(\phi \circ \rho_\tau \circ (\rho_1 \circ \Psi'))(a) = \Lambda'(a)(1)(\tau) = \Lambda(a)(1)(\tau) = \Phi''(a)(1) = \phi((\rho_1 \circ \Phi')(a)) = \phi(id_A(a)) = \phi(a)$ and $(\phi \circ \rho_1 \circ (\rho_t \circ \Psi'))(a) = \Lambda'(a)(t)(1) = \Lambda(a)(max(2t-1, 0))(1) = \phi(\Phi'(a)(1)) = \phi(a)$.

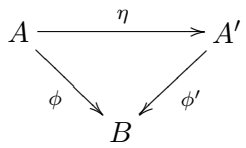
Thus we have obtained the $*$-homotopy equivalence over B, $\eta \circ \xi \overset{h}{\sim}_B id_A$, which means that ξ is a $*$-homotopy right inverse of η over B. □

Lemma 5.2. *If $\phi : A \to B$ and $\phi' : A' \to B$ are $*$-weak cofibrations, $\eta : A \to A'$ is a $*$-homomorphism over B and η admits a $*$-homotopy right inverse, then η admits a $*$-homotopy right inverse over B.*

Proof. Let ξ be a $*$-homotopy right inverse for η. We have $\phi' \circ \eta = \phi$ and $\phi \circ \xi = (\phi' \circ \eta) \circ \xi \overset{h}{\sim} \phi'$. By Prop. 4.2 there exists $\xi' : A' \to A$, a $*$-homomorphism over B such that $\xi' \overset{h}{\sim} \xi$. The $*$-homomorphism $\theta = \eta \circ \xi'$ is over B an $\theta \overset{h}{\sim} id_A$. By Lemma 5.1, θ admits a $*$-homotopy right inverse θ' over B. Then $\xi' \circ \theta'$ is a $*$-homomorphism over B and $\eta \circ (\xi' \circ \theta') \overset{h}{\sim}_B id_A$.□

We conclude with the following Dold type theorem, [3].

Theorem 5.3. *Let a commutative diagram of C^*-algebras and $*$-homomorphisms be given:*



If ϕ and ϕ' are $*$-weak cofibrations and η is a simple $*$-homotopy equivalence, then η is a $*$-homotopy equivalence over B.

Proof. The $*$-homomorphism η is over B and admits a $*$-homotopy right inverse. By Prop. 5.2, η admits a $*$-homotopy right inverse η' over B. In fact η' is a $*$-homotopy equivalence since η is a $*$-homotopy equivalence. Now by the same argument, η' admits a $*$-homotopy right inverse η'' over B. Thus, η' both admits a $*$-homotopy right inverse η'' over B and a $*$-homotopy left inverse η over B. Hence η' is a $*$-homotopy equivalence over B and so η itself is a $*$-homotopy equivalence over B. □

Remark 5.4. A categorical approach of weak cofibrations and Dold type theorems was given in [5].

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Faculty of Mathematics,
"Al.I. Cuza" University,
700505-Iasi,
ROMÂNIA
tofan_alina 2006@yahoo.com