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# STRONG RADICAL CLASS IN SEMIRINGS

BY

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**Abstract.** In this paper we introduce and study the notion of left strong radical class in semiring. We have also shown that Jacobson, Levitzki, nil and prime radical classes are left strong.

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**Key words:** Left strong radical class, Jacobson radical class, Levitzki radical class, nil radical class and prime radical class.

## 1. Introduction.

In [12], VANDIVER introduced a new type of algebraic system which is commonly known as Semiring. Semiring is a common generalization of associative rings and distributive lattices. A semiring is an algebraic system consisting of a nonempty set together with two binary operations, called addition and multiplication, which forms a commutative semigroup relative to addition, a semigroup relative to multiplication and the left, right distributive laws hold. In [10], OLSON and JENKINS defined radical class in semiring. In ring theory [5], it is well known that a radical class $\rho$ is said to be left strong if for a ring $R$, $\rho(R)$ contains all left $\rho$-ideals of $R$. In this paper we introduce the notion of left strong radical class in semiring and characterize them. We have also shown that Jacobson, Levitzki, nil and prime radical classes are left strong. Our some earlier works on semiring may be found in [3] and [4].

## 2. Preliminaries

**Definition 2.1.** *A nonempty set $S$ is said to form a semiring with respect to two binary compositions, addition $(+)$ and multiplication $(.)$ defined*

*on it, if the following conditions are satisfied.*
(1) $(S,+)$ *is a commutative semigroup with zero,*
(2) $(S,.)$ *is a semigroup,*
(3) *for any three elements* $a,b,c \in S$
*the left distributive law* $a.(b+c) = a.b + a.c$ *and*
*the right distributive law* $(b+c).a = b.a + c.a$ *both hold and*
(4) $s.0 = 0.s = 0$ *for all* $s \in S$.

**Definition 2.2.** *A nonempty subset* $I$ *of a semiring* $S$ *is called a left ideal of* $S$ *if (i)* $a,b \in I$ *implies* $a+b \in I$ *and (ii)* $a \in I$, $s \in S$ *implies* $s.a \in I$.

Similarly we can define right ideal of a semiring. A nonempty subset $I$ of a semiring $S$ is an ideal if it is a left as well as a right ideal of $S$.

**Definition 2.3.** *The k-closure* $\widehat{I}$ *of an ideal* $I$ *of a semiring* $S$, *defined by* $\widehat{I} = \{a \in S : a + i \in I \text{ for some } i \in I\}$.

**Definition 2.4 ([7]).** *An ideal* $I$ *of a semiring* $S$ *is called a k-ideal if* $b \in S$, $a+b \in I$ *and* $a \in I$ *implies* $b \in I$.

**Definition 2.5 ([6]).** *Let* $I$ *be a proper ideal of a semiring* $S$. *Then the congruence on* $S$, *denoted by* $\rho_I$ *and defined by* $s\rho_I s'$ *if and only if* $s + a_1 = s' + a_2$ *for some* $a_1, a_2 \in I$, *is called the Bourne congruence on* $S$ *defined by the ideal* $I$.

We denote the Bourne congruence ($\rho_I$) class of an element $r$ of $S$ by $r/\rho_I$ or simply by $r/I$ and denote the set of all such congruence classes of $S$ by $S/\rho_I$ or simply by $S/I$.

It should be noted that for any $s \in S$ and for any proper ideal $I$ of $S$, $s/I$ is not necessarily equal to $s + I = \{s + a : \ a \in I\}$ but surely contains it.

**Definition 2.6 ([6]).** *For any proper ideal* $I$ *of* $S$ *if the Bourne congruence* $\rho_I$, *defined by* $I$, *is proper i.e.* $0/I \neq S$ *then we define the addition and multiplication on* $S/I$ *by* $a/I + b/I = (a+b)/I$ *and* $(a/I)(b/I) = (ab)/I$ *for all* $a,b \in S$. *With these two operations* $S/I$ *forms a semiring and is called the Bourne factor semiring or simply the factor semiring.*

**Definition 2.7.** *An ideal* $I$ *of a semiring* $S$ *is said to be an essential ideal of* $S$ *if* $I \cap H \neq 0$ *for every nonzero ideal* $H$ *of* $S$.

**Definition 2.8 ([6]).** *An ideal $I$ of a semiring $S$ is called a semiprime ideal if $A^2 \subseteq I$ implies $A \subseteq I$, where $A$ is an ideal of $S$.*

**Definition 2.9.** *A semiring $S$ is called a semiprime semiring if $\{0\}$ is a semiprime ideal of $S$.*

## 3. Strong radical class in semiring

**Definition 3.1 ([10], Definition 1).** *A nonempty class $\rho$ of semirings is called a radical class if:*

$(R1)$ *$\rho$ is homomorphically closed and*

$(R2)$ *if $S \notin \rho$, then $S$ contains a proper $k$-ideal $K$ such that $S/K$ has no nonzero $\rho$-ideals (ideals which is as semirings are in the class $\rho$).*

**Proposition 3.2 ([10], Lemma 1).** *For a radical class $\rho$, if $I$ and $J$ are $\rho$-ideals of a semiring $S$, then so is $I + J$.*

**Theorem 3.3 ([10], Theorem 2).** *If $\rho$ is a radical class, then each semiring $S$ contains a maximal $\rho$-ideal which contains all other $\rho$-ideals of $S$.*

The maximal $\rho$-ideal of $S$ will be called $\rho$-radical of $S$ and denoted by $\rho(S)$.

**Theorem 3.4 ([10], Theorem 3).** *$\rho(S)$ is a $k$-ideal of $S$.*

**Theorem 3.5 ([10], Theorem 4).** *For any semiring $S$, $\rho(S/\rho(S)) = 0$.*

**Definition 3.6.** *If $\rho$ is a radical class then the class $\mathcal{S}\rho = \{S : \rho(S) = 0\}$ is called a semisimple class.*

**Definition 3.7.** *A class $\mathcal{M}$ of semirings is called (left) hereditary if $I$ is an (left) ideal of a semiring $S$ and $S \in \mathcal{M}$ implies $I \in \mathcal{M}$.*

**Proposition 3.8 ([8], Theorem 6.3).** *For a radical class $\rho$, the semisimple class $\mathcal{S}\rho$ is hereditary.*

**Definition 3.9.** *A surjective morphism of semirings $\phi : R \to S$ is called semi-isomorphism if and only if $ker\phi = 0$. Here $R$ is called semi-isomorphic to $S$.*

**Theorem 3.10.** *If $I$ and $J$ are ideals of a semiring $S$, then $I/I \cap J^*$ is semi-isomorphic to $(I+J)/J$, where $J^*$ is the $k$-closure of $J$.*

**Definition 3.11.** *A radical class $\rho$ is said to be left strong radical class if $L$ is a left ideal of a semiring $S$ and $L \in \rho$ implies $L \subseteq \rho(S)$.*

**Proposition 3.12.** *The following conditions are equivalent:*

(*i*) *$\rho$ is a left strong radical class;*

(*ii*) *If $L$ is a nonzero left ideal of $S$ and $S \in \mathcal{S}\rho$ then $L \notin \rho$.*

**Proof.** (*i*) $\Rightarrow$ (*ii*) Suppose $\rho$ is a left strong radical class. Let $L$ be a nonzero left ideal of $S$ where $S \in \mathcal{S}\rho$. If possible, let $L \in \rho$. Then $L \subseteq \rho(S) = 0$ i.e. $L = 0$, a contradiction. Therefore $L \notin \rho$.

(*ii*) $\Rightarrow$ (*i*) Let $L \in \rho$ where $L$ is a left ideal of $S$. If possible, let $L \not\subseteq \rho(S)$. Then $(L + \rho(S))/\rho(S)$ is a nonzero left ideal of $S/\rho(S)$. Now by Theorem 3.5, $S/\rho(S) \in \mathcal{S}\rho$. Therefore by (ii), $(L + \rho(S))/\rho(S) \notin \rho$. Let us define a mapping $\phi : L \to (L + \rho(S))/\rho(S)$ by $\phi(l) = l/\rho(S)$. Then $\phi$ is a homomorphism. So $(L + \rho(S))/\rho(S)$ is a homomorphic image of $L$. Since $L \in \rho$, so $(L + \rho(S))/\rho(S) \in \rho$, a contradiction. Hence $L \subseteq \rho(S)$. Thus $\rho$ is a left strong radical class. □

**Definition 3.13.** *A class $\mathcal{M}$ of semirings is called regular* (*left regular*)*, if for every semiring $S \in \mathcal{M}$, every nonzero ideal* (*left ideal*) *of $S$ has a nonzero homomorphic image in $\mathcal{M}$.*

**Theorem 3.14.** *The following four conditions are equivalent:*

(*i*) *$\rho$ is a left strong radical class;*

(*ii*) *$\rho$ is a radical class whose semisimple class $\mathcal{S}\rho$ is left regular;*

(*iii*) *$\rho$ is homomorphically closed class and if every nonzero homomorphic image of a semiring $S$ has a nonzero left ideal in $\rho$ then $S \in \rho$;*

(*iv*) *$\rho$ is a radical class and if $L$ is a left ideal of $S$, $L \in \rho$ then the ideal $L + LS$ of $S$ is in $\rho$.*

**Proof.** (*i*) $\Rightarrow$ (*ii*) Let $L$ be a nonzero left ideal of $S \in \mathcal{S}\rho$. Then by proposition 3.12, $L \notin \rho$. So $\rho(L) \neq L$ i.e $L/\rho(L) \neq 0$. So $L/\rho(L)$ is a

nonzero homomorphic image of $L$ and $L/\rho(L) \in \mathcal{S}\rho$. Therefore $\mathcal{S}\rho$ is left regular.

$(ii) \Rightarrow (iii)$ Since $\rho$ is a radical class, so $\rho$ is homomorphically closed class. If possible let $S$ be a semiring such that every nonzero homomorphic image of $S$ has a nonzero left ideal in $\rho$ but $S \notin \rho$. So $S/\rho(S)$ is a nonzero homomorphic image of $S$ and by assumption $S/\rho(S)$ has a nonzero left ideal $L/\rho(S)$ in $\rho$. By (ii) $L/\rho(S)$ has a nonzero homomorphic image in $\mathcal{S}\rho$, which is a contradiction, since $\rho$ is a radical class.

$(iii) \Rightarrow (iv)$ Here $\rho$ satisfied the condition (R1) of the definition of radical class. Let $S \notin \rho$. Then by given condition $S$ has a homomorphic image $S_1$ such that $S_1$ has no nonzero ideal in $\rho$. Let $\phi$ be the homomorphism. Then $S/ker\phi$ is semi-isomorphic to $S_1$. Let $\psi$ be the semi-isomorphism. If possible, let $S/ker\phi$ have a nonzero $\rho$-ideal $H/ker\phi$ (say). Since $\rho$ is homomorphically closed, so $\psi(H/ker\phi)$ is a nonzero $\rho$-ideal of $S_1$, a contradiction. Therefore $S/ker\phi$ has no nonzero ideal in $\rho$. Hence for the $k$-ideal $ker\phi$ of $S$, $S/ker\phi$ has no nonzero $\rho$-ideals i.e. $\rho$ is a radical class.

Suppose $L$ be a nonzero left ideal of $S$ such that $L \in \rho$. Then $L + LS$ is an ideal of $S$. Let us consider a nonzero homomorphic image $(L + LS)/K$ of $L + LS$, where $K$ is a $k$-ideal of $L + LS$.

If $L \not\subseteq K$, then $L/L \cap K$ is a nonzero homomorphic image of $L$. So $L/L \cap K \in \rho$. Now $L/L \cap K$ is semi-isomorphic to $(L + K)/K$ which is a left ideal of $(L + LS)/K$. Therefore for the nonzero homomorphic image $(L+LS)/K$ of $L+LS$ has a nonzero left ideal $(L+K)/K$ in $\rho$. So by (iii), $L + LS \in \rho$.

If $L \subseteq K$, then $LS \not\subseteq K$ (otherwise $(L + LS)/K = 0$). Therefore there exists an element $s \in S$ such that $Ls \not\subseteq K$. Consider the mapping $f : L \to (Ls + K)/K$ defined by $f(l) = ls/K$ for all $l \in L$. So $f(l_1 + l_2) = (l_1+l_2)s/K = l_1s/K+l_2s/K = f(l_1)+f(l_2)$, $f(l_1l_2) = l_1l_2s/K \in (K(LS)+K)/K = 0$ (since $L \subseteq K$) and $f(l_1)f(l_2) = (l_1s/K)(l_2s/K) = l_1sl_2s/K \in (L(SL)S + K)/K \subseteq (K(LS) + K)/K = 0$. So $f(l_1 + l_2) = f(l_1) + f(l_2)$ and $f(l_1l_2) = f(l_1)f(l_2)$ for all $l_1, l_2 \in L$. Clearly $f$ is surjective. Therefore $(Ls + K)/K$ is a homomorphic image of $L$. Since $\rho$ is homomorphically closed and $L \in \rho$, so $(Ls + K)/K \in \rho$. Hence by (iii), $L + LS \in \rho$.

$(iv) \Rightarrow (i)$ Let $L$ be a left ideal of $S$ and $L \in \rho$. Then by (iv), $L + LS$ is an ideal of $S$ in $\rho$ i.e. $\rho(L + LS) = L + LS$. Since $\rho(S)$ is the largest $\rho$-ideal of $S$. So $L + LS \subseteq \rho(S)$. i.e. $L \subseteq \rho(S)$. Hence $\rho$ is left strong class. □

**Proposition 3.15.** *A radical class $\rho$ is left strong if $\rho$ satisfies the*

*following two conditions:*

(1) *For any two left ideals $L$ and $K$ of a semiring $S$ and $L, K \in \rho$ implies $L + K \in \rho$.*

(2) *If $L \in \rho$ is a left ideal of a semiring $S$ and $a \in S$ then $La \in \rho$.*

**Proof.** From condition (1) we get sum of finitely many left $\rho$-left ideal of a semiring is a $\rho$-left ideal. Let $L$ be left ideal of a semiring $S$ and $L \in \rho$. Let $x \in L + LS$. Then $x = l + \sum_{finite} l_i s_i$ ( $l, l_i \in L; s \in S$). So $x \in L + \sum_{finite} Ls_i$. Since $L \in \rho$, so by (2) $Ls_i \in \rho$. Therefore the finite sum of $\rho$-left ideals $L + \sum_{finite} Ls_i$ is $\rho$-left ideal. Hence by Theorem 3.14 (iv), $\rho$ is left strong. □

**Definition 3.16 ([2], Definition 3).** *A semiring $S$ is said to be left semiregular, if for every pair of elements $i_1$, $i_2$ in $S$ there exist elements $j_1$ and $j_2$ in $S$ such that $i_1 + j_1 + j_1 i_1 + j_2 i_2 = i_2 + j_2 + j_2 i_1 + j_1 i_2$.*

**Theorem 3.17 ([10], Theorem 13).** *The class $\mathcal{J}$ of all left semiregular semirings is a radical class.*

$\mathcal{J}$ is called Jacobson radical class.

**Lemma 3.18.** *Sum of two left semiregular ideals of a semiring is a left semiregular ideal.*

**Proof.** Let $L$ and $K$ be two left semiregular left ideals of a semiring $S$.

Let $a_1 + b_1, a_2 + b_2 \in L + K$ where $a_1, a_2 \in L$ and $b_1, b_2 \in K$.

Since $L$ is left semiregular, there exist two elements $c_1, c_2$ in $L$ such that

$$a_1 + c_1 + c_1 a_1 + c_2 a_2 = a_2 + c_2 + c_1 a_2 + c_2 a_1 \tag{1}$$

Now $b_1 + c_1 b_1 + c_2 b_2, b_2 + c_1 b_2 + c_2 b_1 \in K$. Since $K$ is a left semiregular there exist $d_1, d_2$ in $K$ such that

$$\begin{aligned} &(b_1 + c_1 b_1 + c_2 b_2) + d_1 + d_1(b_1 + c_1 b_1 + c_2 b_2) \\ &+d_2(b_2 + c_1 b_2 + c_2 b_1) = (b_2 + c_1 b_2 + c_2 b_1) \\ &+d_2 + d_2(b_1 + c_1 b_1 + c_2 b_2) + d_1(b_2 + c_1 b_2 + c_2 b_1). \end{aligned} \tag{2}$$

Therefore

$$
\begin{aligned}
&(a_1+b_1)+(c_1+d_1+d_1c_1+d_2c_2)+(c_1+d_1+d_1c_1+d_2c_2)(a_1+b_1)\\
&+(c_2+d_2+d_2c_1+d_1c_2)(a_2+b_2)\\
&=(a_1+c_1+c_1a_1+c_2a_2)+[(b_1+c_1b_1+c_2b_2)\\
&+d_1+d_1(b_1+c_1b_1+c_2b_2)+d_2(b_2+c_1b_2+c_2b_1)]\\
&+d_1(a_1+c_1+c_1a_1+c_2a_2)+d_2(a_2+c_2+c_2a_1+c_1a_2)\\
&=(a_2+c_2+c_2a_1+c_1a_2)+[(b_2+c_1b_2+c_2b_1)\\
&+d_2+d_2(b_1+c_1b_1+c_2b_2)+d_1(b_2+c_1b_2+c_2b_1)]\\
&+d_1(a_2+c_2+c_1a_2+c_2a_1)+d_2(a_1+c_1+c_1a_1+c_2a_2)\\
&[\text{using (1) and (2)}]\\
&=(a_2+b_2)+(c_2+d_2+d_2c_1+d_1c_2)\\
&+(c_2+d_2+d_2c_1+d_1c_2)(a_1+b_1)+(c_1+d_1+d_1c_1+d_2c_2)(a_2+b_2).
\end{aligned}
$$

Since $c_1+d_1+d_1c_1+d_2c_2, c_2+d_2+d_2c_1+d_1c_2$ are in $L+K$, so $L+K$ is a left semiregular left ideal of $S$. □

**Lemma 3.19.** *If $L$ is a left semiregular left ideal of a semiring $S$ then for any element $a \in S$, the left ideal $La$ is a left semiregular left ideal.*

**Proof.** Let $x_1, x_2 \in La$. Then there exist $a_1, a_2$ in $L$ such that $x_1 = a_1a$ and $x_2 = a_2a$. Now $aa_1, aa_2 \in L$. Since $L$ is a left semiregular, there exist $c_1, c_2$ in $L$ such that

$$aa_1 + c_1 + c_1aa_1 + c_2aa_2 = aa_2 + c_2 + c_2aa_1 + c_1aa_2. \tag{1}$$

Multiplying $a_1$ from left and $a$ from right of equation (1), we get

$$x_1x_1 + a_1d_1 + a_1d_1x_1 + a_1d_2x_2 = x_1x_2 + a_1d_2 + a_1d_2x_1 + a_1d_1x_2 \tag{2}$$

where $c_1a = d_1$ and $c_2a = d_2$. Again multiplying $a_2$ from left and $a$ from right of equation (1), we get

$$x_2x_1 + a_2d_1 + a_2d_1x_1 + a_2d_2x_2 = x_2x_2 + a_2d_2 + a_2d_2x_1 + a_2d_1x_2. \tag{3}$$

Adding (2) and (3), we get

$$x_1x_1 + a_1d_1 + a_1d_1x_1 + a_1d_2x_2 + x_2x_2 + a_2d_2 + a_2d_2x_1 + a_2d_1x_2$$
$$= x_1x_2 + a_1d_2 + a_1d_2x_1 + a_1d_1x_2 + x_2x_1 + a_2d_1 + a_2d_1x_1 + a_2d_2x_2$$

or

$$x_2 + (x_1 + a_1d_1 + a_2d_2) + (x_1 + a_1d_1 + a_2d_2)x_1$$
$$+(x_2 + a_1d_2 + a_2d_1)x_2 = x_1 + (x_2 + a_1d_2 + a_2d_1)$$
$$+(x_2 + a_1d_2 + a_2d_1)x_1 + (x_1 + a_1d_1 + a_2d_2)x_2.$$

(Adding $x_1+x_2$ to both sides of the equation.) Let $x_1+a_1d_1+a_2d_2 = p_2$ and $x_2+a_1d_2+a_2d_1 = p_1$ then $p_1, p_2 \in La$ (since $x_1, x_2, d_1 = c_1a, d_2 = c_2a \in La$) and $x_1 + p_1 + p_1x_1 + p_2x_2 = x_2 + p_2 + p_2x_1 + p_1x_2$. Since $x_1$ and $x_2$ are arbitrary elements of $La$, so $La$ is left semiregular. □

**Theorem 3.20.** *The radical class $\mathcal{J}$ is left strong.*

**Proof.** By Lemma 3.18, 3.19 and Proposition 3.15, we get the class $\mathcal{J}$ is left strong. □

**Definition 3.21 ([1]).** *A semiring $S$ is said to be locally nilpotent if every finite subset $F$ of $S$ generates a nilpotent semiring i.e. there exists an integer $N$ such that every product $x_{i1}....x_{iN} = 0$ for $x_{ij}$ in $F$.*

**Theorem 3.22 ([10], Theorem 12).** *The class $\mathcal{L}$ of all locally nilpotent semirings is a radical class.*

$L$ is called Levitzki radical class.

**Lemma 3.23 ([1], Lemma 3).** *If $L$ and $K$ are locally nilpotent ideals of a semiring $S$ then $L + K$ is a locally nilpotent ideal of $S$.*

**Lemma 3.24.** *If $L$ is a locally nilpotent left ideal of a semiring $S$ then for any element $a \in S$, the left ideal $La$ is a locally nilpotent left ideal.*

**Proof.** Let $F = \{x_1, x_2, ...., x_n\}$ be a finite subset of $La$. Then $x_1 = l_1a$, $x_2 = l_2a$, ...., $x_n = l_na$ for some $l_1, l_2, ..., l_n \in L$. Let $y_1 = al_1$, $y_2 = al_2$, ...., $y_n = al_n$ and $G = \{y_1, y_2, ...., y_n\}$. Then $G$ is a finite subset of $L$. Since $L$ is locally nilpotent, there exists a positive integer $N$ such that every product $y_{i1}....y_{iN} = 0$ for $y_{ij}$ in $G$ i.e. $al_{i1}....al_{iN} = 0$. So $ax_{i1}....ax_{iN} = 0$, which shows every product $x_{j1}....x_{jN+1} = 0$ for $x_{jk}$ in $F$. Hence $La$ is locally nilpotent. □

**Theorem 3.25.** *$\mathcal{L}$ is left strong.*

**Proof.** By Lemma 3.23, 3.24 and Proposition 3.15, we can say class $\mathcal{L}$ is left strong. □

**Definition 3.26.** *A semiring $S$ is called a nil semiring if for each $a \in S$ there exists a positive integer $n$ such that $a^n = 0$.*

**Theorem 3.27 ([11], Theorem 31).** *The class of all nil semirings $\mathcal{N}$ is a radical class.*

$\mathcal{N}$ is called nil radical class.

**Lemma 3.28.** *If $L$ and $K$ are two ideals of a semiring $S$ such that $L, K \in \mathcal{N}$ then $L + K \in \mathcal{N}$.*

**Proof.** Let $x \in L + K$. So $x = l + k$ for some $l \in L$ and $k \in K$. Since $L$ and $K$ are nil semirings, there exist positive integers $m, n$ such that $l^m = 0$ and $k^n = 0$. Now $x^{m+n} = (l+k)^{m+n} = l^{m+n} +^n c_1 l^{m+n-1}k +^n c_2 l^{m+n-2}k^2 + .... + k^{m+n} = 0$. Therefore $L + K \in \mathcal{N}$. □

**Lemma 3.29.** *If $L \in \mathcal{N}$ is a left ideal of a semiring $S$ and $a \in S$ then $La \in \mathcal{N}$.*

**Proof.** Let $x \in La$. Then $x = la$ for some $l \in L$. Since $L$ is a left ideal of $S$, $al \in L$. Again $L$ is a nil semiring, so there exists a positive integer $n$ such that $(al)^n = 0$. Therefore $(la)^{n+1} = 0$ i.e $x^{n+1} = 0$. Hence $La \in \mathcal{N}$. □

**Theorem 3.30.** *$\mathcal{N}$ is left strong.*

**Proof.** By Lemma 3.28, 3.29 and Proposition 3.15, we can say class $\mathcal{N}$ is left strong. □

**Theorem 3.31 ([11], Theorem 17).** *If $\mathcal{M}$ is a regular class of semirings, then the class $\mathcal{UM} = \{S : S$ has no nonzero homomorphic image in $\mathcal{M}\}$ is a radical class.*

**Theorem 3.32.** *If $\mathcal{M}$ is a left regular class of semirings, then the upper radical $\rho = \mathcal{UM}$ is left strong.*

**Proof.** Here $\rho$ is homomorphically closed. Let $S$ be a semiring such that $S \notin \rho$. Then $S$ has a nonzero homomorphic image $S_1$ in $\mathcal{M}$. Let $L$ be a nonzero left ideal of $S_1$. Since $\mathcal{M}$ is left regular class, $L$ has a nonzero

homomorphic image in $\mathcal{M}$. Therefore $L \notin \rho$. So $S_1$ has no nonzero left ideal in $\rho$. This implies that if every nonzero homomorphic image of $S$ has a nonzero left ideal in $\rho$, then $S$ is itself in $\rho$. Therefore by Theorem 3.14 (iii), $\rho$ is left strong radical class. □

Let $\mathcal{M}$ be a homomorphically closed class of semirings. Define $\mathcal{M}_1 = \mathcal{M}$, $\mathcal{M}_\alpha$=$\{S$: every nonzero homomorphic image of the semiring $S$ contains a nonzero left ideal in $\mathcal{M}_\beta$ for some $\beta < \alpha\}$ for ordinals $\alpha \geq 1$, $\mathcal{M}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$ for limit ordinal $\lambda$ and $LS(\mathcal{M}) = \bigcup_\alpha \mathcal{M}_\alpha$.

**Proposition 3.33.** *For a homomorphically closed class $\mathcal{M}$ of semirings:*

(*i*) *$\mathcal{M}_\alpha$ is homomorphically closed for every ordinal $\alpha$.*

(*ii*) *If $\beta < \alpha$ then $\mathcal{M}_\beta \subseteq \mathcal{M}_\alpha$.*

**Proof.** Let $S \in \mathcal{M}_\alpha$ and $S_1$ be a homomorphic image of $S$. If $S_1 = 0$ then clearly $S_1 \in \mathcal{M}_\alpha$. If $S_1 \neq 0$, then suppose $S_2$ be a nonzero homomorphic image of $S_1$. Then $S_2$ is a nonzero homomorphic image of $S$. Since $S \in \mathcal{M}_\alpha$, so $S_2$ contains a nonzero left ideal in $\mathcal{M}_\beta$ for some $\beta < \alpha$. Therefore $S_1 \in \mathcal{M}_\alpha$. Hence $\mathcal{M}_\alpha$ is homomorphically closed for every ordinal $\alpha$.

(ii) Let $S \in \mathcal{M}_\beta$ and $S_1$ be a nonzero homomorphic image of $S$. Then $S_1$ contains a nonzero left ideal in $\mathcal{M}_\delta$ for some $\delta < \beta$ i.e. for some $\delta < \alpha$. Therefore $S \in \mathcal{M}_\alpha$. Hence $\mathcal{M}_\beta \subseteq \mathcal{M}_\alpha$. □

**Theorem 3.34.** *$LS(\mathcal{M})$ is the smallest left strong radical class containing $\mathcal{M}$.*

**Proof.** By Proposition 3.33, each $\mathcal{M}_\alpha$ is homomorphically closed, so $LS(\mathcal{M})$ is homomorphically closed. Let $S$ be a semiring such that every nonzero homomorphic image $S_1$ of $S$ has a nonzero left ideal $L_1$ in $LS(\mathcal{M})$. Then $L_1$ belongs to $\mathcal{M}_\alpha$ for some ordinal $\alpha$. For various homomorphic images we get various $\alpha$'s. Let $\delta$ be an ordinal number which is larger than or equal to all of these $\alpha$'s. Then for each $S_1$, $L_1 \in \mathcal{M}_\alpha \subseteq \mathcal{M}_\delta$. So $S \in \mathcal{M}_{\delta+1}$ and therefore $S \in LS(\mathcal{M})$. Hence by Theorem 3.14 (iii), $LS(\mathcal{M})$ is a left strong radical class.

Clearly $LS(\mathcal{M})$ contains $\mathcal{M}$. To prove that $LS(\mathcal{M})$ is the smallest left strong radical class containing $\mathcal{M}$, let $\rho$ be a left strong radical class

containing $\mathcal{M}$. we shall prove by method of induction. Suppose $\mathcal{M}_\beta \subseteq \rho$ for $\beta < \alpha$. Let $S \in \mathcal{M}_\alpha$. Then every nonzero homomorphic image of $S$ contains a nonzero left ideal in $\mathcal{M}_\beta$ for some $\beta < \alpha$. These left ideals also belongs to $\rho$. Since $\rho$ is left strong radical class by Theorem 3.14 (iii), $S \in \rho$. Also $\mathcal{M} \subseteq \rho$. Therefore by induction $LS(\mathcal{M}) = \bigcup_\alpha \mathcal{M}_\alpha \subseteq \rho$. Hence proved. □

## 4. Prime radical class

**Proposition 4.1.** *If $P$ is a semiprime ideal of a semiring $S$ and $I$ is an ideal of $S$ then $P \cap I$ is a semiprime ideal of $I$.*

**Proof.** Let $aIa \subseteq P \cap I$, where $a \in I$. Then $aIa \subseteq P$. Let $K = IaI$. So $K$ is an ideal of $S$ and $K^2 = IaIIaI \subseteq IaIaI \subseteq P$. Since $P$ is a semiprime ideal of $S$, $K \subseteq P$. Now $< a >^3 \subseteq IaI$. So $< a >^3 \subseteq P$, which implies that $a \in P$ i.e. $a \in P \cap I$. Hence $P \cap I$ is a semiprime ideal of $I$. □

**Proposition 4.2.** *The class of semiprime semirings is hereditary.*

**Proof.** Suppose $S$ is a semiprime semiring and $I$ is an ideal of $S$. So $\{0\}$ is semiprime ideal of $S$. Therefore by proposition 4.1, $\{0\}$ is a semiprime ideal of $I$ i.e. $I$ is a semiprime semiring. Hence the proposition. □

**Definition 4.3.** *A semiring $S$ is said to be an essential extension of a semiring $R$, if $R$ is an essential ideal of $S$.*

**Definition 4.4.** *A class $\rho$ of semirings is called closed under essential extension if $I$ is an essential ideal of a semiring $S$ and $I \in \rho$ implies $S \in \rho$.*

**Proposition 4.5.** *The class of semiprime semirings is closed under essential extensions.*

**Proof.** Suppose $I$ is an essential ideal of a semiring $S$ and $I$ is a semiprime semiring. Let $A$ be an ideal of $S$ such that $A^2 = 0$. Let $A_1 = A \cap I$. Then $A_1^2 \subseteq A^2 = 0$. Since $I$ is a semiprime semiring, $A_1 = 0$. Since $I$ is an essential ideal of $S$, So $A = 0$. Therefore $S$ is a semiprime semiring. Hence the proposition. □

**Definition 4.6.** *Let $A$ be a non-empty subset of a semiring $S$. Annihilator of $A$ in $S$, denoted by $ann_S(A)$, is defined by $ann_S(A) = \{s \in S : As = 0 = sA\}$.*

**Definition 4.7.** *A class $\mathcal{M}$ of semirings will be called a weakly special class if $\mathcal{M}$ is a hereditary class of semiprime semirings satisfying:*

(1) *If $\phi$ is a semi-isomorphism of $S$ onto $T$ and $T \in \mathcal{M}$, then $S \in \mathcal{M}$.*

(2) *If $I \in \mathcal{M}$ and $I$ is an ideal of a semiring $S$, then $S/ann_S(I) \in \mathcal{M}$.*

**Lemma 4.8 ([9], Lemma 13).** *If $\mathcal{M}$ is a hereditary class of semiprime semirings which satisfies properties: "if $S \in \mathcal{M}$ and $S$ is semi-isomorphic to $T$ then $T \in \mathcal{M}$" then the following are equivalent:*

(1) *If $A \in \mathcal{M}$ and $A$ is an ideal of $S$, then $S/ann_S(A) \in \mathcal{M}$;*

(2) *If $A \in \mathcal{M}$ with $A$ is an ideal of $S$ and $ann_S(A) = 0$, then $S \in \mathcal{M}$;*

(3) *If $A \in \mathcal{M}$ and $A$ is an essential ideal of $S$, then $S \in \mathcal{M}$.*

**Proposition 4.9.** *If a semiring $S$ is semi-isomorphic to a semiring $T$ and $T$ is semiprime, then $S$ is semiprime.*

**Proof.** Let $\phi$ be the semi-isomorphism i.e $\phi(S) = T$. Suppose $A$ is an ideal of $S$ such that $A^2 = 0$. Therefore $\phi(A^2) = 0$ i.e. $(\phi(A))^2 = 0$ . Since $T$ is a semiprime semiring, so $\phi(A) = 0$. Again since $\phi$ is a semi-isomorphism, so $A = 0$. Hence $S$ is a semiprime semiring. □

**Proposition 4.10.** *If a semiring $S$ is semi-isomorphic to a semiring $T$ and $S$ is semiprime, then $T$ is semiprime.*

**Proof.** Let $\phi$ be the semi-isomorphism i.e $\phi(S) = T$. Suppose $A'$ is an ideal of $T$ such that $(A')^2 = 0$. So there exist an ideal $A$ of $S$ such that $\phi(A) = A'$. Therefore $(\phi(A))^2 = 0$ i.e. $\phi(A^2) = 0$. Since $\phi$ is a semi-isomorphism, so $A^2 = 0$. Again $S$ is a semiprime semiring implies $A = 0$. Hence $A' = 0$. Thus $T$ is a semiprime semiring. □

**Theorem 4.11.** *The class of all semiprime semirings $\wp$ is a weakly special class.*

**Proof.** By Proposition 4.2, $\wp$ is hereditary. By Proposition 4.9, $\wp$ satisfies the condition (1) of the definition of weakly special class. Also by Proposition 4.10, Lemma 4.8 and by Proposition 4.5, $\wp$ satisfies the condition (2) of the definition of weakly special class. Hence $\wp$ is a weakly special class. □

**Definition 4.12.** *An ideal $I$ of a semiring $S$ is said to be nilpotent if there exists a positive integer $n$ such that $I^n = 0$.*

Let $\mathcal{B} = \{S$ : each nonzero homomorphic image of the semiring $S$ contains a nonzero nilpotent ideal $\}$.

**Theorem 4.13 ([10], Theorem 5).** *$\mathcal{B}$ is a radical class.*

$\mathcal{B}$ is called prime radical class.

**Theorem 4.14.** *$\mathcal{B} = \mathcal{U}_\wp$ i.e $\mathcal{B} = \{S$ : no nonzero homomorphic image of the semiring $S$ is a semiprime semiring$\}$.*

**Proof.** Let $S \in \mathcal{U}_\wp$ and $S_1$ be a nonzero homomorphic image of $S$. So $S_1$ is not a semiprime semiring. Therefore there exists a nonzero ideal $I$ such that $I^2 = 0$ i.e. $S_1$ contains a nonzero nil ideal. So $S \in \mathcal{B}$. Thus $\mathcal{U}_\wp \subseteq \mathcal{B}$.

Conversely, let $T \in \mathcal{B}$ and $T_1$ be a nonzero homomorphic image of $T$. If possible let $T_1$ is a semiprime semiring. Since $T \in \mathcal{B}$, $T_1$ contains a nonzero nilpotent ideal $I$. So there exist a positive integer $n$ such that $I^n = 0$. Also $I^n = 0$ implies $I = 0$ (as $T_1$ is a semiprime semiring), a contradiction. Therefore no nonzero homomorphic image of $T$ belongs to $\wp$ i.e. $T \in \mathcal{U}_\wp$. So $\mathcal{B} \subseteq \mathcal{U}_\wp$. Hence $\mathcal{U}_\wp = \mathcal{B}$. □

**Proposition 4.15.** *$\wp = \mathcal{SB}$ i.e $\wp = \{S$ : for each nonzero ideal $I$ of $S$ there exists a nonzero homomorphic image $I_1$ of $I$ such that $I_1$ does not contain any nonzero nilpotent ideal $\}$.*

**Proof.** Let $S \in \wp$ and $I$ be any nonzero ideal of $S$. Since $\wp$ is hereditary, $S \in \wp$ implies $I \in \wp$. We now prove that $I$ does not contain any nonzero nilpotent ideal. If possible, let $K$ be a nonzero nilpotent ideal of $I$. So there exists a positive integer $n > 1$ such that $K^n = 0$. Since $I$ is a semiprime semiring $K^n = 0$ implies $K = 0$, a contradiction. Therefore $I$ does not contain any nonzero nilpotent ideal. So $S \in \mathcal{SB}$.

Let $S \in \mathcal{SB}$. If possible, let $S \notin \wp$ i.e. $S$ is not a semiprime semiring. So there exists a nonzero ideal $I$ of $S$ such that $I^2 = 0$. Therefore for the nonzero ideal $I$, each nonzero homomorphic image of $I$ contains a nonzero nilpotent ideal, a contradiction. So $S \in \wp$. Hence $\wp = \mathcal{SB}$. □

**Theorem 4.16.** *Prime radical class $\mathcal{B}$ is left strong.*

**Proof.** If possible, let $\mathcal{B}$ is not a left strong radical class. So by proposition 3.12, there exists a nonzero left ideal $L$ of $S \in \mathcal{SB}$ such that $L \in \mathcal{B}$. Since $\mathcal{B} = \mathcal{U}_{\wp}$, $L$ is not a semiprime semiring. Therefore there exists a nonzero ideal of $L$ which is a zero semiring. Applying Zorn's lemma to the set of all ideals of $L$ which are zero semirings, we can find an ideal $K$ of $L$ which is maximal with respect to the property $K^2 = 0$. Now $LK + LKS$ is an ideal of $S$ and $(LK + LKS)^2 = (LK)^2 + (LK)(LKS) + (LKS)(LK) + (LKS)^2 = (LK)^2 + (LK)^2S + (LK)(SLK) + (LK)(SLK)S \subseteq (LK)^2 + (LK)^2S \subseteq K^2 + K^2S = 0$ (as $LK \subseteq K$). Since $S$ is a semiprime semiring, so $LK + LKS = 0$ i.e. $LK \subseteq LK + LKS = 0$. Therefore $LK = 0$. Again $L^2 \neq 0$, if not, then $L + LS$ is an ideal of $S$ and $(L + LS)^2 \subseteq L^2 + L^2S = 0$ implies $L + LS = 0$ i.e. $L = 0$, a contradiction. Therefore $L^2 \neq 0$ and $LK = 0$ implies $K \neq L$. So $L/K$ is a nonzero homomorphic image of $L$. Since $\mathcal{B}$ is homomorphically closed and $L \in \mathcal{B}$, $L/K \in \mathcal{B}$. Hence there exists a nonzero ideal $I/K$ of $L/K$ such that $I/K$ is a zero semiring i.e. $I^2 \subseteq K$. Then $(LI)^2 = LILI \subseteq LII \subseteq LK = 0$. Again $LI + LIS$ is an ideal of $S$ and $(LI + LIS)^2 \subseteq (LI)^2 + (LI)^2S = 0$. Since $S$ is semiprime, so $LI + LIS = 0$ i.e. $LI = 0$. Therefore $I^2 \subseteq LI = 0$ i.e. $I^2 = 0$. By maximality condition of $K$ implies that $I = K$ which is a contradiction, since $I/K \neq 0$. Hence $L \notin \mathcal{B}$ i.e. $\mathcal{B}$ is left strong. □

## REFERENCES


1. Barbut, E. – *On nil semirings with ascending chain conditions*, Fundamenta Mathematicae, 68(1970), 261-264.
2. Bourne, S. – *The Jacobson radical of a semiring*, Proc. Nat. Acad. Sci., 37(1951), 163-170.
3. Dutta, T.K.; Das, M.L. – *On uniformly strongly prime semiring*, Int. J. Math. Anal., 2(2006), 73-82.
4. Dutta, T.K.; Das M.L. – *On strongly prime semiring* (to appear).
5. Gardner, B.J.; Wiegandt, R. – *Radical Theory of Rings*, Marcel Dekker, 2004.
6. Golan, Jonathan S. – *Semirings and their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.
7. LaTorre, D.R. – *On h-ideals and k-ideals in hemirings*, Publ. Math. Debrecen, 12(1965), 219-226.

8. Morak, Bettina – *On the radical theory for semirings*, Contribution to Algebra and Geometry, 40(1999), 533-549.

9. Olson, D.M.; Heyman, G.A.P.; LeRoux, H.J. – *Weakly special classes of hemirings*, Quaestiones Mathematicae, 15(1992), 119-126.

10. Olson, D.M.; Jenkins, T.L. – *Radical theory for hemirings*, J. Natur. Sci. and Math., 23(1983), 23-32.

11. Olson, D.M.; Nance A.C. – *A note on radicals for hemirings*, Quaestiones Mathematicae, 12(1989), 307-314.

12. Vandiver, H.S. – *Note on a simple type of algebras in which the cancellation law of addition does not hold*, Bull. Am. Math. Soc. 40(1934), 916-920.

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