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MATEMATICĂ, Tomul LV, 2009, f.1

A GENERALIZED GOULD TYPE INTEGRAL WITH RESPECT TO A SUBMEASURE I

BY

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Abstract. In the paper [5] written jointly with A. Gavrilut we have introduced and studied a Gould type integral for a bounded real valued function with respect to a submeasure of finite variation. In the present paper we define a Gould type integral in a more general case for an unbounded function with respect to a submeasure which may be not of finite variation. We point out several basic properties of this integral and we give a characterization of the integrability in generalized sense.

Mathematics Subject Classification 2000: 28B20, 28C15, 49J53.
Key words: Gould type integral, submeasure, multisubmeasure.

1. Introduction. In [7], Gould has introduced and studied a special type of integral for a real valued function with respect to a finite additive set function taking values in a Banach space. In [8] and [9], Precupanu and Croitoru have defined and studied a Gould type integral for a real valued function with respect to a multimeasure with values in the family of all nonvoid, compact, convex sets of a Banach space. Following ideas from [8], Gavriluţ has studied in [3] and [4] a Gould type integral for a bounded, real valued function with respect to a multisubmeasure of finite variation. In [5] Gavriluţ and Petcu have investigated a Gould type integral for a bounded, real valued function with respect to a real submeasure of finite variation.

In this paper we introduce and study a Gould type integral in a more general case for an arbitrary real function with respect to a submeasure which may be not of finite variation.

In the following, we shall use the terminology from [8] and [9].

Let T be an abstract, nonvoid set, $\mathcal{A}$ an algebra of subsets of T and $f : T \to \mathbb{R}$ a function.

We denote by $\mathcal{P}$ the class of all finite partitions of T by sets from $\mathcal{A}$.

All over this paper we assume that $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ is a submeasure in Drewnowski's sense [2].

Let us also consider the following set functions associated to m:

$$\overline{m}(A) = \sup\{\sum_{i=1}^{n} m(A_i)\}, \text{ for every } A \subset T, \tag{1}$$

where the supremum is extended over all finite partitions $\{A_i\}_{i=\overline{1,n}}$ of A, called *the variation* of m; and

$$\widetilde{m}(A) = \inf\{\overline{m}(B); A \subseteq B, B \in \mathcal{A}\}, \text{ for every } A \subseteq T. \tag{2}$$

We note that $\widetilde{m}(A) = \overline{m}(A)$, for every $A \in \mathcal{A}$ and $\overline{m}$ is a finitely additive set function on $\mathcal{A}$.

Definition 1.1. We say that *a property* (P) *holds* m*-almost everywhere* if the property (P) is valid on $T\backslash A$, with $\widetilde{m}(A) = 0$.

Definition 1.2. A submeasure $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ is said to be:

i) $\sigma-$*subadditive* if, for any sequence $(A_n)_{n\in\mathbb{N}^*}$ of pairwise disjoint sets in $\mathcal{A}$ ($\mathbb{N}^* = \mathbb{N}\backslash\{0\}$), such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, we have $m(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m(A_n)$;

ii) $\varepsilon-\delta-$ *absolutely continuous* with respect to a set function $\nu : \mathcal{A} \mapsto \mathbb{R}$, if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for every $A \in \mathcal{A}$ with $\overline{m}(A) < \delta$, we have $|\nu(A)| < \varepsilon$, denoted by $m \ll_I \nu$;

iii) *absolutely continuous* with respect to a set function $\nu : \mathcal{A} \mapsto \mathbb{R}$, if for every $A \in \mathcal{A}$ with $\nu(A) = 0$ we have $m(A) = 0$, denoted by $m \ll_{II} \nu$.

Definition 1.3.

I) A set $A \in \mathcal{A}$ is said to be $m-$*bounded* if $\{m(B); B \subseteq A, B \in \mathcal{A}\}$ is a bounded set in $\mathbb{R}$.

II) The space $(T, \mathcal{A}, m)$ is said to be *non-degenerate* if for any m-unbounded set $A \in \mathcal{A}$, there exists $\alpha > 0$ such that every $m-$unbounded set $B \in \mathcal{A}$, $B \subseteq A$, contains a $m-$bounded set $C \in \mathcal{A}$, $C \subseteq B$ such that $m(C) > \alpha$.

Remark 1.4. A set $A \in \mathcal{A}$ is bounded if and only if $\overline{m}(A) < \infty$.

Definition 1.5. I) A function $f : T \to \mathbb{R}$ is said to be *$\overline{m}$-totally-measurable on* $(T, \mathcal{A}, m)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

i) $\overline{m}(A_0) < \varepsilon$ and

ii) $\sup_{t,s\in A_i} |f(t) - f(s)| = osc(f, A_i) < \varepsilon$, for every $i = \overline{1, n}$.

II) f is said to be *$\overline{m}$-totally-measurable on* $B \in \mathcal{A}$ if its restriction $f|_B$ of f to B is $\overline{m}$-totally measurable on $(B, \mathcal{A}_B, m_B)$, where $m_B = m|_{\mathcal{A}_B}$ and $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$.

2. Definition of the Gould g-m-integral. Without any special assumptions, $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$is a submeasure with total variation $\overline{m}$, $(T, \mathcal{A},\overline{m})$ is a non-degenerate space and $f : T \to \mathbb{R}$ is a real valued function.

We define:

$$M(f) = \{K \in \mathcal{A}; \overline{m}(K) < \infty \text{ and } f \text{ is bounded over } K\} \tag{3}$$

and

$$M_B(f) = M(f) \cap \mathcal{A}_B, \forall B \in \mathcal{A}, \text{ where } \mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}. \tag{4}$$

Since $\overline{m}$ is a finitely additive set function, using Remark 3.1 [9] for the special case in which the multimeasure is real valued, we obtain:

Remark 2.1. If $f : T \to \mathbb{R}$ is $\overline{m}$-totally-measurable, then $M(f) \neq \emptyset$.

Also, we give the following lemma concerning the $\overline{m}$-totally-measurability of f:

Lemma 2.2. *If $f : T \to \mathbb{R}$ is $\overline{m}$-totally-measurable, then for every $\varepsilon > 0$ there exist $N(\varepsilon) > 0$ and $A_N \in \mathcal{A}$ such that $\overline{m}(T\backslash A_N) < \varepsilon$ and $\sup_{t\in A_N} |f(t)| < N$.*

Proof. Since f is $\overline{m}-$ totally measurable for every $\varepsilon > 0$, there exists a partition $P = \{A_i\}_{i=\overline{0,n}}$ of T such that $\overline{m}(A_0) < \varepsilon$ and $\sup_{s,t\in A_i} |f(s) - f(t)| < \varepsilon$, for every $i = \overline{1, n}$. Because f is bounded on each A_i, for every $i = \overline{1, n}$, f is bounded on $A_N = \bigcup_{i=1}^n A_i$. Then there exists $N > 0$ such that $\sup_{t\in A_N} |f(t)| < N$ and $\overline{m}(T\backslash A_N) < \varepsilon$, as claimed. □

We now define the Gould m-integrability of a function in general case.

Definition 2.3. I) We say that f is *generalized Gould $m-$integrable* (briefly, *g-m-integrable*) *on* T if the following conditions are satisfied:

i) f is $\overline{m}$-totally-measurable;

ii) there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for any $K \in M(f)$ with $K \supseteq K_\varepsilon$, we have:

$$\left| \int_K^* f dm - I \right| < \varepsilon,$$

where $\int_K^* f dm$ is the integral in the sense of Definition 2.3 [5].

The real number I is called the *generalized Gould integral of* f (briefly, *g-m-integral*). We denote it by $\int_T f dm$.

II) f is said to be *$g - m$-integrable on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is *$g - m$-integrable* on $(B, \mathcal{A}_B, m_B)$.

Remark 2.4. I) If $f : T \to \mathbb{R}$ is $g - m-$integrable, then $\int_T f dm$ is uniquely determined.

II) If $f : T \to \mathbb{R}$ is a bounded $g - m-$integrable function on T and $\overline{m}(T) < \infty$, then $\int_T^* f dm = \int_T f dm$.

Thus we can use the same symbol $\int_T f dm$ for the Gould integral in both situations "bounded" or "unbounded".

In the following we give a characterization of the $g - m-$integrability.

Theorem 2.5. *A $\overline{m}$-totally-measurable function $f : T \to \mathbb{R}$ is $g - m-$integrable on T if and only if for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for any $K \in M(f)$ with $K \cap K_\varepsilon = \emptyset$, we have*:

$$\left| \int_K f dm \right| < \varepsilon.$$

Proof. The if part.

From Definition 2.3. there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for any $K \in M(f)$ with $K \supseteq K_\varepsilon$, we have:

$$\left| \int_K f dm - I \right| < \frac{\varepsilon}{2}. \tag{5}$$

Let us consider $K \in M(f)$ with $K \cap K_\varepsilon = \emptyset$. Since f is g$-m-$integrable we have

$$\left| \int_{K \cup K_\varepsilon} f dm - I \right| < \frac{\varepsilon}{2}. \tag{6}$$

According to Theorem 2.12 [5] and using successively (6) and (5), we obtain:

$$\begin{aligned}\left|\int_K f\,dm\right| &= \left|\int\limits_K f\,dm + \int\limits_{K_\varepsilon} f\,dm - \int\limits_{K_\varepsilon} f\,dm\right| \\ &\le \left|\int\limits_{K\cup K_\varepsilon} f\,dm - I\right| + \left|\int\limits_{K_\varepsilon} f\,dm - I\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

The only if part.

By hypothesis for $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for any $K \in M(f)$ with $K \cap K_\varepsilon = \emptyset$, we have:

$$\left|\int_K f\,dm\right| < \varepsilon. \tag{7}$$

For every $K \in M(f)$ we consider the net $\{(\int_A f\,dm)_{K\subseteq A}, A \in M(f)\}$, directed by inclusion relation. Let us take $K = K_\varepsilon$. We shall prove that the $g - m-$integral is the limit of the net $\{(\int_A f\,dm)_{K_\varepsilon\subseteq A},\ A \in M(f)\}$.

With this end in mind we show that the net $(\int_A f\,dm)$ is Cauchy in $\mathbb{R}$.

Indeed, using Theorem 2.12 [5] and from (7), for every $A_1, A_2 \in M(f)$ with $K_\varepsilon \subseteq A_1, K_\varepsilon \subseteq A_2$ we get that

$$\begin{aligned}&\left|\int_{A_1} f\,dm - \int_{A_2} f\,dm\right| = \left|\int_{K_\varepsilon\cup(A_1\setminus K_\varepsilon)} f\,dm - \int_{K_\varepsilon\cup(A_2\setminus K_\varepsilon)} f\,dm\right| \\ &= \left|\int_{K_\varepsilon} f\,dm + \int_{A_1\setminus K_\varepsilon} f\,dm - \int_{K_\varepsilon} f\,dm - \int_{A_2\setminus K_\varepsilon} f\,dm\right| \\ &= \left|\int_{A_1\setminus K_\varepsilon} f\,dm - \int_{A_2\setminus K_\varepsilon} f\,dm\right| \le \left|\int_{A_1\setminus K_\varepsilon} f\,dm\right| + \left|\int_{A_2\setminus K_\varepsilon} f\,dm\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

that is $(\int\limits_A f\,dm)$ is a Cauchy net in $\mathbb{R}$.

Thus, it converges to a real number I, which means, according to Definition 2.3, that f is g-m-integrable on T.

The proof is completed. □

Remark 2.6. According to Theorem 2.15 [5], we observe that $f : T \to \mathbb{R}$ is $g - m-$integrable on T if and only if is g-$\overline{m}-$integrable on T.

Theorem 2.7. *If $f = 0$ m-almost everywhere, then f is $g-m-$integrable on T and $\int_T f dm = 0$.*

Proof. First, we shall prove that f is $\overline{m}$-totally-measurable.

Indeed, since $f = 0$ m-almost everywhere there exists $A \subset T$ such that $\widetilde{m}(A) = 0$ and $f(t) = 0$ for $t \in T \backslash A$. Using (2), from $\widetilde{m}(A) = 0$ it results that for every $\varepsilon > 0$ there exists $B_\varepsilon \in \mathcal{A}$ such that $A \subseteq B_\varepsilon$ and $\overline{m}(B_\varepsilon) < \varepsilon$.

We consider the partition $P_\varepsilon = \{B_\varepsilon, T \backslash B_\varepsilon\}$ of T.

Since $\overline{m}(B_\varepsilon) < \varepsilon$ and $osc(f, T\backslash B_\varepsilon) = \sup_{t,s \in T\backslash B_\varepsilon} |f(t) - f(s)| = 0 < \varepsilon$, we get that f is $\overline{m}$-totally-measurable.

From Remark 2.1 we see that $M(f) \neq \emptyset$. Let $K_0 \in M(f)$ fixed and $K \in M(f)$ arbitrary with $K \supseteq K_0$.

Since $K \in M(f)$ we have $\overline{m}(K) < \infty$ and f is bounded on K.

According to Theorem 2.6 [5] we obtain $\int\limits_K f dm = 0$.

Consequently, for every $\varepsilon > 0$ there is $K_0 \in M(f)$ such that for every $K \in M(f)$ with $K \supseteq K_0$ $\left|\int_K f dm - 0\right| = |0 - 0| = 0 < \varepsilon$, that is f is $g - m-$integrable on T and $\int_T f dm = 0$.

3. Properties of the g-m-integral.

In this section we point out some basic properties of the $g - m-$integral concerning its behavior with respect to the function. More precisely, homogeneity, additivity and monotonicity properties of this type of integral are obtained.

Theorem 3.1. *If f is a $g - m-$integrable function on T and $\alpha \in \mathbb{R}$, then αf is $g - m-$integrable on T and $\int_T \alpha f dm = \alpha \int_T f dm$.*

Proof. The case $\alpha = 0$ is trivial. Let $\alpha \neq 0$.

We easily see that αf is $\overline{m}$-totally-measurable.

Now we prove ii) of Definition 2.3.

Let be $M(\alpha f) = \{K \in \mathcal{A}; \overline{m}(K) < \infty \text{ and } \alpha f \text{ is bounded over } K\}$.

Because f is g-m-integrable on T for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for any $K \in M(f)$ with $K \supseteq K_\varepsilon$ we have:

$$\left|\int_K f dm - \int_T f dm\right| < \frac{\varepsilon}{|\alpha|}. \tag{8}$$

Let $K \in M(\alpha f)$ such that $K \supseteq K_\varepsilon$. We observe that $K \in M(f)$. By virtue of Theorem 2.7-i) [5] and from (8) we obtain:

$$\begin{aligned}\left|\int_K (\alpha f) dm - \alpha \int_T f dm\right| &= \left|\alpha \int_K f dm - \alpha \int_T f dm\right| \\ &= |\alpha| \left|\int_K f dm - \int_T f dm\right| < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon,\end{aligned}$$

which means that αf is g-m-integrable on T and $\int_T \alpha f dm = \alpha \int_T f dm$.

Theorem 3.2. *If f is g$-m$-integrable on T, then:*
i) f is g$-m$-integrable on every set $A \in \mathcal{A}$;
ii) The set function $I : \mathcal{A} \to \mathbb{R}$, defined by

$$I(A) = \int_A f dm, \text{ for every } A \in \mathcal{A}, \tag{9}$$

is finite additive;
iii) $I \ll_{II} m$.

Proof. i) We easily obtain that f is $\overline{m}$-totally-measurable on A.
Using Theorem 2.5 we have that for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for every $K \in M(f)$ with $K \cap K_\varepsilon = \emptyset$:

$$\left|\int_K f dm\right| < \varepsilon. \tag{10}$$

We consider $K_\varepsilon^{'} = K_\varepsilon \cap A \in M_A(f)$. For every $K^{'} \in M_A(f)$ with $K^{'} \cap K_\varepsilon^{'} = \emptyset$ we have $(K^{'} \cap A) \cap K_\varepsilon = \emptyset$. Hence,

$$\left|\int_{K'} f dm\right| = \left|\int_{K' \cap A} f dm\right| < \varepsilon.$$

ii) Let $A_1, A_2 \in \mathcal{A}$ with $A_1 \cap A_2 = \emptyset$.
From i) it follows that f is g-m-integrable on A_1, A_2 and $A_1 \cup A_2$.
So, for every $\varepsilon > 0$ there exists $K_\varepsilon^1 \subseteq A_1, K_\varepsilon^1 \in M_{A_1}(f)$ such that for every $K \in M_{A_1}(f)$ with $K \supseteq K_\varepsilon^1$ we obtain

$$\left|\int_{A_1} f dm - \int_K f dm\right| < \frac{\varepsilon}{3} \tag{11}$$

$K_\varepsilon^2 \subseteq A_2, K_\varepsilon^2 \in M_{A_2}(f)$ such that for every $K \in M_{A_2}(f)$ with $K \supseteq K_\varepsilon^2$ we have

$$\left|\int_{A_2} f dm - \int_K f dm\right| < \frac{\varepsilon}{3} \tag{12}$$

and $K_\varepsilon^3 \subseteq A_1 \cup A_2, K_\varepsilon^3 \in M_{A_1\cup A_2}(f)$ such that for every $K \in M_{A_1\cup A_2}(f)$ with $K \supseteq K_\varepsilon^3$ we have

$$\left|\int_{A_1\cup A_2} f dm - \int_K f dm\right| < \frac{\varepsilon}{3}. \tag{13}$$

Since $A_1 \cap A_2 = \emptyset$ it implies that $K_\varepsilon^1 \cap K_\varepsilon^2 = \emptyset$.

Now, we consider $K_1 = K_\varepsilon^1 \cup (K_\varepsilon^3 \cap A_1) \in M_{A_1}(f)$, $K_2 = K_\varepsilon^2 \cup (K_\varepsilon^3 \cap A_2) \in M_{A_2}(f)$. We observe that $K_1 \supset K_\varepsilon^1, K_2 \supset K_\varepsilon^2, K_1 \cap K_2 = \emptyset$, $K_1 \cup K_2 \in M_{A_1\cup A_2}(f)$ and $K_1 \cup K_2 \supset K_\varepsilon^3$.

According to Theorem 2.12 [5] and using succesively (13), (11) and (12), we get that

$$\begin{aligned}
&\left|\int_{A_1\cup A_2} f dm - \int_{A_1} f dm - \int_{A_2} f dm\right| \le \left|\int_{A_1\cup A_2} f dm - \int_{K_1\cup K_2} f dm\right| \\
&+ \left|\int_{K_1\cup K_2} f dm - \int_{A_1} f dm - \int_{A_2} f dm\right| < \frac{\varepsilon}{3} + \left|\int_{A_1} f dm - \int_{K_1} f dm\right| \\
&+ \left|\int_{A_2} f dm - \int_{K_2} f dm\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

that is I is a finite additive set function on$\mathcal{A}$.

iii) Because f is g-m-integrable on T, according to Theorem 2.5. for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for every $B \in M(f)$ with $B \cap K_\varepsilon = \emptyset$ holds:

$$\left|\int_B f dm\right| < \frac{\varepsilon}{2}. \tag{14}$$

First, we prove that for every $C \in \mathcal{A}$ with $C \cap K_\varepsilon = \emptyset$ we have $\left|\int_C f dm\right| < \varepsilon$.

Indeed, let $C \in \mathcal{A}$ be such that $C \cap K_\varepsilon = \emptyset$. From i) it follows that f is g-m-integrable on C. Thus, there exists $\widetilde{K_\varepsilon} \in M_C(f)$ such that for every $K \in M_C(f)$ with $K \supseteq \widetilde{K_\varepsilon}$,

$$\left|\int_K f dm - \int_C f dm\right| < \frac{\varepsilon}{2}. \tag{15}$$

Since $C \cap K_\varepsilon = \emptyset$ it results that $K \cap K_\varepsilon = \emptyset$. From (14) and (15), we obtain

$$\left|\int_C f dm\right| \leq \left|\int_K f dm - \int_C f dm\right| + \left|\int_K f dm\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{16}$$

Let us take $A \in \mathcal{A}$ with $m(A) = 0$. We can write $A = (A \cap K_\varepsilon) \cup (A \backslash K_\varepsilon)$.

Since, from i) f is $g - m-$integrable on A, using ii) we have

$$\int_A f dm = \int_{A \cap K_\varepsilon} f dm + \int_{A \backslash K\varepsilon} f dm.$$

Then, by virtue of Corollary 2.13-i) [5] and taking $C = A \backslash K_\varepsilon$ in (16), we get that

$$\left|\int_A f dm\right| = \left|\int_{A \cap K_\varepsilon} f dm + \int_{A \backslash K\varepsilon} f dm\right| \leq \left|\int_{A \cap K_\varepsilon} f dm\right| + \left|\int_{A \backslash K\varepsilon} f dm\right| < \varepsilon.$$

The proof is finished. □

Corollary 3.3. *Let f be a $g - m$-integrable function on T.*

i)If m is $\sigma-$subadditive and $\overline{m}(T) < \infty$, then the set function I defined by (9) *is a $\sigma-$additive set function on $\mathcal{A}$;*

ii) $I \ll_I m$.

Proof. i) We consider a mutually disjoint sequence $(A_j)_j \subset \mathcal{A}$ with $A = \bigcup_{j=1}^\infty A_j \in \mathcal{A}$.

Because m is $\sigma-$subadditive on $\mathcal{A}$ and $\overline{m}(T) < \infty$, then $\overline{m}$ is σ-additive, according to Remark, p. 37 [1].

So, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\overline{m}(\bigcup_{j=n+1}^{\infty} A_j) < \varepsilon, \text{ for every } n \geq n_0. \tag{17}$$

Since f is g-m-integrable on T, from Theorem 3.2-i) f is g-m-integrable on $\bigcup_{j=n_0+1}^\infty A_j$ and A.

Consequently, for every $\varepsilon > 0$ there exists $K_\varepsilon^1 \subseteq \bigcup_{j=n_0+1}^\infty A_j$ with f bounded on K_ε^1 such that for every $K \subseteq \bigcup_{j=n_0+1}^\infty A_j$ with f bounded on K and $K \supseteq K_\varepsilon^1$, we have

$$\left|\int_K f dm - I(\bigcup_{j=n_0+1}^{\infty} A_j)\right| < \varepsilon. \tag{18}$$

and there exists $K_\varepsilon^2 \subseteq A$ with f bounded on K_ε^2 such that for every $K \subseteq A$ with f bounded on K and $K \supseteq K_\varepsilon^2$, we get that

$$\left| \int_K f dm - I(A) \right| < \ \varepsilon.$$

Let us take $K = (K_\varepsilon^1 \cup K_\varepsilon^2) \cap \bigcup_{j=n_0+1}^\infty A_j$. We see that $K \subseteq \bigcup_{j=n_0+1}^\infty A_j$ with f bounded on K and $K \supseteq K_\varepsilon^1$. Hence, holds (18).

Let us denote by $M = \sup_{t\in K} |f(t)|$.

Now, using successively (18), Theorem 2.7-iii) from [5] and (17), we have

$$\begin{aligned} |I(\bigcup_{j=n_0+1}^\infty A_j)| &\leq \left| \int_K f dm - I(\bigcup_{j=n_0+1}^\infty A_j) \right| + \left| \int_K f dm \right| \\ &< \varepsilon + M\overline{m}(\bigcup_{j=n_0+1}^\infty A_j) < \varepsilon(1+M), \end{aligned}$$

which means that I is $\sigma-$additive on $\mathcal{A}$.

ii) Because f is $g-m-$integrable on T, using an analog of the proof of Theorem 3. 2-ii) for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for every $C \in \mathcal{A}$ with $C \cap K_\varepsilon = \emptyset$ we have

$$\left| \int_C f dm \right| < \frac{\varepsilon}{2}. \tag{19}$$

On the other hand, according to Theorem 2.10 [6], for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $K \in M(f)$ with $\overline{m}(K) < \delta$ we have

$$\left| \int_K f dm \right| < \frac{\varepsilon}{2}. \tag{20}$$

Let us consider $A \in \mathcal{A}$ with $\overline{m}(A) < \delta$. We can write $A = (A \cap K_\varepsilon) \cup (A \backslash K_\varepsilon)$. By virtue of Theorem 3.2-ii) we get that

$$\int_A f dm = \int_{A\cap K_\varepsilon} f dm + \int_{A\backslash K\varepsilon} f dm.$$

Taking $C = A \backslash K_\varepsilon$ in (19) we obtain

$$\left| \int_{A\backslash K_\varepsilon} f dm \right| < \frac{\varepsilon}{2}.$$

Since $\overline{m}(A \cap K_\varepsilon) \leq \overline{m}(A) < \delta$ and f is bounded on $A \cap K_\varepsilon$ holds (20). Consequently,

$$\left|\int_A f dm\right| \leq \left|\int_{A\cap K_\varepsilon} f dm\right| + \left|\int_{A\setminus K\varepsilon} f dm\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that I is $\varepsilon - \delta-$absolutely continuous with respect to m.

Remark 3.4. We observe that, according to "the only if part" of Proposition 13, p. 148 from [1], the $\varepsilon - \delta-$absolut continuity implies the absolut continuity.

Theorem 3.5. *Suppose that* $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ *is a submeasure and* $f, g : T \mapsto \mathbb{R}$ *are bounded g-m-integrable functions on* T. *Then* $f + g$ *is g-m-integrable on* T *and*

$$\int_T (f+g)dm = \int_T f dm + \int_T g dm. \tag{21}$$

Proof. To prove that $f + g$ is $g - m-$integrable we shall use Theorem 2.5.

It is easy to see that $f + g$ is $\overline{m}-$totally measurable.

Since f, g are g-$m-$integrable on T, according to Theorem 2.5, for every $\varepsilon > 0$ there exists $K_\varepsilon^1 \in \mathcal{A}$ with $\overline{m}(K_\varepsilon^1) < \infty$ such that for every $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$ and $K \cap K_\varepsilon^1 = \emptyset$ holds

$$\left|\int_K f dm\right| < \frac{\varepsilon}{2} \tag{22}$$

and respectively $K_\varepsilon^2 \in \mathcal{A}$ with $\overline{m}(K_\varepsilon^2) < \infty$ such that for every $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$ and $K \cap K_\varepsilon^2 = \emptyset$ we have

$$\left|\int_K g dm\right| < \frac{\varepsilon}{2}. \tag{23}$$

Let us consider $K_\varepsilon^0 = K_\varepsilon^1 \cup K_\varepsilon^2 \in \mathcal{A}$ and $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$ such that $K \cap K_\varepsilon^0 = \emptyset$.

By virtue of Theorem 2.7-iv) [5] we have

$$\int_K (f+g)dm = \int_K f dm + \int_K g dm.$$

Then, applying (22) and (23), we obtain

$$\left|\int_K (f+g)dm\right| \leq \left|\int_K fdm\right| + \left|\int_K gdm\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

that is $f+g$ is g-m-integrable on T.

We now prove the equality (21).

Indeed, because f, g and $f+g$ are g-m-integrable on T we get that

for every $\varepsilon > 0$ there exist $K^1_\varepsilon \in \mathcal{A}$ with $\overline{m}(K^1_\varepsilon) < \infty$ such that for every $K \in \mathcal{A}$ with $\overline{m}(K^1_\varepsilon) < \infty$ and $K \supseteq K^1_\varepsilon$

$$\left|\int_K fdm - \int_T fdm\right| < \frac{\varepsilon}{3}, \tag{24}$$

$K^2_\varepsilon \in \mathcal{A}$ with $\overline{m}(K^2_\varepsilon) < \infty$ such that for every $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$ and $K \supseteq K^2_\varepsilon$

$$\left|\int_K gdm - \int_T gdm\right| < \frac{\varepsilon}{3}, \tag{25}$$

and $K_\varepsilon \in \mathcal{A}$ with $\overline{m}(K_\varepsilon) < \infty$ such that for every $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$ and $K \supseteq K_\varepsilon$

$$\left|\int_K (f+g)dm - \int_T (f+g)dm\right| < \frac{\varepsilon}{3}. \tag{26}$$

Let us take $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$ and $K \supseteq K^1_\varepsilon \cup K^2_\varepsilon \cup K_\varepsilon$.

Then, from (26) and applying successively Theorem 2.7-iv) [5], (24) and (25), we have

$$\begin{aligned}
&\left|\int_T (f+g)dm - \int_T fdm - \int_T gdm\right| \leq \left|\int_T (f+g)dm - \int_K (f+g)dm\right| \\
&+ \left|\int_K (f+g)dm - \int_T fdm - \int_T gdm\right| \\
&< \frac{\varepsilon}{3} + \left|\int_K fdm + \int_K gdm - \int_T fdm - \int_T gdm\right| \\
&< \frac{\varepsilon}{3} + \left|\int_K fdm - \int_T fdm\right| + \left|\int_K gdm - \int_T gdm\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which completes the proof. $\square$

In what follows we give a property of monotonicity with respect to the function.

Theorem 3.6. *Let $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ be a submeasure and $f, g : T \mapsto \mathbb{R}$ be bounded, g- m-integrable functions on T so that $f \leq g$ on T. Then*

$$\int_T f dm \leq \int_T g dm.$$

Proof. Because f and g are g-m-integrable on T, for every $\varepsilon > 0$ there exists $K_\varepsilon^1 \in \mathcal{A}$ with $\overline{m}(K_\varepsilon^1) < \infty$ such that for every $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$, $K \supseteq K_\varepsilon^1$ we get that

$$\left|\int_K f dm - \int_T f dm\right| < \frac{\varepsilon}{2} \tag{27}$$

and respectively $K_\varepsilon^2 \in \mathcal{A}$ with $\overline{m}(K_\varepsilon^2) < \infty$ such that for every $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$, $K \supseteq K_\varepsilon^2$ we have

$$\left|\int_K g dm - \int_T g dm\right| < \frac{\varepsilon}{2}. \tag{28}$$

Let us consider $K_\varepsilon = K_\varepsilon^1 \cup K_\varepsilon^2$ and $K \in \mathcal{A}$ with $\overline{m}(K) < \infty$ such that $K \supseteq K_\varepsilon$.

By virtue of Theorem 2.7-v) [5] we have $\int_K f dm \leq \int_K g dm$. Then, using (27) and (28), we obtain

$$\begin{aligned}
\int_T f dm - \int_T g dm &\leq \int_T f dm - \int_K f dm + \int_K f dm \\
&\quad - \int_K g dm + \int_K g dm - \int_T g dm \\
&\leq \left|\int_T f dm - \int_K f dm\right| + \int_K f dm - \int_K g dm \\
&\quad + \left|\int_K g dm - \int_T g dm\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Corollary 3.7. *If $f : T \mapsto \mathbb{R}_+$ is a bounded g-m-integrable function on T. Then $\int_T f dm \geq 0$.*

Theorem 3.8. *Let $f : T \mapsto \mathbb{R}$, $g : T \to \mathbb{R}_+$ be two functions such that $|f(t)| \leq g(t)$, for every $t \in T$.*

If f is $\overline{m}-$totally measurable and g is g-m-integrable on T, then f is g-m-integrable on T.

Proof. Because g is g-m-integrable on T, according to Theorem 2.5 we get that for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(g)$ such that for every $K \in M(g)$ with $K \cap K_\varepsilon = \emptyset$,

$$\left| \int_K gdm \right| < \frac{\varepsilon}{2}. \tag{29}$$

From $|f(t)| \leq g(t)$, for every $t \in T$, we have that $M(g) \subseteq M(f)$. Let be fixed $K \in M(f)$ with $K \cap K_\varepsilon = \emptyset$.

Now, since g is g-m-integrable on K there exists $K'_\varepsilon \in M_K(g)$ such that for every $A \in M_K(g)$ with $A \supseteq K'_\varepsilon$, we have

$$\left| \int_A gdm - \int_K gdm \right| < \frac{\varepsilon}{2}. \tag{30}$$

On the other hand, from (30) and (29) we have

$$\int_K gdm \leq \left| \int_{K'_\varepsilon} gdm - \int_K gdm \right| + \left| \int_{K'_\varepsilon} gdm \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

According to Theorem 2.16 [5] , f is $g-m-$integrabile on K.

Using successively Theorems 2.17 and 2.7-v) from [5] we obtain:

$$\left| \int_K fdm \right| \leq \int_K |f|\, dm \leq \int_K gdm < \varepsilon.$$

Thus, by virtue of Theorem 2.5, f is g-m-integrable on T.

Corollary 3.9. *If f is bounded $\overline{m}-$totally measurable function and $|f|$ is g-m-integrable on T, then f is g-m-integrable on T and, moreover,*

$$\left| \int_T fdm \right| \leq \int_T |f|\, dm.$$

Theorem 3.10. *Let $f : T \to \mathbb{R}_+$ and $B, C \in \mathcal{A}$, with $B \subseteq C$. If f is a bounded g-m-integrable function on T, then*

$$\int_B fdm \leq \int_C fdm.$$

Proof. According to Theorem 3.2-i), f is g-m-$i$ntegrable on B and C and then for every $\varepsilon > 0$ there exist $K_\varepsilon^B \in \mathcal{A}$, $K_\varepsilon^B \subseteq B$ with $\overline{m}(K_\varepsilon^B) < \infty$ such that for every $K \in \mathcal{A}$, $K \subseteq B$ with $\overline{m}(K) < \infty$ and $K \supseteq K_\varepsilon^B$

$$\left| \int_K f dm - \int_B f dm \right| < \frac{\varepsilon}{2} \tag{31}$$

and $K_\varepsilon^C \in \mathcal{A}, K_\varepsilon^C \subseteq C$ with $\overline{m}(K_\varepsilon^C) < \infty$ such that for every $K \in \mathcal{A}, K \subseteq C$ with $\overline{m}(K) < \infty$ and $K \supseteq K_\varepsilon^C$, we have

$$\left| \int_K f dm - \int_C f dm \right| < \frac{\varepsilon}{2}. \tag{32}$$

Let $K_\varepsilon = K_\varepsilon^B \cup K_\varepsilon^C \in \mathcal{A}$.

We observe that $K_\varepsilon \supseteq K_\varepsilon^B$, $K_\varepsilon \supseteq K_\varepsilon^C$ and $\overline{m}(K_\varepsilon) < \infty$.

Then for every $K \in \mathcal{A}$, $K \subseteq C$ with $\overline{m}(K) < \infty$ such that $K \supseteq K_\varepsilon$ hold (31) and (32).

According to Theorem 2.14 [5] we have $\int_{K_\varepsilon^B} f dm \leq \int_K f dm$. Now, using (31) and (32) we obtain

$$\begin{aligned}
&\int_B f dm - \int_C f dm \leq \int_B f dm - \int_{K_\varepsilon^B} f dm + \int_{K_\varepsilon^B} f dm - \int_K f dm \\
&+ \int_K f dm - \int_C f dm \leq \left| \int_B f dm - \int_{K_\varepsilon^B} f dm \right| \\
&+ \int_{K_\varepsilon^B} f dm - \int_K f dm + \left| \int_K f dm - \int_C f dm \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $\int_B f dm \leq \int_C f dm$.

Acknowledgement. The author is indebted to Prof. Dr. A. M. Precupanu for the careful reading of the paper and for the useful suggestions.

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Received: 10.IV.2008

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