ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LV, 2009, f.1

A SUFFICIENT CONDITION FOR THE CONNECTEDNESS OF THE ATTRACTORS OF INFINITE ITERATED FUNCTION SYSTEMS

BY

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Abstract. The aim of the paper is to give a sufficient condition for the connectedness of the attractors of infinite iterated function systems.

Mathematics Subject Classification 2000: 28A80.

Key words: infinite iterated function system, connected set, arcwise connected set, shift space, attractor.

1. Introduction. Iterated function systems (IFS) are one of the most common and general ways to generate fractals. IFS were conceived in their present form by HUTCHINSON [4] and are popularized by BARNSLEY [1]. There is a current effort to extend Hutchinson's classical framework for fractals to more general spaces and infinite IFSs and to study them. For example, in [6], it was provided such a general framework where attractors are non-empty closed and bounded subsets of complete metric spaces and where the IFSs may be infinite, in contrast with the classical theory ([1], [2], [3] or [4]), where only attractors that are compact metric spaces and IFSs that are finite are considered. In the classical case there exists sufficient and necessary conditions for the connectedness of the attractor of an IFS (see Theorem 1.3 below). These conditions are not equivalent for IIFSs. We give in this paper a sufficient condition for the connectedness of the attractors of infinite iterated function systems.

The paper is divided in three parts. The first part is the introduction. The second part contains the main results. The last one contains some examples.

We start by a short presentation of an infinite iterated function system, an IIFS for short, and by fixing the notations.

Let (X, d_X) and (Y, d_Y) be two metric spaces. By $\mathcal{C}(X, Y)$ we will understand the set of continuous functions from X to Y. On $\mathcal{C}(X, Y)$ we will consider the generalized metric $\overline{d} : \mathcal{C}(X, Y) \times \mathcal{C}(X, Y) \to \overline{\mathbb{R}}_+$, defined by $\overline{d}(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$, where $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$.

For $f_n, f \in \mathcal{C}(X, Y)$, $f_n \overset{s}{\to} f$ denotes the pointwise convergence, $f_n \overset{u.c}{\to} f$ denotes the uniform convergence on compact sets and $f_n \overset{u}{\to} f$ denotes the uniform convergence, that is the convergence in the generalized metric $\overline{d}$.

A family of functions $(f_i)_{i \in I} \subset \mathcal{C}(X, Y)$ is said to be bounded if for every bounded set $A \subset X$ the set $\bigcup_{i \in I} f_i(A)$ is bounded.

For a function $f : X \to X$ let us denote by $\mathbf{Lip}(f) \in [0, +\infty]$ the Lipschitz constant associated to f, $\mathbf{Lip}(f) = \sup_{x,y \in X; x \neq y} \frac{d(f(x), f(y))}{d(x,y)}$.

Definition 1.1. f is a *Lipschitz function* if $\mathbf{Lip}(f) < +\infty$ and a *contraction* if $\mathbf{Lip}(f) < 1$.

Let $Con(X) := \{f : X \to X \mid \mathbf{Lip}(f) < 1\}$.

For a set X, $\mathcal{P}(X)$ denotes the subsets of X and $\mathcal{P}^*(X) = \mathcal{P}(X) - \{\emptyset\}$. For a subset A of $\mathcal{P}(X)$, by A^* we mean $A - \{\emptyset\}$.

Let (X, d) be a metric space, $\mathcal{K}(X)$ be the set of compact subsets of X and $\mathcal{B}(X)$ the set of bounded closed subsets of X. It is obvious that $\mathcal{K}(X) \subset \mathcal{B}(X) \subset \mathcal{P}(X)$.

Definition 1.2. The *generalized Hausdorff-Pompeiu semidistance* is an application $h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to [0, +\infty]$ defined by $h(A, B) = \max(d(A, B), d(B, A)) = \inf\{r \in [0, \infty] \mid A \subset B(B, r) \text{ and } B \subset B(A, r)\}$, where $d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A}(\inf_{y \in B} d(x, y))$.

Concerning the Hausdorff-Pompeiu semidistance we have the following important properties:

Proposition 1.1 ([7] Ch1 Sec1 Theorem 1.1 and [6] Sec 3 proof of Lemma 1)**.** *Let (X, d_X) and (Y, d_Y) be two metric spaces. Then:*

1) *If H and K are two nonempty subsets of X then $h_X(H, K) = h_X(\overline{H}, \overline{K})$, where h_X is the Hausdorff-Pompeiu semidistance associated with the distance d_X.*

2) *If $(H_i)_{i \in I}$ and $(K_i)_{i \in I}$ are two families of nonempty subsets of X then*

$$h_X(\bigcup_{i \in I} H_i, \bigcup_{i \in I} K_i) = h_X(\overline{\bigcup_{i \in I} H_i}, \overline{\bigcup_{i \in I} K_i}) \leq \sup_{i \in I} h_X(H_i, K_i).$$

3) *If H and K are two nonempty subsets of X and $f : X \to Y$ is a function then*

$$h_X(f(K), f(H)) \leq \mathbf{Lip}(f) \cdot h_Y(K, H).$$

4) *If $(H_n)_{n\geq 1} \subset \mathcal{P}(X)$ is a sequence of sets and $H \in \mathcal{P}(X)$ is a set such that $h_X(H, H_n) \to 0$, then an element $x \in X$ is in H if and only if there exists $x_n \in H_n$ such that $x_n \to x$.*

Theorem 1.1 (see [1] Ch2 Sec 7 Theorem 1, [6] Sec 3 Lemma 1 and [7] Ch1 Sec2 Theorems 1.3, 1.4 and 1.5)**.** *Let (X, d) be a metric space and $h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to [0, \infty]$ the Hausdorff-Pompeiu semidistance. Then:*

1) *$(\mathcal{B}^*(X), h)$ and $(\mathcal{K}^*(X), h)$ are metric spaces with $(\mathcal{K}^*(X), h)$ closed in $\mathcal{B}^*(X)$.*

2) *If (X, d) is complete then $(\mathcal{B}^*(X), h)$ and $(\mathcal{K}^*(X), h)$ are complete metric spaces.*

3) *If (X, d) is compact then $(\mathcal{K}^*(X), h)$ is compact and in this case $\mathcal{B}^*(X) = \mathcal{K}^*(X)$.*

4) *If (X, d) is separable then $(\mathcal{K}^*(X), h)$ is separable.*

An infinite iterated function system on X consists of a bounded family of contractions $(f_i)_{i\in I}$ on X and it is denote by $\mathcal{S} = (X, (f_i)_{i\in I})$.

For an IIFS, $\mathcal{S} = (X, (f_i)_{i\in I})$, $F_{\mathcal{S}} : \mathcal{B}^*(X) \to \mathcal{B}^*(X)$ is the function defined by $F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)}$.

The function $F_{\mathcal{S}}$ is a contraction with $\mathbf{Lip}(F_{\mathcal{S}}) \leq \sup_{i\in I} \mathbf{Lip}(f_i)$.

By using the Banach's contraction theorem we can prove the following:

Theorem 1.2 ([6] Sec 3 Theorem 1)**.** *Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_i)_{i\in I})$ be an* IIFS *with $c = \sup_{i\in I} \mathbf{Lip}(f_i) < 1$. Then there exists a unique $A(\mathcal{S}) \in \mathcal{B}^*(X)$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$. Moreover, for any $H_0 \in \mathcal{B}^*(X)$ the sequence $(H_n)_{n\geq 0}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n)$ is convergent to $A(\mathcal{S})$. For the speed of the convergence we have the following estimation: $h(H_n, A(\mathcal{S})) \leq \frac{c^n}{1-c} h(H_0, H_1)$.*

The particular case for IFSs can be found in [1] (Ch 3 Sec 7 Theorem 1 pag 82), [3] (Ch 9 Sec 1 Theorem 9.1), [4], or [5] (Ch. 1 Sec 1 Theorem 1.1.7) and so on.

Definition 1.3. Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_i)_{i\in I})$ be an IIFS with $c = \sup_{i\in I} \mathbf{Lip}(f_i) < 1$. The unique invariant set, $A(S) = F_{\mathcal{S}}(A(S))$, from Theorem 1.2., is named the *attractor* of the IIFS $\mathcal{S} = (X, (f_i)_{i\in I})$.

Definition 1.4. Let (X,d) be a metric space and $(A_i)_{i\in I}$ be a family of nonempty subsets of X. The family $(A_i)_{i\in I}$ is said to be *connected* if for every $i,j\in I$ there exists $\{i_0,i_1,...,i_n\}\subset I$ such that $i_0=i$, $i_n=j$ and $A_k\cap A_{k+1}\neq\emptyset$ for every $k=0,1,...,n-1$.

It is well-known the following result:

Lemma 1.1. *Let (X,d) be a metric space and $(A_i)_{i\in I}$ a connected family of connected subset of X. Then $\bigcup_{i\in I}A_i$ is connected.*

Definition 1.5. A metric space (X,d) is *arcwise connected* if for every $x,y\in X$ there exists a continuous function $\varphi:[0,1]\to X$ such that $\varphi(0)=x$ and $\varphi(1)=y$.

Theorem 1.3 ([5] Ch 1 Sec 6 Theorem 1.6.2, [8] Ch 2 Sec 2 Theorem 2.5)**.** *Let (X,d) be a complete metric space, $\mathcal{S}=(X,(f_k)_{k=\overline{1,n}})$ be an* IFS *with $c=\max_{k=\overline{1,n}}\mathbf{Lip}(f_k)<1$ and $A(\mathcal{S})$ the attractor of $\mathcal{S}$. The following are equivalent:*

1) *The family $(A_i)_{i=\overline{1,n}}$ is connected where $A_i=f_i(A(\mathcal{S}))$ for all i.*

2) *$A(\mathcal{S})$ is arcwise connected.*

3) *$A(\mathcal{S})$ is connected.*

This theorem is not true for IIFS as we can see from Example 3.1. We want to study the case of IIFS.

2. The main result. We present now a sufficient condition for the connectedness of the attractors of infinite iterated function systems.

Theorem 2.1. *Let (X,d) be a complete metric space and $\mathcal{S}$=$(X,(f_i)_{i\in I})$ an* IIFS*. Let $(I_j)_{j\in J}\subset I$ such that:*

1) *$I=\bigcup_{j\in J}I_j$.*

2) *B_j is connected, where B_j:=$A(\mathcal{S}_j)$ is the attractor of $\mathcal{S}_j$=$(X,(f_i)_{i\in I_j})$ for all j.*

3) *$\bigcup_{j\in J}B_j$ is connected.*

Then $A(\mathcal{S})$ is connected.

Proof. Let $C=\bigcup_{j\in J}B_j$ and $(C_n)_n$ be a sequence of subsets of X defined by $C_0:=C$ and $C_n:=F_{\mathcal{S}}(C_{n-1})$, for every $n\in\mathbb{N}^*$ where $F_{\mathcal{S}}(H)$ is the function we already introduced, for any $H\subset X$.

The proof will be made into two steps.

Firstly, we prove that $C_n \subset C_{n+1}$ and $\overline{\bigcup_{n\geq 1} C_n} = A(\mathcal{S})$. Secondly we will prove by induction that C_n is connected for all n. From the basic properties of connected sets will result that $A(\mathcal{S})$ is connected.

We prove by induction that $C_n \subset C_{n+1}$.

For the first step we have: $B_j = F_{\mathcal{S}_j}(B_j) \subset F_{\mathcal{S}}(B_j)$ for every $j \in J$. It results that $C = C_0 = \bigcup_{j\in J} B_j \subset \bigcup_{j\in J} F_{\mathcal{S}}(B_j) \subset F_{\mathcal{S}}(\bigcup_{j\in J} B_j) = F_{\mathcal{S}}(C_0) = C_1$. Thus $C_0 \subset C_1 = F_{\mathcal{S}}(C_0)$.

By induction, suppose that $C_n \subset C_{n+1}$. It follows that $F_{\mathcal{S}}(C_n) \subset F_{\mathcal{S}}(C_{n+1})$. That is $C_{n+1} \subset C_{n+2}$.

Let $D = \bigcup_{n\geq 1} C_n$. Because $C_n = F_{\mathcal{S}}(C_{n-1})$ for every $n \in \mathbb{N}^*$ we have from Theorem 1.2. that $C_n \to A(\mathcal{S})$.

Let $n_0 \in \mathbb{N}^*$ and $x \in C_{n_0}$. We consider the sequence $(x_n)_{n\geq n_0}$, defined by $x_n = x$. For every $n \geq n_0$, we have $x_n \in C_n$, because $C_{n_0} \subset C_n$. Then from Proposition 1.1 (4), $x = \lim x_n \in \lim C_n = A(\mathcal{S})$. Because x was arbitrarily chosen in C_{n_0}, we have $C_{n_0} \subset A(\mathcal{S})$. Then $D = \bigcup_{n\geq 1} C_n \subset A(\mathcal{S})$ and so $\overline{D} \subset \overline{A(\mathcal{S})} = A(\mathcal{S})$, because $A(\mathcal{S})$ is closed.

Now, let $a \in A(\mathcal{S})$. From Proposition 1.1 (4) there exists $(x_n)_{n\geq 1}$ such that $x_n \in C_n \subset D = \bigcup_{n\geq 1} C_n$, for all $n \in \mathbb{N}^*$ and $x_n \to a$ (this is true because $C_n \to A(\mathcal{S})$). Then $A(\mathcal{S}) \subset \overline{D}$ and it follows that $A(\mathcal{S}) = \overline{D}$.

Secondly, we prove that C_n is connected for all n. We will prove that by induction. $C_0 = C$ is connected by hypothesis. Suppose now that C_n is connected for an $n > 0$.

We have $C_{n+1} = \overline{\bigcup_{i\in I} f_i(C_n)} = \overline{C_0 \cup (\bigcup_{i\in I} f_i(C_n))}$. To prove that C_{n+1} is connected it is sufficient to prove that $C_0 \cup (\bigcup_{i\in I} f_i(C_n))$ is connected, because the closure of every connected set is connected. We will prove that $C_0 \cap f_i(C_n) \neq \emptyset$, for all $i \in I$.

Let $i \in I = \bigcup_{j\in J} I_j$. It follows that there exists $l(i) \in J$ such that $i \in I_{l(i)}$.

Because $B_{l(i)} = A(\mathcal{S}_{l(i)}) \subset C_0 \subset C_n$, for every $i \in I$ we have $f_i(B_{l(i)}) \subset f_i(C_n)$.

But, $f_i(B_{l(i)}) \subset F_{\mathcal{S}_{I_{l(i)}}}(B_{l(i)}) = B_{l(i)} \subset C_0$.

Then $\emptyset \neq f_i(B_{l(i)}) \subset C_0 \cap f_i(C_n)$, for every $i \in I$. We have proved that the set $C_0 \cap f_i(C_n)$ is nonempty.

Because C_n is connected it results that $f_i(C_n)$ is connected and then from Lemma 1.1, using the fact that $C_0 \cap f_i(C_n)$ is nonvoid, it follows that $C_0 \cup (\bigcup_{i\in I} f_i(C_n))$ is connected. With this we have proved that C_{n+1} is connected.

Thus, $\bigcap_{n\geq 0} C_n = C_0$ and C_n are connected for every $n \in \mathbb{N}$. This imply that $D = \bigcup_{n\geq 0} C_n$ is connected ant so $A(S) = \overline{D}$ is connected. □

Corollary 2.1. *Let (X, d) be a complete metric space and $\mathcal{S}=(X, (f_i)_{i\in I})$ an IIFS. Let $(I_j)_{j\in J} \subset I$ such that:*
1) $I = \bigcup_{j\in J} I_j$.
2) *B_j is connected, where $B_j:=A(\mathcal{S}_j)$ is the attractor of $\mathcal{S}_j=(X, (f_i)_{i\in I_j})$ for all j.*
3) *$(B_j)_{j\in J}$ is connected.*
Then $A(\mathcal{S})$ is a connected set.

Proof. Because $(B_j)_{j\in J}$ is a connected family of connected sets, it follows from the Lemma 1.1. that $\bigcup_{j\in J} B_j$ is connected. Therefore, by the Theorem 2.1, $A(\mathcal{S})$ is connected. □

Corollary 2.2. *Let (X, d) be a complete metric space and $\mathcal{S}=(X, (f_i)_{i\in I})$ an IIFS. Let $(I_j)_{j\in J} \subset I$ be such that:*
1) $I = \bigcup_{j\in J} I_j$.
2) *I_j are finite for all $j \in J$.*
3) *The families of sets $(f_i(B_j))_{i\in I_j}$ are connected, where $B_j := A(\mathcal{S}_j)$ is the attractor of $\mathcal{S}_j = (X, (f_i)_{i\in I_j})$ for all j.*
4) *$(B_j)_{j\in J}$ is connected.*
Then $A(\mathcal{S})$ is a connected set.

Proof. Let us consider a $j \in J$. Because I_j is finite then $\mathcal{S}_j=(X, (f_i)_{i\in I_j})$ is an IFS. Because the family $(f_i(B_j))_{i\in I_j}$ is connected, where $B_j := A(\mathcal{S}_j)$ is the attractor of $\mathcal{S}_j$,then by the Theorem 1.3 we have that B_j is connected. This is true for all $j \in J$, because j was arbitrarily chosen, thus, by the Corollary 2.1, $A(\mathcal{S})$ is connected. □

3. Examples

Example 1. Let (X, d) be a complete metric space and $A \in \mathcal{B}^*(X)$. For an element $a \in X$, f_a will denote the constant function of values a, that is $f_a : X \to X$ and $f_a(x) = a$ for every $x \in X$. Then A is the attractor of the IIFS $\mathcal{S} = (X, (f_a)_{a\in A})$ if A is infinite or IFS $\mathcal{S} = (X, (f_a)_{a\in A})$ if A is finite. Also A is the attractor of the IIFS $\mathcal{S}_B = (X, (f_a)_{a\in B})$ for any dense set B in A. In particular if A is separable and B is a countable dense set in A then A is the attractor of the CIFS (countable IFS) $\mathcal{S}_B = (X, (f_a)_{a\in B})$.

This happens, in particular for any compact set A. Because a closed (or compact) set could be connected but not arcwise connected it follows that the conditions 2) and 3) are not equivalent for a general IIFS. Also because the family of sets $(A_a = f_a(A) = \{a\})_{a\in A}$ is not connected for every set A (in fact $A_a \cap A_b = \emptyset$) it follows that points 1) and 2) from Theorem 1.3 are not equivalent for a general IIFS.

Example 2. (see [6] Ch 1, Ch 4 and 5 for details) First let us fix some notations. Let A be an arbitrary set, z a point of A and $A' = A - \{z\}$. The points of $l^2(A)$ are collections of real numbers indexed by the points of A' such that $\sqrt{\sum_a (x_a)^2} < +\infty$. The topology of $l^2(A)$ is induced from the metric $d(x,y) = \sqrt{\sum_a (x_a - y_a)^2}$, where we think x_a as the a-th coordinate of x. Let us also consider, for the case when A is an arbitrary set with the discrete topology, the Baire space $N(A)$ which is the topological product of countably many copies A_n of A. Hence the points of $N(A)$ consist of all sequences $\mathbf{a} = a_1a_2...a_n...$, with $a_n \in A$. Moreover $N(A)$ is a metric space with the metric

$$d(\mathbf{a}, \mathbf{a}') = \begin{cases} \frac{1}{k}, & \text{if } k \text{ is the first index where } a_i \neq a_i' \\ 0, & \text{if } \mathbf{a} = \mathbf{a}'. \end{cases}$$

Lipscomb's space $L(A)$ is a quotient space of $N(A)$ such that each equivalence class consists of either a single point or two points. Those classes with two points come from identifying $a_1a_2...a_{k-2}a_{k-1}a_ka_k...$ with the point $a_1a_2...a_{k-2}a_ka_{k-1}a_{k-1}...$, where $a_{k-1} \neq a_k$.

Let ϖ^A be the imbedded version of $L(A)$ endowed with the $l^2(A)$-induced topology. More precisely, let us consider the function $s : L(A) \to l^2(A)$ given by $s(\alpha) = (\alpha_b)_{b\in A}$, where $\alpha = a_1a_2.... \in N(A)$ and

$$\alpha_b = \begin{cases} \sum\limits_{k \text{ with } a_k = b} \frac{1}{2^k}, & \text{if there exists } k \text{ such that } a_k = b \\ 0, & \text{if there exists no } k \text{ such that } a_k = b. \end{cases}$$

Then ϖ^A is $s(L(A))$.

We describe now an iterated function system. Let us consider the complete metric space $(l^2(A), d)$. For each $a \in A \setminus \{z\}$, let $f_a : l^2(A) \to l^2(A)$ be the function given by $f_a(x) = \frac{1}{2}(x + u_a)$, for all $x \in l^2(A)$ where $u_z = 0_{l^2(A)}$ and, for $a \in A \setminus \{z\}$, $u_a = (\alpha_j)_{j\in A} \in l^2(A)$ is described by

$\alpha_j = 0$, for $j \neq a$ and $\alpha_a = 1$. Let us note that, for each $a \in A$, f_a is a $\frac{1}{2}$-contraction. Moreover, the family $(f_a)_{a\in A}$ is bounded. Indeed, if M is a bounded subset of $l^2(A)$, then there exists $r > 0$ such that $M \subseteq B_r(0)$. Then $f_a(B_r(0)) \subseteq B_{\frac{r+1}{2}}(0)$, for each $a \in A$, so $\bigcup_{a\in A} f_a(M) \subseteq B_{\frac{r+1}{2}}(0)$, i.e. $\bigcup_{a\in A} f_a(M)$ is bounded. Let $\mathcal{S} = (l^2(A), (f_a)_{a\in A})$. According to Theorem 1.2, there exists B a bounded closed subset of $l^2(A)$ such that $B = \overline{\cup_{a\in A} f_a(B)}$. It has been proved in [6] Ch 5 that B is ϖ^A.

We suppose that A contains at least two elements. If A contains one element then $\varpi^A = \{0_{l^2(A)}\}$. Let us consider a pair (a, b) of different elements from A. $A_{(a,b)}$, the attractor of the IFS $\mathcal{S}_{(a,b)} = (l^2(A), (f_a, f_b))$, is connected. Indeed, the fixed points of f_a and f_b are u_a and u_b. Then $f_a(u_b) = f_b(u_a) = \frac{1}{2}(u_a + u_b) \in f_a(A_{(a,b)}) \cap f_b(A_{(a,b)})$. From Theorem 1.3 it follows that $A_{(a,b)}$ is connected. Let $J = P_2(A)$ and $I_{\{a,b\}} = \{a, b\}$. Then $A = \bigcup_{\{a,b\}\in P_2(A)} \{a, b\}$, the sets $A_{(a,b)} = A(\mathcal{S}_{(a,b)})$ are connected and because the family of sets $(A_{(a,b)})_{\{a,b\}\in P_2(A)}$ is connected from Lemma 1.1 it follows that the set $\bigcup_{\{a,b\}\in P_2(A)} A_{(a,b)}$ is connected. From Theorem 2.1 the set $M = A(\mathcal{S})$ is connected.

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Received: 29.IX.2007
Revised: 8.XI.2007

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