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WEAK FIBRATIONS FOR C^*-ALGEBRAS

BY

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Abstract. In this paper we define a notion of weak fibration for C*-algebras and as an application we prove a Dold's type theorem [3].

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Key words: C^*-algebra, $*$-homomorphism, homotopy for $*$-homomorphisms, weak fibration, *WHLP*, semistationary homotopy, weak fibration of C^*-algebras, fibration of C^*-algebras, homotopy equivalence under, shrinkable $*$-homomorphism, Dold's type theorem.

1. Introduction. The notion of weak fibration for topological spaces was introduced by Dold [3] and Weinzweig [6] in connection with the study of so called 'spaces over'. A continuous map $p : E \to B$ is called a *space over* B. If $p : E \to B$, $p' : E' \to B$ are spaces over B then a continuous map $f : E \to E'$ is called a *map over* B if $p' \circ f = p$. The spaces over B and the maps over B under ordinary composition form a category $\mathcal{C}_B$. A homotopy $\Theta : E \times I \to E'$ is called a *homotopy overB* or a *vertical homotopy* if $\Theta_t : E \to E'$, $\Theta_t(e) = \Theta(e,t)$ is a map over B for every $t \in I = [0,1]$. Two maps $f_0, f_1 : E \to E'$ are *vertical homotopic*, $f_0 \simeq_B f_1$, if there exists a vertical homotopy Θ with $\Theta_0 = f_0, \Theta_1 = f_1$. By identifying equivalent maps, one gets a new category $\overline{\mathcal{C}}_B$ whose objects are those of $\mathcal{C}_B$ and whose morphisms are vertical homotopy classes of maps over B. A map $f : E \to E'$ over B whose class in $\overline{\mathcal{C}}_B$ is an equivalence (i.e., has left and right inverses) is called a *fibre homotopy equivalence*. A *weak fibration* $p : E \to B$ satisfies a weak form of the homotopy lifting (covering) property (*WHLP* or *WCHP*). This property can be formulated in two equivalent ways (see [4], Ch. 6, and [2], Ch. II). Either way a space X, a map

$g : X \to E$ and a homotopy $\overline{G} : X \times I \to B$ with $(p \circ g)(x) = \overline{G}(x, 0)$ are given. In the first formulation the property is that there exists a homotopy $G : X \times I \to E$ with $p \circ G = \overline{G}$ and such that $G_0 : X \to E$, $G_0(x) = G(x, 0)$, is vertically homotopic to g, $G_0 \simeq_B g$. In the second formulation one requires that $\overline{G}$ is *semistationary*(in the sense that $\overline{G}(x, .)$ is stationary on $[0, \frac{1}{2}]$ for each point $x \in X$) and then the property is that there exists a homotopy $H : X \times I \to E$ with $p \circ H = \overline{G}$ and $H(x, 0) = g(x)$. One can also prove that a map $p : E \to B$ is a weak fibration if and only if it is vertically homotopic with a (Hurewicz) fibration $p' : E' \to B$, $p \simeq_B p'$. This fact shows that *WCHP* is essentially a local property, just like the covering homotopy property (*CHP*). The main application of weak fibrations is the following Dold's theorem: Let $p : E \to B$, $p' : E' \to B$ be spaces over B which have *WCHP*. Then a map $f : E \to E'$ over B is a fibre homotopy equivalence if and only if it is an ordinary homotopy equivalence ([3], Th. 6. 1).

The subject of the present paper is a translation of the notion of weak fibration together with some properties of this and also a Dold's type theorem in the language of so called noncommutative homotopy theory , i.e., in the language of the homotopy theory of C^*- algebras. Naturally the objects which appear here are C^*-*algebras under*, i.e., $*$-homomorphisms ($\phi : A \to B$, $\phi' : A \to B'$) and $*$-*homomorphisms under* ($\theta : B \to B'$ with $\theta \circ \phi = \phi'$).

In the definition which we propose for the notion of weak fibration of C^*-algebras we illustrate the three characterizations from the topological case: 1) *WHLP* (Definition 3.1), 2) *HLP* for semistationary homotopies (Theorem 3.3 and Proposition 3.4), 3) The vertical homotopy equivalence with a fibration (Definition 4.6 and Proposition 4.7).

A lot of examples prove the consistency of our definition.

Other properties refer to a weak fibration by a push-out diagram (Proposition 4.4) and to a weak fibration by a domination (Proposition 5.1 and Corollary 5.3).

Finally, some properties (Propositions 5.4, 5.5 and Lemma 5.6) prepare the proof of our Dold type results (Theorem 5.7 and Corollary 5.8).

2. The notion of homotopy for $*$-homomorphisms. By a $*$-homomorphism we shall mean a homomorphism of C^*-algebras. We recall from [1] the definition of the homotopy relation between $*$-homomorphisms.

Given a C^*-algebra A and a (locally) compact space Y, denote by AY the C^*-algebra of (vanishing at infinity) continuous functions on Y into A. If $Y = I := [0,1]$ then, for every $t \in I$, denote by $\rho_t : AI \to A$ the $*$-homomorphism defined by $\rho_t(\alpha) = \alpha(t), \forall \alpha \in AI$. If $\phi : A \to B$ is a $*$-homomorphism, then ϕ induces a $*$-homomorphism $\phi I : AI \to BI$ given by $\phi I(\alpha) = \phi \circ \alpha, \forall \alpha \in AI$.

Definition 2.1. (i) *The $*$-homomorphisms $\eta, \theta : A \to B$ are homotopic if there exists a $*$-homomorphism $\Phi : A \to BI$ satisfying $\rho_0 \circ \Phi = \eta$ and $\rho_1 \circ \Phi = \theta$. We shall use the notation $\eta \overset{h}{\sim} \theta$ and Φ will be called a $*$-homotopy.*

(ii) *Two $*$-homomorphisms $\eta : A \to B$ and $\theta : B \to A$ establish a homotopy equivalence if $\eta \circ \theta \overset{h}{\sim} id_B$ and $\theta \circ \eta \overset{h}{\sim} id_A$.*

Definition 2.2. (i) *Let $\phi : A \to B$ and $\phi' : A \to B'$ be some $*$-homomorphisms. A $*$-homomorphism $\eta : B \to B'$ such that $\eta \circ \phi = \phi'$ is called under A.*

(ii) *Let $\eta, \theta : B \to B'$ be some $*$-homomorphisms under A. The $*$-homomorphisms η and θ are homotopic under A if there exists a $*$-homotopy $\Phi : B \to B'I$ satisfying $\rho_0 \circ \Phi = \eta$,$\rho_1 \circ \Phi = \theta$ and $(\rho_t \circ \Phi) \circ \phi = \phi', \forall t \in I$.*

Definition 2.3. *A $*$-homomorphism $\phi : A \to B$ is called shrinkable if one of the following equivalent properties holds:*

(i) *ϕ is a fibre homotopy equivalence*;

(ii) *ϕ dominates id_A*;

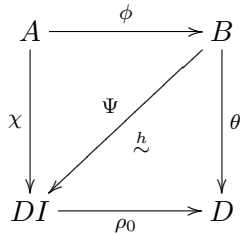
(iii) *there exists a $*$-homomorphism $\psi : B \to A$ such that $\psi \circ \phi = id_A$ and $\phi \circ \psi \overset{h}{\sim} id_B$ under A.*

3. Weak fibrations.

Definition 3.1. *A $*$-homomorphism $\phi : A \to B$ is a weak fibration if it has a weak lifting homotopy property $*$- (WHLP) in the following sense: for any $*$-homomorphism $\theta : B \to D$ and any $*$-homotopy $\chi : A \to DI$, with $\rho_0 \circ \chi = \theta \circ \phi$, there exists a $*$-homotopy $\Psi : B \to DI$, with the properties $\rho_0 \circ \Psi \overset{h}{\sim} \theta$ under A, and $\Psi \circ \phi = \chi$.*

Remark 3.2. This property is schematically illustrated by the following

diagram



Theorem 3.3. *Let $\phi : A \to B$ be a $*$-homomorphism and $0 < \varepsilon < 1$. The following statements are equivalent:*

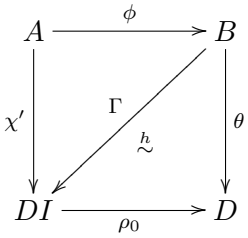
(i) *ϕ is a weak fibration*

(ii) *for an arbitrary C^*-algebra D, an arbitrary $*$-homomorphism $\theta : B \to D$ and a "semistationary" $*$-homotopy $\chi : A \to DI$, satisfying $\rho_t \circ \chi = \theta \circ \phi, \forall t \in [0, \varepsilon]$, there exists a $*$-homotopy $\Psi : B \to DI$, such that $\rho_0 \circ \Psi = \theta$ and $\Psi \circ \phi = \chi$.*

Proof. For (i)$\Rightarrow$ (ii), let $\phi : A \to B$ be a weak fibration. Let $\theta : B \to D$ and $\chi : A \to DI$ be some $*$-homomorphisms such that $\rho_t \circ \chi = \theta \circ \phi, \forall t \in [0, \varepsilon]$.

We can define a $*$-homotopy $\chi' : A \to DI$ by $\chi'(a)(t) = \chi(a)((1-\varepsilon)t+\varepsilon)$. Then $\rho_0 \circ \chi' = \rho_\varepsilon \circ \chi = \theta \circ \phi$ and, by Def. 3.1, there exists a $*$-homotopy $\Gamma : B \to DI$ such that $\rho_0 \circ \Gamma \overset{h}{\sim} \theta$ under A and $\Gamma \circ \phi = \chi'$.

Let $\Lambda : B \to DI$ be a $*$-homotopy under A, satisfying $\rho_0 \circ \Lambda = \rho_0 \circ \Gamma$, $\rho_1 \circ \Lambda = \theta$ and $\rho_t \circ \Lambda \circ \phi = \theta \circ \phi, \forall t \in I$.



Denote $J = [0, \varepsilon]$ and $J' = [\varepsilon, 1]$. For the $*$-homomorphism $\Psi' : B \to DJ$, defined by $\Psi'(b)(t) = \Lambda(b)(\frac{\varepsilon - t}{\varepsilon})$, we have $\Psi'(b)(\varepsilon) = \Lambda(b)(0) = \Gamma(b)(0)$ and $\Psi'(b)(0) = \Lambda(b)(1) = \theta(b), \forall b \in B$. Moreover, $(\rho_t \circ \Psi') \circ \phi = \rho_{\frac{\varepsilon - t}{\varepsilon}} \circ \Lambda \circ \phi = \rho_0 \circ \chi' = \rho_\varepsilon \circ \chi = \rho_t \circ \chi, \forall t \in [0, \varepsilon]$.

Let $\Psi'' : B \to DJ'$ be the $*$-homomorphism defined by $\Psi''(b)(t) = \Gamma(b)(\frac{t-\varepsilon}{1-\varepsilon})$. We obtain the relations $\Psi''(b)(\varepsilon) = \Gamma(b)(0), \forall b \in B$ and $(\rho_t \circ \Psi'') \circ \phi = \rho_{\frac{t-\varepsilon}{1-\varepsilon}} \circ \Gamma \circ \phi = \rho_{\frac{t-\varepsilon}{1-\varepsilon}} \circ \chi' = \rho_t \circ \chi, \forall t \in [\varepsilon, 1]$.

There exists a $*$-homomorphism $\Psi : B \to DI$ well defined by $\Psi(b)/J = \Psi'(b)$ and $\Psi(b)/J' = \Psi''(b)$, $\forall b \in B$, since $\rho_\varepsilon \circ \Psi' = \rho_\varepsilon \circ \Psi''$. It follows immediately that $\rho_0 \circ \Psi = \theta$ and $\Psi \circ \phi = \chi$.

For (ii)$\Rightarrow$ (i) it is sufficient to consider $\varepsilon = \frac{1}{2}$. If $\theta : B \to D$ and $\chi : A \to DI$ are $*$-homomorphisms such that $\rho_0 \circ \chi = \theta \circ \phi$, then we can define $\chi' : A \to DI$ by $\chi'(a)(t) = \chi(a)(max(2t-1, 0))$. For any $t \in [0, \frac{1}{2}]$, we have $2t - 1 \leq 0$ and $\rho_t \circ \chi' = \rho_0 \circ \chi = \theta \circ \phi$. We can apply (ii) for θ and χ'.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle \chi'}\downarrow & \overset{\Psi'}{\swarrow} & \downarrow{\scriptstyle \theta} \\ DI & \xrightarrow[\rho_0]{} & D \end{array}$$

Thus there exists $\Psi' : B \to DI$, a $*$-homotopy, with the properties $\rho_0 \circ \Psi' = \theta$ and $\Psi' \circ \phi = \chi'$. Let $\Psi : B \to DI$ be the $*$-homotopy defined by $\Psi(b)(t) = \Psi'(b)(\frac{1+t}{2})$. Then $\Psi \circ \phi = \chi$, since $(\Psi \circ \phi)(a)(t) = \Psi'(\phi(a))(\frac{1+t}{2}) = \chi'(a)(\frac{1+t}{2}) = \chi(a)(max(t, 0)) = \chi(a)(t), \forall a \in A, \forall t \in I$. It remains to see that $\rho_0 \circ \Psi \overset{h}{\sim} \theta$ under A. For this purpose, consider the $*$-homotopy $\Lambda : B \to DI$ defined by $\Lambda(b)(t) = \Psi'(b)(\frac{1-t}{2})$. It follows, by a simple computation, that $\rho_0 \circ \Lambda = \rho_{\frac{1}{2}} \circ \Psi' = \rho_0 \circ \Psi$ and $\rho_1 \circ \Lambda = \rho_0 \circ \Psi' = \theta$. Moreover, $(\rho_t \circ \Lambda) \circ \phi = \rho_{\frac{1-t}{2}} \circ \Psi' \circ \phi = \rho_{\frac{1-t}{2}} \circ \chi' = \rho_0 \circ \chi = \theta \circ \phi$. Thus Λ is a $*$-homotopy for $\rho_0 \circ \Psi \overset{h}{\sim} \theta$ under A and the proof is complete. □

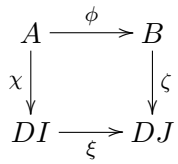
Proposition 3.4. *Let $\phi : A \to B$ be a $*$-homomorphism and $0 < \varepsilon < 1$. Denote $J = [0, \varepsilon]$ and let $\xi : DI \to DJ$ be the $*$-homomorphism defined by $\xi(\alpha) = \alpha/J$. Then the following statements are equivalent:*

(i) *ϕ is a weak fibration*

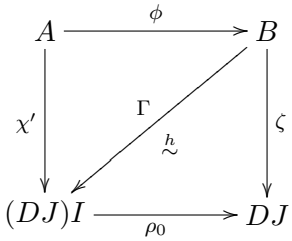
(ii) *for any C^*-algebra D and $*$-homomorphisms $\zeta : B \to DJ$ and $\chi : A \to DI$, satisfying $\xi \circ \chi = \zeta \circ \phi$, there exists a $*$-homotopy $\Psi : B \to DI$ with the properties $\Psi \circ \phi = \chi$ and $\rho_0 \circ \Psi = \rho_0 \circ \zeta$.*

Proof. (i)$\Rightarrow$ (ii).

Suppose that the following commutative diagram is given



Define $\chi' : A \to (DJ)I$, by $\chi'(a)(t)(\tau) = \chi(a)(1-(1-t)(1-\tau)), \forall a \in A, \forall t \in I, \forall \tau \in J$. For this we have $(\rho_0 \circ \chi')(a)(\tau) = \chi'(a)(0)(\tau) = \chi(a)(\tau) = (\xi \circ \chi)(a)(\tau) = (\zeta \circ \phi)(a)(\tau), \forall a \in A, \forall \tau \in J$. Thus $\rho_0 \circ \chi' = \zeta \circ \phi$. By the definition of a weak fibration, there exists a $*$-homotopy $\Gamma : B \to (DJ)I$ such that $\rho_0 \circ \Gamma \overset{h}{\sim} \zeta$ under A and $\Gamma \circ \phi = \chi'$.



Consider a $*$-homotopy, $\Lambda : B \to (DJ)I$, satisfying $\rho_0 \circ \Lambda = \zeta$, $\rho_1 \circ \Lambda = \rho_0 \circ \Gamma$ and $\rho_t \circ \Lambda \circ \phi = \zeta \circ \phi$.

Let $\Psi : B \to DI$ be the $*$-homotopy defined by:

$$\Psi(b)(t) = \begin{cases} \Lambda(b)(\frac{t}{\varepsilon})(t), & \text{if } 0 \leq t \leq \varepsilon, \\ \Gamma(b)(\frac{t-\varepsilon}{1-\varepsilon})(\varepsilon), & \text{if } \varepsilon \leq t \leq 1. \end{cases}$$

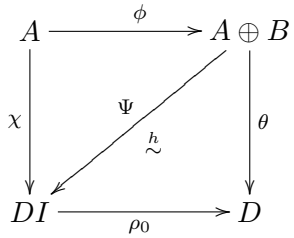
For $t = \varepsilon$, the expressions are equal since $\rho_1 \circ \Lambda = \rho_0 \circ \Gamma$. Therefore Ψ is a well defined $*$- homomorphism. We have $\Psi(b)(0) = \Lambda(b)(0)(0) = \zeta(b)(0), \forall b \in B$. Thus $\rho_0 \circ \Psi = \rho_0 \circ \zeta$. It remains to prove that $\Psi \circ \phi = \chi$. If $t \in [0, \varepsilon]$, then $\Psi(\phi(a))(t) = (\rho_{\frac{t}{\varepsilon}} \circ (\Lambda \circ \phi))(a)(t) = (\zeta \circ \phi)(a)(t) = (\xi \circ \chi)(a)(t) = \chi(a)_{/J}(t) = \chi(a)(t)$. If $t \in [\varepsilon, 1]$, then $\Psi(\phi(a))(t) = (\Gamma \circ \phi)(a)(\frac{t-\varepsilon}{1-\varepsilon})(\varepsilon) = \chi'(a)(\frac{t-\varepsilon}{1-\varepsilon})(\varepsilon) = \chi(a)(t)$. Thus indeed we have $\Psi \circ \phi = \chi$.

For (ii)$\Rightarrow$ (i), let D be an arbitrary C^*- algebra and let $\chi : A \to DI$, $\theta : B \to D$ be $*$- homomorphisms such that $\rho_t \circ \chi = \theta \circ \phi$, for any $t \in [0, \varepsilon]$. We define $\zeta : B \to DJ$ by $\zeta(b)(t) = \theta(b), \forall b \in B, \forall t \in J$. Then $(\zeta \circ \phi)(a)(t) = \theta(\phi(a)) = \chi(a)(t), \forall t \in [0, \varepsilon]$. We apply Theorem 3.3. Then there exists a $*$-homomorphism $\Psi : B \to DI$ satisfying $\rho_0 \circ \Psi = \theta = \rho_0 \circ \zeta$ and $\Psi \circ \phi = \chi$. □

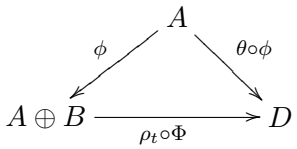
4. Examples

Example 4.1. The $*$- homomorphism $\phi : A \to A \oplus B$, defined by $\phi(a) = (a, 0)$, is a weak fibration.

If D is an arbitray C^* -algebra and $\theta : A \oplus B \to D$, $\chi : A \to DI$ are $*$-homomorphisms such that $\rho_0 \circ \chi = \theta \circ \phi$, then we define $\Psi : A \oplus B \to DI$, by $\Psi((a, b)) = \chi(a)$. It follows immediately that $\Psi \circ \phi = \chi$.



Let $\Phi : A \oplus B \to DI$ be the $*$-homotopy defined by $\Phi(a, b)(t) = \theta(a, tb)$. We have: $(\rho_0 \circ \Phi)((a, b)) = \theta((a, 0)) = (\theta \circ \phi)(a) = (\rho_0 \circ \chi)(a) = \Psi((a, b))(0)$ and $(\rho_1 \circ \Phi)((a, b)) = \theta((a, b)), \forall (a, b) \in A \oplus B$. Thus, $\Phi : \rho_0 \circ \Psi \overset{h}{\sim} \theta$. Moreover, this homotopy is under A



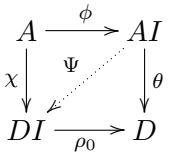
since $((\rho_t \circ \Phi) \circ \phi)(a) = \Phi((a, 0))(t) = \theta((a, 0)) = (\theta \circ \phi)(a)$.

Example 4.2. Let $\phi : A \to AI$ be the $*$-homomorphism defined by $\phi(a) = \alpha_a$, with α_a the constant path $\alpha_a : I \to A, \alpha_a(t) = a, \forall t \in I$.

Then ϕ is a weak fibration.

Consider an arbitrary C^*-algebra and two $*$-homomorphisms $\theta : AI \to D$ and $\chi : A \to DI$ satisfying $\rho_0 \circ \chi = \theta \circ \phi$.

Let $\Psi : AI \to DI$ be the $*$-homotopy defined by $\Psi(\alpha) = \chi(\alpha(0))$. Then $(\Psi \circ \phi)(a) = \Psi(\alpha_a) = \chi(\alpha_a(0)) = \chi(a), \forall a \in A$, therefore $\Psi \circ \phi = \chi$.



We define $\Phi : AI \to DI$ by $\Phi(\alpha)(t) = \theta(\beta_t)$, where $\beta_t \in AI$ is given by $\beta_t(\tau) = \alpha(t\tau), \forall \tau \in I$. We have $(\rho_0 \circ \Phi)(\alpha) = \theta(\beta_0) = \theta(\alpha_a) = (\theta \circ \phi)(a) = (\rho_0 \circ \chi)(a) = \chi(a)(0) = \chi(\alpha(0))(0) = \Psi(\alpha)(0), \forall \alpha \in AI$, since $\beta_0(\tau) = \alpha(0) = a = \alpha_a(\tau), \forall \tau \in I$. Thus $\rho_0 \circ \Phi = \rho_0 \circ \Psi$. It's obvious that $\rho_1 \circ \Phi = \theta$. Moreover, $((\rho_t \circ \Phi) \circ \phi)(a) = \Phi(\alpha_a)(t) = \theta(\beta_t) = \theta(\alpha_a) = (\theta \circ \phi)(a)$, since $\beta_t(\tau) = \alpha_a(t\tau) = a = \alpha_a(\tau), \forall \tau \in I$. It follows that Φ is a homotopy under A.

Remark 4.3. Example 4.2 is inspired by the trivial fibration $p : B \times I \to B$, using the isomorphism $C(B \times I) \simeq C(B)I$.

Proposition 4.4. *In a push-out diagram,*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\overline{\eta}} \\ C & \xrightarrow[\overline{\phi}]{} & B\star_A C \end{array}$$

if $\phi : A \to B$ is a weak fibration and $\eta : A \to C$ is an arbitrary $*$-homomorphism, then $\overline{\phi}$ is a weak fibration.

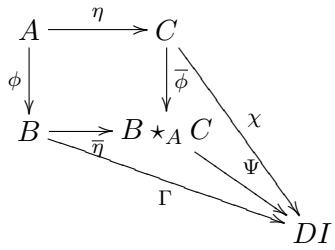
Proof. We verify condition (ii) from Th.3.3.

Let D be an arbitrary C^*-algebra and $0 < \varepsilon < 1$. Let $\theta : B \star_A C \to D$ and $\chi : C \to DI$ be $*$-homomorphisms satisfying the condition $\rho_t \circ \chi = \theta \circ \overline{\phi}$, for any $t \in [0, \varepsilon]$. Then we have $\varrho_t \circ (\chi \circ \eta) = (\theta \circ \overline{\phi}) \circ \eta = \theta \circ (\overline{\phi} \circ \eta) = \theta \circ (\overline{\eta} \circ \phi) = (\theta \circ \overline{\eta}) \circ \phi$.

Since ϕ is a weak fibration, by Th.3.3, there exists a $*$-homotopy, $\Gamma : B \to DI$, such that $\Gamma \circ \phi = \chi \circ \eta$ and $\rho_0 \circ \Gamma = \theta \circ \overline{\eta}$.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle\chi\circ\eta}\downarrow & \overset{\Gamma}{\swarrow} & \downarrow{\scriptstyle\theta\circ\overline{\eta}} \\ DI & \xrightarrow[\rho_0]{} & D \end{array}$$

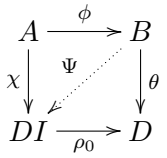
We have $\Gamma \circ \phi = \chi \circ \eta$, thus Γ and χ give us a push-out coherent pair of $*$-homomorphisms. Then there exists a unique $*$-homomorphism $\Psi : B \star_A C \to DI$ satisfying $\Psi \circ \overline{\phi} = \chi$ and $\Psi \circ \overline{\eta} = \Gamma$.



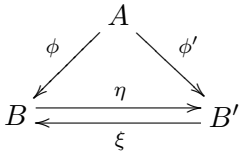
Now we have to prove that $\rho_0 \circ \Psi = \theta$. By the push-out property, $\rho_0 \circ \Psi$ is the unique $*$-homomorphism with the properties $(\rho_0 \circ \Psi) \circ \overline{\eta} = \rho_0 \circ \Gamma$ and $(\rho_0 \circ \Psi) \circ \overline{\phi} = \rho_0 \circ \chi$. But we know that $\theta \circ \overline{\phi} = \rho_0 \circ \chi$ and $\theta \circ \overline{\eta} = \rho_0 \circ \Gamma$. Therefore $\rho_0 \circ \Psi = \theta$ and this ends the proof. □

Remark 4.5. The existence of push-out diagrams in the category of C^*-algebras is proved in [5], p. 247.

Definition 4.6. *A $*$-homomorphism $\phi : A \to B$ is a fibration if for any C^*-algebra D and two $*$-homomorphisms $\theta : B \to D$, $\chi : A \to DI$, satisfying $\rho_0 \circ \chi = \theta \circ \phi$, there exists a $*$-homomorphism $\Psi : B \to DI$ with the following properties:$\Psi \circ \phi = \chi$ and $\rho_0 \circ \Psi = \theta$.*

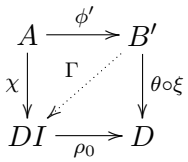


Proposition 4.7. *Let a commutative diagram of $*$-homomorphisms be given.*



If ϕ' is a fibration and the $*$-homomorphisms η and ξ establish a homotopy equivalence under A, then ϕ is a weak fibration.

Proof. Let D be a C^*-algebra. Let $\theta : B \to D$ and $\chi : A \to DI$ be $*$-homomorphisms such that $\rho_0 \circ \chi = \theta \circ \phi$. By the commutativity of the given diagram, we have $\xi \circ \phi' = \phi$. Then $(\theta \circ \xi) \circ \phi' = \rho_0 \circ \chi$.



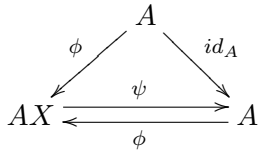
Since ϕ' is a fibration, there exists a $*$-homotopy $\Gamma : B' \to DI$ satisfying $\Gamma \circ \phi' = \chi$ and $\rho_0 \circ \Gamma = \theta \circ \xi$. We define $\Psi : B \to DI$, by $\Psi = \Gamma \circ \eta$. Then $\Psi \circ \phi = (\Gamma \circ \eta) \circ \phi = \Gamma \circ (\eta \circ \phi) = \Gamma \circ \phi' = \chi$ and $\rho_0 \circ \Psi = (\rho_0 \circ \Gamma) \circ \eta = (\theta \circ \xi) \circ \eta = \theta \circ (\xi \circ \eta) \overset{h}{\sim} \theta \circ id_B = \theta$ under A. □

Corollary 4.8. *Let $\phi : A \to B$ be a $*$-homomorphism and let $\phi' : A \to B'$ be a $*$-isomorphism. If $\eta : B' \to B$ is a $*$-homotopy equivalence under A, then ϕ is a weak fibration.*

Example 4.9. Let A be a C^*-algebra and X a contractible topological space.

Let $\phi : A \to AX$ be the $*$-homomorphism defined by $\phi(a) = f_a$, with $f_a : X \to A$ the constant map $f_a(x) = a, \forall x \in X$. Then ϕ is a weak fibration.

We prove that A and AX have the same $*$-homotopy type. Let x_0 be a fixed point in X. By the property of being a contractible space, the constant map $\theta : X \to X, \theta(x) = x_0$ and id_X are homotopic, i.e., there exists $H : X \times I \to X$ such that $H(x, 0) = x_0$ and $H(x, 1) = x$. We define $\psi : AX \to A$ by $\psi(f) = f(x_0)$. We prove that ϕ and ψ establish a $*$-homotopy equivalence. First, $(\psi \circ \phi)(a) = \psi(f_a) = f_a(x_0) = a = id_A(a), \forall a \in A$. On the other hand, $\phi \circ \psi \overset{h}{\sim} id_{AX}$, by the $*$-homotopy $\Phi : AX \to (AX)I$, defined by $\Phi(f)(t) = g_t$, where $g_t : X \to A$ is given by $g_t(x) = f(H(x, t))$. Moreover, this homotopy is under A.

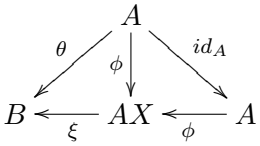


Indeed, we have $((\rho_t \circ \Phi) \circ \phi)(a) = \Phi(\phi(a))(t) = g_t = f_a = \phi(a), \forall a \in A$, since $g_t(x) = f_a(H(x, t)) = a = f_a(x), \forall x \in X$.

Thus, there exists a fibration , $id_A : A \to A$, and ϕ, ψ are $*$-homotopy equivalences under A. Then by Prop. 4.7, ϕ is a weak fibration.

Remark 4.10. This $*$-homomorphism $\phi : A \to AX$ is a $*$-homotopy equivalence under A, hence ϕ is shrinkable.

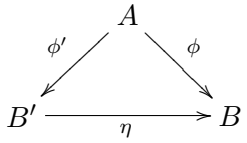
Remark 4.11. If $\theta : A \to B$ is a $*$-homorphism and $\eta : B \to AX, \xi : AX \to B$ establish a $*$-homotopy equivalence under A, then $\theta = \xi \circ \phi$ is a fibre $*$-homotopy equivalence, therefore θ is shrinkable.



Moreover, θ is a weak fibration, we'll prove this in Corollary 5.3.

5. Properties

Proposition 5.1. *Let*

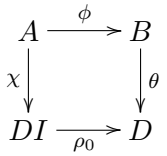


be a commutative diagram of $*$-homomorphisms.

If ϕ' is a weak fibration and η admits a $*$-homotopy right inverse under A, then ϕ is a weak fibration.

Proof. Let $\eta' : B' \to B$ be a $*$- homotopy right inverse of η under A. Thus, $\eta \circ \eta' \overset{h}{\sim} id_B$ by a $*$-homotopy $\Gamma : B \to BI$, with $\rho_0 \circ \Gamma = \eta \circ \eta', \rho_1 \circ \Gamma = id_B$ and $(\rho_t \circ \Gamma) \circ \phi = \phi$.

Let D be an arbitrary C^*-algebra. Let $\chi : A \to DI$ and $\theta : B \to D$ be $*$-homomorphisms such that $\rho_0 \circ \chi = \theta \circ \phi$. By the commutativity of the diagram mentioned in the statement, we have $\eta \circ \phi = \phi'$, therefore $(\theta \circ \eta) \circ \phi' = \theta \circ (\eta \circ \phi') = \theta \circ \phi = \rho_0 \circ \chi$. Since ϕ' is a weak fibration, there exists a $*$-homotopy $\Psi' : B' \to DI$, with the properties $\Psi' \circ \phi' = \chi$ and $\rho_0 \circ \Psi' \overset{h}{\sim} \theta \circ \eta$ under A.

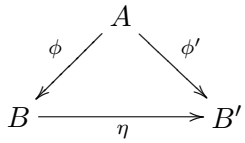


Let $\Phi : B' \to DI$ be such a $*$-homotopy, i.e., $\rho_0 \circ \Phi = \rho_0 \circ \Psi', \rho_1 \circ \Phi = \theta \circ \eta$ and $(\rho_t \circ \Phi) \circ \phi' = \theta \circ (\eta \circ \phi') = \theta \circ \phi$.

We can define $\Psi : B \to DI$, by $\Psi = \Psi' \circ \eta'$. It follows, by a simple computation, that $\Psi \circ \phi = (\Psi' \circ \eta') \circ \phi = \Psi' \circ (\eta' \circ \phi) = \Psi' \circ \phi' = \chi$. Moreover, $\rho_0 \circ \Psi = (\rho_0 \circ \Psi') \circ \eta' \overset{h}{\sim} (\theta \circ \eta) \circ \eta'$ under A, by the $*$-homotopy $\Phi \circ \eta'$, and $\theta \circ (\eta \circ \eta') \overset{h}{\sim} \theta$ under A, by the $*$-homotopy $\theta I \circ \Gamma$. Thus $\rho_0 \circ \Psi \overset{h}{\sim} \theta$ under A. □

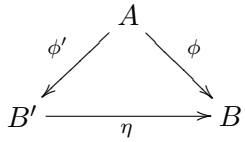
Remark 5.2. As we saw in Prop.4.7, fibrations are not invariant under $*$-fibre homotopy equivalence. Corollary 5.3 from below shows that the notion of weak fibration is adequate in questions dealing with $*$-fibre homotopy equivalence.

Corollary 5.3. *Let a commutative diagram of $*$-homomorphisms be given:*



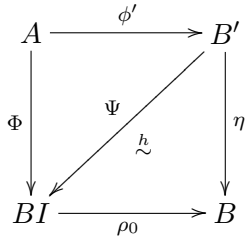
If ϕ is a weak fibration and η is a homotopy equivalence under A, then ϕ' is a weak fibration.

Proposition 5.4. *Let a $*$-homotopy commutative diagram be given:*



If ϕ' is a weak fibration, then there exists a $*$-homomorphism $\zeta : B' \to B$ under A, such that $\eta \overset{h}{\sim} \zeta$.

Proof. Let $\Phi : A \to BI$ be a $*$-homotopy satisfying $\rho_0 \circ \Phi = \eta \circ \phi'$ and $\rho_1 \circ \Phi = \phi$. Since ϕ' is a weak fibration, there exists a $*$-homotopy $\Psi : B' \to BI$, such that $\Psi \circ \phi' = \Phi$ and $\rho_0 \circ \Psi \overset{h}{\sim} \eta$.



We consider $\zeta = \rho_1 \circ \Psi$. Then $\zeta \overset{h}{\sim} \rho_0 \circ \Psi \overset{h}{\sim} \eta$. We have $\zeta \circ \phi' = \rho_1 \circ (\Psi \circ \phi') = \rho_1 \circ \Phi = \phi$, thus ζ is also a $*$-homomorphism under A. □

Proposition 5.5. *Let $\phi : A \to B$ be a weak fibration. If ϕ admits a left inverse up to a $*$-homotopy, then ϕ admits a left inverse.*

Proof. This statement is a special case of Proposition 5.4. Let ψ be a $*$-homotopy left inverse of ϕ. Then $\psi \circ \phi \overset{h}{\sim} id_A$. It follows that there exists $\psi' : B \to A$ a $*$-homomorphism under A. Therefore $\psi' \circ \phi = id_A$. □

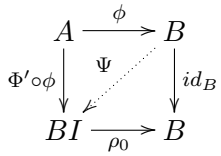
Lemma 5.6. *Let $\phi : A \to B$ be a weak fibration. If $\eta : B \to B$ is a $*$-homomorphism under A and $\eta \overset{h}{\sim} id_B$, then there exists $\xi : B \to B$ a $*$-homomorphism under A, such that $\xi \circ \eta \overset{h}{\sim} id_B$ under A.*

Proof. Let $\Phi : B \to BI$ be a corresponding $*$-homotopy, satisfying $\rho_0 \circ \Phi = \eta$ and $\rho_1 \circ \Phi = id_B$.

We can consider that Φ is semistationary, otherwise we define $\Phi' : B \to BI$, by:

$$\Phi'(b)(t) = \begin{cases} \eta(b), & \text{if } 0 \le t \le \frac{1}{2}, \\ \Phi(b)(2t-1), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

It follows immediately that $\Phi' \circ \phi : A \to BI$ is a semistationary $*$-homotopy, which satisfies the relations $\rho_0 \circ (\Phi' \circ \phi) = \eta \circ \phi = \phi$ and $\rho_1 \circ (\Phi' \circ \phi) = id_B \circ \phi = \phi$. Since $\rho_t \circ (\Phi' \circ \phi) = id_B \circ \phi$, for any $t \in [0, \frac{1}{2}]$ an ϕ is a weak fibration, there exists a $*$-homotopy $\Psi : B \to BI$, with the properties $\Psi \circ \phi = \Phi' \circ \phi$ and $\rho_0 \circ \Psi = id_B$.



Let $\xi = \rho_1 \circ \Psi$. We have $\xi \circ \phi = \rho_1 \circ (\Psi \circ \phi) = \rho_1 \circ (\Phi' \circ \phi) = \phi$. Therefore ξ is a $*$-homomorphism under A. Let $\Gamma : B \to BI$ be the $*$-homotopy defined by:

$$\Gamma(b)(t) = \begin{cases} \Psi(\eta(b))(1-2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \Phi'(b)(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then Γ is well defined, since $\Psi(\eta(b))(0) = \eta(b)$ and $\Phi'(b)(0) = \eta(b)$. Moreover, $(\rho_0 \circ \Gamma)(b) = \Psi(\eta(b))(1) = ((\rho_1 \circ \Psi) \circ \eta)(b) = (\xi \circ \eta)(b)$ and $(\rho_1 \circ \Gamma)(b) = \Phi'(b)(1) = \Phi(b)(1) = b, \forall b \in B$. But this $*$-homotopy $\Gamma : \xi \circ \eta \overset{h}{\sim} id_B$ is not under A.

Let $\Lambda : A \to (BI)I$ be the $*$-homotopy defined by:

$$\Lambda(a)(t)(\tau) = \begin{cases} \Phi'(\phi(a))(1-2\tau(1-t)), & \text{if } 0 \leq \tau \leq \frac{1}{2}, t \in I \\ \Phi'(\phi(a))(1-2(1-\tau)(1-t)), & \text{if } \frac{1}{2} \leq \tau \leq 1, t \in I. \end{cases}$$

It follows, by a simple computation, that for $\tau = \frac{1}{2}$ the expressions are equal. Thus Λ is well defined. Moreover, Λ has the following properties:

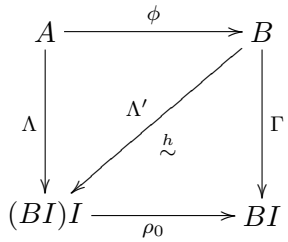
(i) $\Lambda(a)(0)(\tau) = \Gamma(\phi(a))(\tau), \forall \tau \in I, \forall a \in A$, i.e., $\rho_0 \circ \Lambda = \Gamma \circ \phi$;

(ii) $\Lambda(a)(t)(0) = \Phi'(\phi(a))(1) = \phi(a), \forall t \in I, \forall a \in A$, i.e., $\rho_0 \circ (\rho_t \circ \Lambda) = \phi$;

(iii) $\Lambda(a)(1)(\tau) = \Phi'(\phi(a))(1) = \phi(a), \forall \tau \in I, \forall a \in A$, i.e., $\rho_\tau \circ (\rho_1 \circ \Lambda) = \phi$;

(iv) $\Lambda(a)(t)(1) = \Phi'(\phi(a))(1) = \phi(a), \forall t \in I, \forall a \in A$, i.e., $\rho_1 \circ (\rho_t \circ \Lambda) = \phi$.

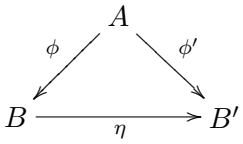
Since $\rho_0 \circ \Lambda = \Gamma \circ \phi$ and ϕ is a weak fibration, there exists a $*$-homotopy $\Lambda' : B \to (BI)I$ such that $\Lambda' \circ \phi = \Lambda$ and $\rho_0 \circ \Lambda' \overset{h}{\sim} \Gamma$ under A.



Let $\chi : B \to (BI)I$ be a $*$-homotopy satisfying $\rho_0 \circ \chi = \rho_0 \circ \Lambda', \rho_1 \circ \chi = \Gamma$ and $\rho_t \circ \chi \circ \phi = \Gamma \circ \phi$. Then $\xi \circ \eta = \rho_0 \circ \Gamma \overset{h}{\sim} \rho_0 \circ (\rho_0 \circ \Lambda') \overset{h}{\sim} \rho_0 \circ (\rho_1 \circ \Lambda') \overset{h}{\sim} \rho_1 \circ (\rho_1 \circ \Lambda') \overset{h}{\sim} \rho_1 \circ (\rho_0 \circ \Lambda') = \rho_1 \circ \Gamma = id_B$. The first $*$-homotopy of this sequence is under A, since $\rho_0 \circ (\rho_t \circ \chi) \circ \phi = \rho_0 \circ (\Gamma \circ \phi) = \rho_1 \circ \Phi \circ \phi = \phi$.

By the properties (ii)-(iv) and the relation $\Lambda' \circ \phi = \Lambda$, we obtain that the other $*$-homotopies are under A. The proof is complete. □

Theorem 5.7. *Let a commutative diagram of $*$-homomorphisms be given:*



If ϕ and ϕ' are weak fibrations and η is an ordinary $*$- homotopy equivalence, then η is a $*$-homotopy equivalence under A.

Proof. Let $\theta : B' \to B$ be a $*$-homotopy inverse of η, as an ordinary $*$-homomorphism. Then $\theta \circ \phi' = \theta \circ (\eta \circ \phi) \overset{h}{\sim} \phi$. Hence $\theta \overset{h}{\sim} \zeta$, for $\zeta : B' \to B$ a $*$-homomorphism under A. Since $\zeta \circ \eta \overset{h}{\sim} \theta \circ \eta \overset{h}{\sim} id_B$ and $\zeta \circ \eta$ is a $*$-homomorphism under A, there exists, by Lemma 5.6, a $*$-homomorphism $\xi : B \to B$ under A, such that $\xi \circ (\zeta \circ \eta) \overset{h}{\sim} id_B$ under A. Thus η admits a $*$-homotopy right inverse $\eta' := \xi \circ \zeta$ under A.

Now η' is a $*$-homotopy equivalence, since η is a $*$-homotopy equivalence. The same argument, applied to η' instead of η, shows that η' admits a $*$-homotopy right inverse η'' under A. Thus η' admits both a $*$-homotopy left inverse η under A and a $*$-homotopy right inverse η'' under A. Hence η' is a $*$-homotopy equivalence under A and so η itself is a $*$-homotopy equivalence under A. □

Corollary 5.8. *If $\theta : A \to B$ is a weak fibration, then ϕ is shrinkable if and only if ϕ is an ordinary $*$-homotopy equivalence.*

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REFERENCES

1. Gracia-Bondía, J.M.; Várilly, F.C.; Figueroa, H. – *Elements of Noncommutative Geometry,* Birkhäuser, Boston, 2001.

2. tom Dieck, T.; Kamps, K.H.; Puppe, D. – *Homotopietheorie,* Lection Notes in Math., No. 157, 1970.

3. Dold, A. – *Partitions of unity in the theory of fibrations,* Annals of Math., Vol. 78, No. 2, 223-255 (1963).

4. James, I.M. – *General Topology and Homotopy Theory,* Springer Verlag, New York Inc., 1984.

5. Pedersen, G.P. – *Pullback and Pushout Constructions in C^*-algebra Theory,* J. Funct. Analysis,Vol. 167, No. 2, 243-344 (1999).

6. Weinzweig, A.I. – *Fibre spaces and fibre homotopy equivalence,* Colloq. Algebraic Topology, Aarhus 1962, 72-78 (1962).

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