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ON THE ALGEBRAICALLY COMPACT $\aleph_1$-EMBEDDING ABELIAN GROUPS

BY

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Abstract. We prove that every algebraically compact $\aleph_1$-embedding abelian group (in particular every algebraically compact $\aleph_1$-separable group or algebraically compact Fuchs five-group, respectively) is a direct sum of a divisible group and a bounded group. This parallels to a result due to Rangaswamy (Bull. Soc. Math. France, 1964).

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Throughout this brief article all groups into consideration are assumed to be arbitrary abelian groups written additively as is the custom when regarding group theory. For such a group G, the letter G^1 will always denote the first Ulm subgroup of G, that is, the subgroup of G consisting of all elements of infinite height. A group G is termed separable if $G^1 = 0$. As usual, $|M|$ denotes the cardinality of a set M. All other unexplained notions and notation are standard and follow essentially those from [2] and [3]. For instance, for any set S, the symbol $|S|$ denotes the cardinality of S.

Before proceed by proving the main assertion, motivated the writing of this paper, we provide the reader with some more classical terminology.

Definition 1 ([3]). A group G is called *an* $\aleph_1$*-separable group* if each its countable subgroup can be embedded in a countable direct summand of G which is a direct sum of cyclic groups.

It is straightforward that $\aleph_1$-separable groups are separable, and it is well-known the non-trivial fact that a direct summand of an $\aleph_1$-separable

group is again $\aleph_1$-separable. It is also simple checked that direct sums of cycles are $\aleph_1$-separable.

Definition 2 ([4]). A separable group G is said to be *a Q-group* if $|(G/H)^1| \leq |H|$ whenever $H \leq G$ with $|H| \geq \aleph_0$. The condition is obviously equivalent to $|\cap_{n<\omega} (nG + H)| = |H|$.

It was established in [4] the obvious fact that a subgroup of a Q-group is also a Q-group. Likewise, it is not hard to check that every $\aleph_1$-separable group of cardinality $\aleph_1$ is a Q-group whereas the general situation for cardinalities greater than $\aleph_1$ as well as the converse claim are not ever true. Nevertheless, direct sums of cycles are Q-groups.

Definition 3 ([4]). A group G is called *a Fuchs five-group* if each its infinite subgroup can be embedded in a direct summand of G of the same cardinality, itself.

Direct sums of countable groups are themselves Fuchs five. It is also a routine technical matter to see that any separable Fuchs five-group is a Q-group (compare with [4]), while the reciprocal claim is known to be wrong. Likewise, it was proved in ([4], Proposition 2) that both fully invariant subgroups of Fuchs five-groups as well as factor-groups of Fuchs five-groups modulo fully invariant subgroups are again Fuchs five-groups.

We are now ready to state the following new generalization of Definitions 1 and 3.

Definition 4. A group G is said to be *an* $\aleph_1$*-embedding group* if every its countable subgroup can be embedded in a countable direct summand of G.

It is self-evident that each $\aleph_1$-separable group and each Fuchs five-group are themselves $\aleph_1$-embedding. Even more, owing to the second Prüfer's theorem (see [2], Theorem 17.3) Definitions 1 and 4 are equivalent for primary groups but, however, they are totally different for mixed and torsion-free groups. Also, for groups of cardinalities not exceeding $\aleph_1$, Definitions 3 and 4 are tantamount.

Definition 5 ([2]). A group G is called *algebraically compact* if it separates as a direct summand of every group which contains G as a pure subgroup.

Referring to ([2], p. 187 or Corollary 38.3) one can have for a further use that a direct summand of an algebraically compact group is algebraically

compact as well; so in particular the reduced part of any algebraically compact group is algebraically compact too. It is well known that both divisible groups and bounded groups are algebraically compact.

Definition 6 ([5]). A group G is said to be *a* Σ*-group* provided that all its high subgroups (that are subgroups maximal with respect to disjoint intersection with G^1) are direct sums of cyclic groups.

It is easy to argue that every separable Σ-group is a direct sum of cycles since each separable group is a high subgroup itself, and hence Q-groups are Σ-groups precisely when they are direct sums of cycles. Moreover, it was shown in [5] that any algebraically compact Σ-group is the direct sum of a divisible group and a bounded group.

The purpose of the present short note is to find an interrelation between the classes of algebraically compact groups and $\aleph_1$-embedding groups by obtaining in an explicit form their intersection. Actually, we shall deduce a suitable criterion when an algebraically compact group is $\aleph_1$-embedding. Before doing that, we need a few technicalities.

Proposition. (1) *Fully invariant subgroups of* $\aleph_1$*-embedding groups are* $\aleph_1$*-embedding groups;*

(2) *Quotients of* $\aleph_1$*-embedding groups modulo fully invariant subgroups are* $\aleph_1$*-embedding groups.*

Proof. Suppose that F is a fully invariant subgroup of the $\aleph_1$-embedding group G.

(1) Let $H \leq F$ with $|H| = \aleph_0$. Then $G = T \oplus V$, where $T \supseteq H$ and $|T| = |H|$. But $F = (T \cap F) \oplus (V \cap F)$ and $T \cap F \supseteq H$, hence $|T \cap F| \geq |H|$. Moreover, $|T \cap F| \leq |T| = |H|$ which leads us to $|T \cap F| = |H| = \aleph_0$. Thus, F is, by definition, $\aleph_1$-embedding.

(2) Let $L \leq G/F$ with $|L| = \aleph_0$. We can find a subgroup K of G such that $|K| = |L|$ and $(K + F)/F = L$. In fact, L is of the type $L = M/F$ for some $M \leq G$. Certainly, M can be decomposed as $P + F$ for some $P \leq M$ and clearly $|L| \leq |P|$ because $L = (P + F)/F \cong P/(P \cap F)$. Moreover, we decompose $P = P \cap F + K$, and thus $M = K + F$, where K is the group generated by all elements of P which form different coset of P with respect to $P \cap F$, that is, these are the elements of P modulo $P \cap F$, i.e., those which do not belong to $P \cap F$. Since as early noted $L \cong P/(P \cap F)$ is countable, it is self-evident that so is K and the claim sustained. (Note also that if $|P| = \aleph_0$

or $|P \cap F| < |P|$, whence $|P| = |P/(P \cap F)|.|P \cap F| = |P/(P \cap F)| = |L|$, then the choice $K = P$ directly works. However, the case $|P \cap F| = |P| > \aleph_0$ is complicated, so the above construction for K is necessary.)

Then, $G = T \oplus V$ where $T \supseteq K$ and $|T| = |K| = |L|$. But $F = (T \cap F) \oplus (V \cap F)$ and $(T+F) \cap V \subseteq F$. Indeed, given $t+f = v$, where $t \in T$, $f \in F$ and $v \in V$. Therefore, $v - t = f \in (T \oplus V) \cap F = (T \cap F) \oplus (V \cap F)$ and thereby $v \in V \cap F \subseteq F$, which substantiates our claim. Whence, with the help of the modular law from [2], we obtain that $G/F = [(T+F)/F] \oplus [(V+F)/F] \cong [T/(T \cap F)] \oplus [V/(V \cap F)]$ with $(T+F)/F \supseteq (K+F)/F = L$ and $|(T+F)/F| \geq |L|$. Moreover, $|(T+F)/F| = |T/(T \cap F)| \leq |T| = |L|$. This ensures that $|(T+F)/F| = |L| = \aleph_0$. Thus, G/F is, by definition, $\aleph_1$-embedding. □

Problem. Decide whether or not direct summands of $\aleph_1$-embedding groups are $\aleph_1$-embedding.

Specifically, we are now prepared to prove the following statement.

Theorem. *The algebraically compact $\aleph_1$-embedding groups are the direct sums of divisible groups and bounded groups.*

Proof. Letting G be the group in question, we may write $G \cong G_d \oplus (G/G_d)$, where G_d is the maximal divisible subgroup of G. Since G_d is fully invariant in G, by what we have shown in the Proposition combined with our comments alluded to above, we deduce that G/G_d is $\aleph_1$-embedding and algebraically compact. That is why, with no loss of generality, we shall assume that G is reduced. So, we derive with the aid of Definition 4 that $H \subseteq C$ where C is at most countable direct summand of G whenever $H \leq G$ with $|H| \leq \aleph_0$. Consequently, according to the discussion above and ([2], p. 200, Exercise 3(a)), C must be bounded. That is why, any countable subgroup of G has to be bounded. We therefore infer that G does not contain elements of infinite order, i.e., G must be torsion, since G does not possess infinite cycles. Next, an appeal to ([2], Corollary 40.3) assures the wanted conclusion that G is bounded. □

As immediate consequences, we yield the following.

Corollary 1. *Each algebraically compact $\aleph_1$-separable group is bounded.*

Proof. As early observed, $\aleph_1$-separable groups are known to be $\aleph_1$-embedding. Henceforth the Theorem works. □

Corollary 2. *Every algebraically compact Fuchs five-group is a direct sum of a divisible group and a bounded group.*

Proof. As already observed, Fuchs five-groups are known to be $\aleph_1$-embedding. Hereafter the Theorem is applicable. □

We end the study with the following two problems of interest.

Problem 1. Describe those groups which are $\aleph_1$-embedding Σ-groups.

Problem 2. Determine the intersection between the classes of Q-groups and $\aleph_1$-embedding groups.

As observed above, each separable $\aleph_1$-embedding group of cardinality $\aleph_1$ is a Q-group. Thus, inspired by our Theorem and this last problem, we state the following conjecture.

Conjecture. Each algebraically compact Q-group is bounded.

In this aspect, it was proven in [1] that every primary torsion-complete Q-group of cardinality $\aleph_1$ is bounded and there we were currently unable to say whether or not the limitation on the cardinal $\aleph_1$ can be generally removed.

Notice also that the maximal torsion subgroup of a reduced algebraically compact group is torsion-complete ([3], Theorem 68.4). Besides, all the results in this paper remain valid by replacing the word "algebraically compact" with "cotorsion".

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