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A NOTE ON THE NUMBER OF FUZZY SUBGROUPS OF FINITE GROUPS*

BY

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Abstract. In the paper [9] an explicit formula for the number of fuzzy subgroups of a finite cyclic group G is indicated. In the case when $|G|$ has exactly two prime factors, a quadratic form can be associated. We prove that this quadratic form is positive definite. We also count the number of fuzzy subgroups for a particular class of finite hamiltonian groups.

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1. Preliminaries. One of the most important problems in fuzzy group theory is concerned with classifying the fuzzy subgroups of a finite group. This subject has enjoyed a quick evolution in the last few years. Several papers have considered the particular case of finite cyclic groups. Thus, in [3] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [4], [5], [6] and [10] deal with this number for cyclic groups of order $p^n q^m$ (p, q primes). The starting point for our discussion is given by the the paper [9], where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: finite cyclic groups and finite elementary abelian p-groups. For the first class we find an explicit formula of the above number in the most general case. The numbers $f_2(n, m)$ of distinct fuzzy subgroups of cyclic groups of orders $p^n q^m$, $n, m \in$

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$\mathbb{N}$, can be written in a more convenient manner, which allows us to show that the quadratic form associated to them is positive definite. Another goal of our paper is to extend the study of classifying fuzzy subgroups of finite groups, started in [9] for abelian groups, to a class of nonabelian groups: hamiltonian groups. An explicit formula for the number of distinct fuzzy subgroups of a finite hamiltonian group is obtained in a particular case.

First of all, we present some basic notions and results of fuzzy subgroup theory (for more details, see [1]). Let $(G, \cdot, e)$ be a group (e denotes the identity of G) and $\mu : G \to [0,1]$ be a fuzzy subset of G. We say that μ is a *fuzzy subgroup* of G if it satisfies the following two conditions:

a) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$;

b) $\mu(x^{-1}) \geq \mu(x)$, for any $x \in G$.

In this situation we have $\mu(x^{-1}) = \mu(x)$, for any $x \in G$, and $\mu(e) = \max \mu(G)$. For each $\alpha \in [0,1]$, we define the level subset:

$$_{\mu}G_{\alpha} = \{x \in G \mid \mu(x) \geq \alpha\}.$$

These subsets allow us to characterize the fuzzy subgroups of G, in the next way: μ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G.

The fuzzy subgroups of G can be classified up to some natural equivalence relations on the set consisting of all fuzzy subsets of G. One of them (used in [9] and [10], too) is defined by

$$\mu \sim \eta \text{ iff } (\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y), \text{ for all } x, y \in G)$$

and two fuzzy subgroups μ, η of G will be called *distinct* if $\mu \not\sim \eta$. This equivalence relation generalizes that used in Murali's papers [2]-[6]. It is also closely connected to the concept of level subgroup. In this way, suppose that the group G is finite and let $\mu : G \to [0,1]$ be a fuzzy subgroup of G. Put $\mu(G) = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ and assume that $\alpha_1 > \alpha_2 > ... > \alpha_r$. Then μ determines the following chain of subgroups of G which ends in G:

$$(*) \qquad _{\mu}G_{\alpha_1} \subset {_{\mu}G_{\alpha_2}} \subset ... \subset {_{\mu}G_{\alpha_r}} = G.$$

Moreover, for any $x \in G$ and $i = \overline{1, r}$, we have

$$\mu(x) = \alpha_i \Longleftrightarrow i = \max\{j \mid x \in {_{\mu}G_{\alpha_j}}\} \Longleftrightarrow x \in {_{\mu}G_{\alpha_i}} \setminus {_{\mu}G_{\alpha_{i-1}}},$$

where, by convention, we set ${}_{\mu}G_{\alpha_0} = \emptyset$. A necessary and sufficient condition for two fuzzy subgroups μ, η of G to be equivalent with respect to $\sim$ has been identified in [10]: $\mu \sim \eta$ if and only if μ and η have the same set of level subgroups, that is they determine the same chain of subgroups of type $(*)$. This result shows that *there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G which end in G*. So, the problem of counting all distinct fuzzy subgroups of G can be translated into a combinatorial problem on the subgroup lattice $L(G)$ of G: finding the number of all chains of subgroups of G that terminate in G. Even for some particular classes of finite groups, as finite abelian groups, this problem is very difficult. The largest class of groups for which it was completely solved is constituted by finite cyclic groups (see Corollary 4 of [9]). If G is a finite cyclic group of order n (that is $G \cong \mathbb{Z}_n$) and $n = p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$ is the decomposition of n as a product of prime factors, then the number $f_k(n_1, n_2, ..., n_k)$ of all distinct fuzzy subgroups of G satisfies the recurrence relation

$$f_k(n_1, n_2, ..., n_k)$$

$$= 2 \sum_{q=1}^{k} (-1)^{q-1} \sum_{1 \le i_1 < i_2 < ... < i_q \le k} f_k(n_1, ..., n_{i_1} - 1, ..., n_{i_2} - 1, ..., n_{i_q} - 1, ..., n_k)$$

and is given by the equality

$$f_k(n_1, n_2, ..., n_k)$$

$$= 2^{\sum\limits_{q=1}^{k} n_q} \sum_{i_2=0}^{n_2} \sum_{i_3=0}^{n_3} ... \sum_{i_k=0}^{n_k} \left(-\frac{1}{2}\right)^{\sum\limits_{q=2}^{k} i_q} \prod_{q=2}^{k} \binom{n_q}{i_q} \binom{n_1 + \sum\limits_{s=2}^{q} (n_s - i_s)}{n_q},$$

where the above iterated sums are equal to 1 for $k = 1$. Mention that an important step in order to establish a similar explicit formula for finite elementary abelian p-groups is also made in [9] (see Section 3).

Finally, recall that a *hamiltonian group* is a nonabelian group having all subgroups normal. The structure of such a group is strongly connected to that of abelian groups. More precisely, a hamiltonian group can be written as the direct product of a quaternion group of order 8, an elementary abelian 2-group and a torsion abelian group with all elements of odd order.

2. Main results. Let G be a finite cyclic group of order $p^n q^m$. As we have seen above, the number $f_2(n, m)$ of all distinct fuzzy subgroups of

G verifies the following recurrence relation

$$f_2(n,m) = 2\left[f_2(n-1,m) + f_2(n,m-1) - f_2(n-1,m-1)\right]$$

and is given by the formula

$$f_2(n,m) = 2^{n+m}\sum_{r=0}^{m}\left(-\frac{1}{2}\right)^r\binom{m}{r}\binom{n+m-r}{m},$$

where $\binom{a}{b} = 0$ for any $b > a$.

First of all, we indicate another manner to write $f_2(n,m)$.

Proposition 1. *The number $f_2(n,m)$ of distinct fuzzy subgroups of a finite cyclic group G of order $p^n q^m$ (p, q primes) is also given by the equality:*

$$f_2(n,m) = 2^{n+m}\sum_{r=0}^{m}\frac{1}{2^r}\binom{n}{r}\binom{m}{r}.$$

Proof. Let $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be the function defined by:

$$f(n,m) = 2^{n+m}\sum_{r=0}^{m}\frac{1}{2^r}\binom{n}{r}\binom{m}{r}.$$

Then, by a direct calculation, it is easy to see that f satisfies a recurrence relation of the same type as $f_2(n,m)$. Consider the function $\varphi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined through $\varphi(n,m) = f(n,m) - f_2(n,m)$. It results

$$\varphi(n,m) = 2\left[\varphi(n-1,m) + \varphi(n,m-1) - \varphi(n-1,m-1)\right], \tag{1}$$

for all $n, m \in \mathbb{N}^*$. Now, let us suppose $n \in \mathbb{N}$ to be arbitrary and fixed. We will show by induction on $m \in \mathbb{N}$ that $\varphi = 0$. Obviously, we have $\varphi(n,0) = \varphi(0,n) = 0$. Assume that $\varphi(n,m) = 0$ for an arbitrary $m \in \mathbb{N}$. Using (1), one obtains

$$\varphi(n,m+1) = 2\varphi(n-1,m+1) = 2^2\varphi(n-2,m+1) = \cdots = 2^n\varphi(0,m+1) = 0,$$

which shows that indeed $\varphi = 0$. In other words, the functions f and $f_2(n,m)$ are identical. □

Remark. From all quantities $f(n,m)$, $n,m \in \mathbb{N}$, the most interesting are the central numbers $f(n,n)$. They can be written as

$$f(n,n) = 2^{2n} \sum_{r=0}^{n} \frac{1}{2^r} \binom{n}{r}^2,$$

according to Proposition 1. We infer two immediate inequalities verified by these numbers:

$$2^n \binom{2n}{n} \leq f(n,n) \leq 2^{2n} \binom{2n}{n}$$

and

$$2^{2n+1} - 2^n \leq f(n,n) \leq (2^{2n+1} - 2^n) \binom{n}{\left[\frac{n}{2}\right]}^2.$$

Next, for every $n \in \mathbb{N}$, let us consider the matrix $A_n = (a_{ij}) \in M_{n+1}(\mathbb{N})$ defined by $a_{ij} = f(i,j)$, $i,j = \overline{0,n}$. Clearly, A_n is symmetric and so it induces a quadratic form $\sum_{i,j=0}^{n} a_{ij} X^i Y^j$. In order to prove that this quadratic form is positive definite, one needs to compute the principal minors in the top left corner of A_n (that is, to find an explicit expression of $\det A_m$, for $m = 0,1,...,n$).

Theorem 2. *For each* $0 \leq m \leq n$*, the following equality holds:*

$$\det A_m = 2^{\frac{m(m+1)}{2}}.$$

Proof. Let $m \in \{0,1,\cdots,n\}$ be fixed. On account of Proposition 1, the determinant $\det A_m$ is given by:

$$\det \left(2^{i+j} \sum_{r=0}^{m} \frac{1}{2^r} \binom{i}{r} \binom{j}{r} \right)_{i,j=\overline{0,m}}.$$

Hence, for a fixed $k \in \{0,1,\cdots,m\}$, the line L_k of A_m is of the following form:

$$L_k = \Big(2^k \quad 2^{k+1} \left[\frac{1}{2^0} \binom{k}{0} \binom{1}{0} + \frac{1}{2} \binom{k}{1} \binom{1}{1} \right]$$
$$\cdots \quad 2^{k+m} \left[\frac{1}{2^0} \binom{k}{0} \binom{m}{0} + \frac{1}{2^1} \binom{k}{1} \binom{m}{1} + \cdots + \frac{1}{2^k} \binom{k}{k} \binom{m}{k} \right] \Big).$$

We shall apply several transformations on the matrix A_m in order to put it into upper triangular form. Thus, consider the consecutive transformations

$$L'_k = L_k - 2^k \sum_{i=0}^{k-1} \frac{1}{2^i} \binom{k}{i} L'_i,$$

for every $k = \overline{1,m}$, where L'_i is line i of matrix A_m after applying the transformation at the step i. Also, mention that we have denoted by L'_0 the actual line L_0.

For $k \in \{1, 2, \cdots, m\}$, consider the predicate $P(k)$: At the step k of the above algorithm, the new lines are of the form

$$L'_j = \left(\underbrace{0 \quad 0 \quad \cdots \quad 0}_{j} \quad 2^j \binom{j}{j} \quad 2^{j+1} \binom{j+1}{j} \quad \cdots \quad 2^m \binom{m}{j} \right),$$

for all $j = \overline{1,k}$. We will show, by induction on k, that this predicate becomes a true statement for all $k = \overline{1,m}$.

Since $L'_1 = L_1 - 2L_0$, one obtains that

$$L'_1 = \Bigg(2^1 \quad 2^2 \left[\frac{1}{2^0} \binom{1}{0} \binom{1}{0} + \frac{1}{2^1} \binom{1}{1} \binom{1}{1} \right]$$
$$\cdots \quad 2^{m+1} \left[\frac{1}{2^0} \binom{1}{0} \binom{m}{0} + \frac{1}{2^1} \binom{1}{1} \binom{m}{1} \right] \Bigg) - 2 \left(2^0 \quad 2^1 \quad 2^2 \quad \cdots \quad 2^m \right),$$

or equivalently

$$L'_1 = \left(0 \quad 2^1 \binom{1}{1} \quad 2^2 \binom{2}{1} \quad \cdots \quad 2^m \binom{m}{1} \right),$$

which shows that $P(1)$ is true. Next, let $k \in \{1, 2, \cdots, m\}$ be fixed and suppose that $P(k)$ is a true statement. Hence the following equalities hold:

$$L'_1 = \left(0 \quad 2^1 \binom{1}{1} \quad 2^2 \binom{2}{1} \quad \cdots \quad 2^m \binom{m}{1} \right),$$
$$L'_2 = \left(0 \quad 0 \quad 2^2 \binom{2}{2} \quad 2^3 \binom{3}{2} \quad \cdots \quad 2^m \binom{m}{2} \right),$$
$$\cdots$$
$$L'_k = \left(\underbrace{0 \quad 0 \quad \cdots \quad 0}_{k} \quad 2^k \binom{k}{k} \quad 2^{k+1} \binom{k+1}{k} \quad \cdots \quad 2^m \binom{m}{k} \right).$$

At the step $k+1$ one applies the transformation:

$$L'_{k+1} = L_{k+1} - 2^{k+1}\sum_{i=0}^{k}\frac{1}{2^i}\binom{k+1}{i}L'_i.$$

Remark that, after a few straight-forward manipulations, the next identity is revealed:

$$\frac{1}{2^i}\binom{k+1}{i}L'_i = \Bigg(\underbrace{0 \quad 0 \quad \cdots \quad 0}_{i} \quad 2^0\binom{k+1}{i}\binom{i}{i}$$
$$2^1\binom{k+1}{i}\binom{i+1}{i} \quad 2^2\binom{k+1}{i}\binom{i+2}{i} \quad \cdots \quad 2^{m-i}\binom{k+1}{i}\binom{m}{i}\Bigg).$$

Summing up these identities for $i = \overline{0,k}$, we obtain:

$$\sum_{i=0}^{k}\frac{1}{2^i}\binom{k+1}{i}L'_i = \Bigg(2^0\binom{k+1}{0}\binom{0}{0} \; 2^1\binom{k+1}{0}\binom{1}{0} + 2^0\binom{k+1}{1}\binom{1}{1}$$
$$\cdots 2^k\binom{k+1}{0}\binom{k}{0} + 2^{k-1}\binom{k+1}{1}\binom{k}{1} + \cdots + 2^0\binom{k+1}{k}\binom{k}{k}$$
$$2^{k+1}\binom{k+1}{0}\binom{k+1}{0} + 2^k\binom{k+1}{1}\binom{k+1}{1} + \cdots + 2^1\binom{k+1}{k}\binom{k+1}{k}$$
$$\cdots 2^m\binom{k+1}{0}\binom{m}{0} + 2^{m-1}\binom{k+1}{1}\binom{m}{1} + \cdots + 2^{m-k}\binom{k+1}{k}\binom{m}{k}\Bigg).$$

Finally, multiplying this sum by 2^{k+1} and factorizing through 2^{k+w+1} on each column $w = \overline{0,m}$, one arrives at:

$$2^{k+1}\sum_{i=0}^{k}\frac{1}{2^i}\binom{k+1}{i}L'_i$$
$$= \Bigg(2^{k+1}\frac{1}{2^0}\binom{k+1}{0}\binom{0}{0} 2^{k+2}\left[\frac{1}{2^0}\binom{k+1}{0}\binom{1}{0} + \frac{1}{2^1}\binom{k+1}{1}\binom{1}{1}\right]$$
$$\cdots 2^{k+k+1}\left[\frac{1}{2^0}\binom{k+1}{0}\binom{k}{0} + \frac{1}{2^1}\binom{k+1}{1}\binom{k}{1} + \cdots + \frac{1}{2^k}\binom{k+1}{k}\binom{k}{k}\right]$$

$$2^{k+k+2}\left[\frac{1}{2^0}\binom{k+1}{0}\binom{k+1}{0}+\frac{1}{2^1}\binom{k+1}{1}\binom{k+1}{1}\right.$$
$$\left.+\cdots+\frac{1}{2^k}\binom{k+1}{k}\binom{k+1}{k}\right]\cdots$$
$$\left.2^{m+k+1}\left[\frac{1}{2^0}\binom{k+1}{0}\binom{m}{0}+\frac{1}{2^1}\binom{k+1}{1}\binom{m}{1}+\cdots+\frac{1}{2^k}\binom{k+1}{k}\binom{m}{k}\right]\right).$$

On account of how the line L_{k+1} is defined, we get the following identity:

$$L'_{k+1}=\left(\underbrace{0\ \ 0\ \ \cdots\ \ 0}_{k+1}\quad 2^{k+1}\binom{k+1}{k+1}\quad 2^{k+2}\binom{k+2}{k+1}\cdots 2^m\binom{m}{k+1}\right).$$

Therefore $P(k+1)$ is a true statement, and hence, by principle of mathematical induction, $P(k)$ is true for any $k=\overline{1,m}$.

So, after m steps of the above algorithm, the matrix A_m is put into upper triangular form and we have:

$$\det A_m=\begin{vmatrix} 2^0\binom{0}{0} & 2^1\binom{1}{0} & 2^2\binom{2}{0} & \cdots & 2^m\binom{m}{0}\\ 0 & 2^1\binom{1}{1} & 2^2\binom{2}{1} & \cdots & 2^m\binom{m}{1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 2^m\binom{m}{m}\end{vmatrix}=2^{\frac{m(m+1)}{2}}.\quad\square$$

Now, the following two corollaries are obvious in view of Theorem 2.

Corollary 3. *The quadratic form $\sum_{i,j=0}^{n} f(i,j)X^iY^j$ induced by the matrix A_n is positive definite, for all $n\in\mathbb{N}$.*

Corollary 4. *For each $n\in\mathbb{N}$, all eigenvalues of the matrix A_n are positive.*

In the following, for every finite group G, we shall denote by $g(G)$ the number of distinct fuzzy subgroups of G. Our next aim is to determine explicitly this number for finite hamiltonian groups of type $H=Q_8\times\mathbb{Z}_{p^n}$, where Q_8 is the quaternion group of order 8, n is a positive integer and p is an odd prime.

In order to compute the number of elements in the set $\mathcal{C}$ consisting of all chains of subgroups of H that end in H, we shall follow the counting method developed in [9]. It is well-known that both Q_8 and $\mathbb{Z}_{p^n}$ contain an unique minimal subgroup, say M_1 and M_2, respectively. Moreover, we are able to describe the quotients of H with respect to M_1, M_2 and M_1M_2:

$$H/M_1 \cong (Q_8/M_1) \times \mathbb{Z}_{p^n} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p^n},$$

$$H/M_2 \cong Q_8 \times (\mathbb{Z}_{p^n}/M_2) \cong Q_8 \times \mathbb{Z}_{p^{n-1}},$$

$$H/M_1M_2 \cong (Q_8/M_1) \times (\mathbb{Z}_{p^n}/M_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p^{n-1}}.$$

Denote by $\mathcal{C}_i$ the set of all chains of $\mathcal{C}$ contained in the lattice interval $[H/M_i] = \{K \in L(H) \mid M_i \subseteq K \subseteq H\}, i = 1, 2$. Since every such chain $(**)$ determines two chains of $\mathcal{C}$ ($(**)$ itself and the chain obtained from $(**)$ by adding the trivial subgroup of H), it results:

$$g(H) = \mid \mathcal{C} \mid = 2 \mid \mathcal{C}_1 \cup \mathcal{C}_2 \mid = 2(\mid \mathcal{C}_1 \mid + \mid \mathcal{C}_2 \mid - \mid \mathcal{C}_1 \cap \mathcal{C}_2 \mid). \quad (2)$$

We have

$$\mid \mathcal{C}_1 \mid = g(H/M_1) = g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p^n}),$$

$$\mid \mathcal{C}_2 \mid = g(H/M_2) = g(Q_8 \times \mathbb{Z}_{p^{n-1}}),$$

$$\mid \mathcal{C}_1 \cap \mathcal{C}_2 \mid = g(H/M_1M_2) = g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p^{n-1}}),$$

according to the previous isomorphisms. Put $g(n) = g(Q_8 \times \mathbb{Z}_{p^n})$ and $h(n) = g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p^n})$, for all $n \in \mathbb{N}$. Then (2) becomes:

$$g(n) = 2g(n-1) + 2(h(n) - h(n-1)). \quad (3)$$

In this way, one needs to compute first the quantities $h(n)$, $n \in \mathbb{N}$. By applying the same reasoning for the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{p^n}$, it is easy to see that h verifies the recurrence relation

$$h(n) = 2h(n-1) + 2(f(2, n) - f(2, n-1)),$$

where f is the function defined in the proof of Proposition 1. One obtains

$$h(n) = 2h(n-1) + 2^n(3n + 7),$$

implying that

$$h(n) = 2^{n-1}(3n^2 + 17n + 16),$$

for all $n \in \mathbb{N}$. Thus, (3) can be written as

$$g(n) = 2g(n-1) + 2^{n-1}(3n^2 + 23n + 30),$$

which leads to

$$g(n) = 2^n g(0) + \sum_{k=1}^{n} 2^{n-k} 2^{k-1}(3k^2 + 23k + 30)$$

$$= 2^{n+4} + 2^{n-1} \sum_{k=1}^{n} (3k^2 + 23k + 30) = 2^{n-1}(n^3 + 13n^2 + 42n + 32),$$

for all $n \in \mathbb{N}$. Hence we have proved the following theorem.

Theorem 5. *The number $g(n)$ of distinct fuzzy subgroups of a finite hamiltonian group of type $Q_8 \times \mathbb{Z}_{p^n}$, where p is an odd prime, is given by the equality:*

$$g(n) = 2^{n-1}(n^3 + 13n^2 + 42n + 32).$$

Remarks.

1) Even if the groups $Q_8 \times \mathbb{Z}_{p^n}$ and $\mathbb{Z}_{2^3} \times \mathbb{Z}_{p^n}$ have the same order, their numbers of distinct fuzzy subgroups

$$g(n) \text{ and } f_2(3,n) = \frac{2^{n-1}}{3}\,(n^3 + 15n^2 + 8n + 42)$$

are different. So, the commutativity plays an essential role in determining the number of distinct fuzzy subgroups of a group. Moreover, we remark that:

$$\lim_{n\to\infty} \frac{g(n)}{f_2(3,n)} = 3.$$

This is due to the fact that Q_8 possesses three maximal chains of subgroups, in contrast with $\mathbb{Z}_{2^3}$ which has only one.

2) A special class of fuzzy subgroups of a group G consists of all fuzzy subgroups that correspond to maximal chains of subgroups of G (that is, to composition series of G). Let us denote this class by $\mathcal{M}(G)$. By Theorem 2.2 of [7], if G is a finite nilpotent group of order $p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$ $(p_1, p_2, ..., p_m$

distinct primes) and $G = \bigtimes_{i=1}^{k} G_i$ is the decomposition of G as a direct product of its Sylow subgroups, then the numbers $m(G)$ and $m(G_i)$ ($i = 1, 2, ..., k$) of maximal chains of subgroups in G and G_i, respectively, are connected by the following equality

$$m(G) = \binom{n_1 + n_2 + ... + n_k}{n_1, n_2, ..., n_k} \prod_{i=1}^{k} m(G_i),$$

where
$$\binom{n_1 + n_2 + ... + n_k}{n_1, n_2, ..., n_k} = \frac{(n_1 + n_2 + ... + n_k)!}{n_1! n_2! ... n_k!}.$$

In our case, by applying the above result, it results:

$$\mid \mathcal{M}(Q_8 \times \mathbb{Z}_{p^n}) \mid = \binom{3+n}{3, n} m(Q_8) m(\mathbb{Z}_{p^n})$$

$$= 3\binom{n+3}{3} = \frac{1}{2}\,(n^3 + 6n^2 + 11n + 6).$$

Finally, we mention that the method used in proving Theorem 5 can successfully be extended to the most general situation of a finite hamiltonian group of type $H = Q_8 \times \mathbb{Z}_n$, where $n = p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$ with $p_1, p_2, ..., p_k$ odd primes. As for $k = 1$, by applying the well-known Inclusion-Exclusion Principle, it follows that the number $g(H) = g(n_1, n_2, ..., n_k)$ satisfies a certain recurrence relation which depends on the quantity $h(n_1, n_2, ..., n_k) = g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_n)$.

Open problem. Find an explicit formula for the number of distinct fuzzy subgroups of a finite hamiltonian group of type $Q_8 \times \mathbb{Z}_n$, when n is an *arbitrary* positive odd integer.

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