ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LV, 2009, f.1

ON STRONGLY PRIME Γ-SEMIRINGS

BY

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Abstract. The strongly prime Γ-semiring is a special type of Γ-semiring which we study in this paper.

Mathematics Subject Classification 2000: 16Y60.

Key words: Γ-semiring, right strongly prime ideal of a Γ-semiring and right strongly prime Γ-semiring.

1. Introduction.

In 1975, Handelman and Lawrence [8] introduced the notion of (right) strongly prime ring motivated by the notion of primitive group ring and characterized strongly prime rings. In 1988, Booth [2] extended the notions of uniformly strongly prime ideal and uniformly strongly prime ring to Γ-ring. In 2006, Dutta and Das [7] introduced the notion of (right) strongly prime ideal and (right) strongly prime semiring and characterized strongly prime semirings.

In this paper we introduce the notions of right strongly prime ideal and right strongly prime Γ-semiring and study these as G.L. Booth did in case of Γ-ring . Some earlier work on the operator semiring of a Γ-semiring may be found in [4]. We obtain a one-to-one correspondence between the set of all rsp k-ideals of a Γ-semiring and the set of all rsp k-ideals of the operator semiring of the Γ-semiring.

2. Right strongly prime Γ-semirings

Definition 2.1. [4] *Let S and Γ be two additive commutative semigroups. Then S is called a Γ-semiring if there exists a mapping $S\times\Gamma\times S \longrightarrow$*

S (the image to be denoted by $a\alpha b$, for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$;

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$;

(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$;

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$ *for all* $a, b, c \in S$ *and for all* $\alpha, \beta \in \Gamma$.

Every semiring S is a Γ-semiring with $\Gamma = S$ where $a\alpha b$ denotes the product of elements $a, \alpha, b \in S$.

If S contains an element 0 such that $0+x = x = x+0$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in S$, for all $\alpha \in \Gamma$, then 0 is called the zero of S.

Throughout this paper we assume that a Γ-semiring S always contains a zero element 0 and we denote the set $S \setminus \{0\}$ by S^*.

Definition 2.2. [4] *A nonempty subset I of a Γ-semiring S is called an ideal of S if $I + I \subseteq I$, $I\Gamma S \subseteq I$, $S\Gamma I \subseteq I$, where for subsets U, V of S and Δ of Γ,*

$U\Delta V = \{\sum_{i=1}^{n} u_i\gamma_i v_i : u_i \in U, v_i \in V, \gamma_i \in \Delta$ and n is a positive integer$\}$.

Definition 2.3. [4] *An ideal I of a Γ-semiring S is called a k-ideal if for $x, y \in S$, $x + y \in I$ and $y \in I$ implies that $x \in I$.*

Definition 2.4. *A Γ-semiring S is called right strongly prime if for every $x \in S^*$ there exist a finite subset F of S and finite subsets Φ and Δ of Γ such that $x\Phi F\Delta y = \{0\}$ implies that $y = 0$ for all $y \in S$.*

Analogously we define left strongly prime Γ-semirings.

Definition 2.5. [3] *A proper ideal P of a Γ-semiring S is called a prime ideal of S if for any two ideals A, B of S, $A\Gamma B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.*

A Γ-semiring S is called a prime Γ-semiring if (0) is a prime ideal of S.

Proposition 2.6. *A right strongly prime Γ-semiring is prime.*

Proof. Let S be a right strongly prime Γ-semiring. Let $a(\neq 0) \in S$ be such that $a\Gamma S\Gamma b = 0$ for all $b \in S$. Since $a \neq 0$ and S is right strongly prime, there exist a finite subset F of S and finite subsets Φ and Δ of Γ

such that $a\Phi F\Delta b = 0$ implies that $b = 0$. Now $a\Phi F\Delta b \subseteq a\Gamma S\Gamma b = 0$, which implies that $b = 0$. It follows that by Theorem 3.6 of [3], S is a prime Γ-semiring. □

Proposition 2.7. *The following statements about a Γ-semiring S are equivalent:*

(i) S is right strongly prime;

(ii) If I is a nonzero ideal of S, there exist finite subsets F of I and Δ of Γ such that $F\Delta y = \{0\}$ implies that $y = 0$ for all $y \in S$;

(iii) If $x \in S^*$, there exist $\gamma \in \Gamma$ and finite subsets F of S and Δ of Γ such that $x\gamma F\Delta y = \{0\}$ implies that $y = 0$ for all $y \in S$.

Proof. *(i)* $\Rightarrow$ *(ii)* Let I be a nonzero ideal of S. Let $x(\neq 0) \in I$. Then by *(i)* there exist finite subsets F of S and finite subsets Δ, Φ of Γ such that $x\Phi F\Delta y = \{0\}$ implies that $y = 0$ for all $y \in S$. Then $F' = x\Phi F$ is a finite subset of I and $F'\Delta y = 0$ implies that $y = 0$ for all $y \in S$.

(ii) $\Rightarrow$ *(iii)* Let $a(\neq 0) \in S$. Then $< a >$ is a nonzero ideal of S. So by (ii) there exist finite subsets F' of $< a >$ and Φ of Γ such that $F'\Phi y = \{0\}$ implies that $y = 0$ for all $y \in S$. Suppose that $a\Gamma S = \{0\}$. Then $< a > \Gamma S = 0$. Since $F'\Phi a \subseteq < a > \Gamma S = \{0\}$, we have $F'\Phi a = \{0\}$. This implies that $a = 0$, a contradiction. Therefore $a\gamma x \neq 0$ for some $\gamma \in \Gamma$ and $x \in S$. Then $I =< a\gamma x >$ is a nonzero ideal of S. Again by (ii) there exist a finite subset G of I and a finite subset Δ of Γ such that $G\Delta y = \{0\}$ implies that $y = 0$ for all $y \in S$. Since G is a finite subset of I, we have $G = \{na\gamma x + \sum_{i=1}^{m} a\gamma x\beta_i z_i + \sum_{j=1}^{l} s_j\gamma_j a\gamma x + \sum_{t=1}^{k} u_t\lambda_t a\gamma x\mu_t v_t$, where $m, n, l, k \in Z_0^+; z_i, s_j, u_t, v_t \in S$ and $\beta_i, \gamma_j, \lambda_t, \mu_t \in \Gamma\}$. Let $F = \{x,\ x\mu_t v_t,\ x\beta_i z_i : i = 1, 2, ...m;\ t = 1, 2, ...k;\ m, k \in Z_0^+$ and $\beta_i,\ \mu_t \in \Gamma\}$. Let $a\gamma F\Delta y = \{0\}$. Then $G\Delta y = \{0\}$. By (ii) $y = 0$.

(iii) $\Rightarrow$ *(i)* Let $a(\neq 0) \in S$. Then by *(iii)*, there exist $\gamma \in \Gamma$ and finite subsets F of S and Δ of Γ such that $a\gamma F\Delta y = \{0\}$ implies that $y = 0$ for all $y \in S$. Let $\Phi = \{\gamma\}$. Hence there exist finite subsets F of S and Φ, Δ of Γ such that $a\Phi F\Delta y = \{0\}$ implies that $y = 0$, for all $y \in S$. Thus *(i)* holds. □

Examples 2.8. (1) Every strongly prime Γ-ring is a strongly prime Γ-semiring.

(2) Let $S = \{r\omega : r \in \mathbf{Z}\}$ and $\Gamma = \{r\omega^2 : r \in \mathbf{Z}\}$, where ω is a cube root of unity and $\mathbf{Z}$ is the set of all integers. Then S is a Γ-semiring with usual addition and multiplication. Let $r\omega(\neq 0) \in S$. Let $F = \{r\omega\}$ and

$\Phi = \Delta = \{r\omega^2\}$. Then $(r\omega)\Phi F\Delta y = 0$ implies that $y = 0$ for all $y \in S$. Hence by definition, S is a right strongly prime Γ-semiring.

(3) Let S be the set of all 2×3 matrices over $\mathbf{Z_0^+}$ and Γ be the set of all 3×2 matrices over $\mathbf{Z_0^+}$, where $\mathbf{Z_0^+}$ is the set of all nonnegative integers. Then S is a Γ-semiring with usual matrix addition and matrix multiplication. Let I be a nonzero ideal of S. Then I has a nonzero element, say $(a_{ij})_{2\times 3}$. Then $(a_{ij})_{2\times 3}$ has atleast one nonzero element, say a_{rs}. Since I is an ideal of S, $E_{11}E'_{1r}(a_{ij})_{2\times 3}E'_{s1}E_{11} \in I$, where E_{rs}(respectively E'_{rs}) denotes a 2×3(respectively 3×2) matrix whose $(r, s)^{th}$ element is 1 and all others elements are zero. This shows that I has an element, say f_1 whose $(1, 1)^{th}$ element is nonzero and all others elements are all zero. Similarly we can get an element, say f_2 in I whose $(2, 2)^{th}$ element is nonzero and all others elements are all zero. Let $f_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ where $a \in Z^+$ and $f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix}$ where $b \in Z^+$. Let $F = \{f_1,\ f_2\}$ and $\Delta = \{\alpha_1,\ \alpha_2\}$ where $\alpha_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ with $\alpha \in Z^+$ and $\alpha_2 = \begin{pmatrix} 0 & 0 \\ 0 & \beta \\ 0 & 0 \end{pmatrix}$ with $\beta \in Z^+$. Let $F\Delta t = 0$, where $t = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} \in S$. Then $f_1\alpha_1 t = 0 = f_2\alpha_2 t$ which implies that $a\alpha u = a\alpha v = a\alpha w = b\beta x = b\beta y = b\beta z = 0$. Since $a, b, \alpha, \beta \in \mathbf{Z}^+$, we must have $u = 0 = v = w = x = y = z$ i.e. $t = 0$. Hence by above Proposition, S is a right strongly prime Γ-semiring.

(4) Let S be a simple Γ-semiring(i.e. a Γ-semiring S in which $S\Gamma S \neq 0$ and $S, \{0\}$ are the only ideals of S) with left unity(defined in proposition 2.29). Then S is the only nonzero ideal of S. Let $F = \{e_1, e_2, ..., e_m\}$ and $\Delta = \{\delta_1, \delta_2, ..., \delta_m\}$ where $\sum_{i=1}^{m}[e_i, \delta_i]$ is the left unity of S. Let $F\Delta y = 0$, where $y \in S$. Then $e_i\delta_i y = 0$ for all $i = 1, 2, ..., m$ which implies that $y = \sum_{i=1}^{m} e_i\delta_i y = 0$. Hence by the above Proposition, S is a right strongly prime Γ-semiring.

Definition 2.9. *Let A be a nonempty subset of a Γ-semiring S. The right annihilator of A with respect to $\Phi \subseteq \Gamma$ in S, denoted by $r(A,\ \Phi)$, is defined by $r(A,\ \Phi) = \{s \in S : A\Phi s = \{0\}\}$.*

Proposition 2.10. *The right annihilator $r(A,\ \Phi)$ of A with respect to Φ in a Γ-semiring S is a right ideal of S.*

Proof. Since $A\Phi 0 = \{0\}$, $0 \in r(A, \Phi)$; so $r(A, \Phi)$ is nonempty. Let $s, s' \in r(A, \Phi)$. Then $A\Phi s = \{0\} = A\Phi s'$. Then $A\Phi(s + s') = A\Phi s + A\Phi s' = \{0\}+\{0\} = \{0\}$. It follows that $r(A, \Phi)+r(A, \Phi) \subseteq r(A, \Phi)$. Again $A\Phi(s\alpha t) = (A\Phi s)\alpha t = 0\alpha t = 0$ for all $\alpha \in \Gamma$ and for all $t \in S$ i.e $s\alpha t \in r(A, \Phi)$ i.e. $r(A, \Phi)\Gamma S \subseteq r(A, \Phi)$. Thus $r(A, \Phi)$ is a right ideal of S. □

Definition 2.11. *A Γ-semiring S is said to satisfy the descending chain condition (DCC) for left (right) ideals if for each sequence of left (right) ideals $A_1, A_2, A_3, ...$ of S with $A_1 \supseteq A_2 \supseteq A_3 \supseteq .....$ there exists a positive integer n such that $A_n = A_{n+1} = ...$*

Proposition 2.12. *If S is a prime Γ-semiring with DCC on right annihilator ideals then S is a right strongly prime Γ-semiring.*

Proof. Let I be a non zero ideal of S and let $\mathcal{M}$ denote the class of all right annihilators of the form $r(F, \Phi)$, where F and Φ are finite subsets of I and Γ respectively. Then $\mathcal{M}$ contains a minimal element $J = r(F_0, \Phi_0)$, say. Suppose $J \neq 0$. Since S is a prime Γ-semiring, $I\ \Gamma J \neq 0$. Let $x \in I$, $\gamma \in \Gamma$, $y \in J$ be such that $x\gamma y \neq 0$. Let $F' = F_0 \cup \{x\}$ and $\Phi' = \Phi_0 \cup \{\gamma\}$. Since $F_0\Phi_0 s \subseteq F'\Phi' s$, $r(F', \Phi') \subseteq J$. Again $y \in J$ and $x\gamma y \neq 0$ implies that $r(F', \Phi') \subset J$, which contradicts the minimality of J. Hence $J = \{0\}$. Therefore by (ii) of Proposition 2.8, S is a right strongly prime Γ-semiring. □

Definition 2.13. [3] *For an ideal A of a Γ-semiring S, the Γ-congruence on S , denoted by ρ_A and defined as $s\rho_A s'$ if and only if $s + a_1 = s' + a_2$, for some $a_1, a_2 \in A$, is called the Bourne Γ-congruence on S defined by the ideal A.*

For the Bourne Γ-congruence ρ_A we denote the class of an element r of S by r/ρ_A or simply by r/A and denote the set of all such Γ-congruences classes of the elements of the Γ-semiring S by S/ρ_A or simply by S/A .

Definition 2.14. [3] *For an ideal A of a Γ-semiring S if the Bourne Γ-congruence ρ_A, defined by A , is proper i.e. $0/A \neq S$ then we define on S/A the following operations: $s/A + s'/A = (s + s')/A$ and $(s/A)\alpha(s'/A) = (s\alpha s')/A$ for all $\alpha \in \Gamma$. Now S/A is a Γ-semiring with these operations. We call this Γ-semiring the Bourne factor Γ-semiring or simply the factor Γ-semiring of S by A.*

Definition 2.15. *An ideal I of a Γ-semiring S is called a right strongly prime ideal if the factor Γ-semiring S/I is right strongly prime.*

Proposition 2.16. *Let Q be a k-ideal of a Γ-semiring S. Then Q is a right strongly prime ideal of S if and only if, for every ideal I of S not contained in Q, there exist finite subsets F and Φ of I and Γ respectively such that $F\Phi y \subseteq Q$ implies that $y \in Q$ for all $y \in S$.*

Proof. Let Q be a right strongly prime ideal of the Γ-semiring S. Then the factor Γ-semiring S/Q is right strongly prime. Let I be an ideal of S not contained in Q. Then I/Q is a non zero ideal of the right strongly prime factor Γ-semiring S/Q. Hence there exist finite subsets $F' = \{i_1/Q,\ i_2/Q, i_3/Q, ..., i_n/Q\}$ and Φ of I/Q and Γ respectively such that $F'\Phi(y/Q) = 0/Q$ implies that $y/Q = 0/Q$ for all $y/Q \in S/Q$. Let $F = \{i_1,\ i_2, i_3, ..., i_n\}$. Then F is a finite subset of I. Let $F\Phi y \subseteq Q$. Then $(F/Q)\Phi(y/Q) = 0/Q$ i.e. $F'\Phi(y/Q) = 0/Q$, which implies that $y/Q = 0/Q$ for all $y/Q \in S/Q$. Since Q is a k-ideal of S, $y \in Q$ for all $y \in S$.

Conversely, let I/Q be a non zero ideal of S/Q. Then I is an ideal of S not contained in Q. Then by hypothesis there exist finite subsets F and Φ of I and Γ respectively such that $F\Phi y \subseteq Q$ implies that $y \in Q$ for all $y \in S$. Since F is a finite subset of I, F/Q is a finite subset of I/Q. Let $(F/Q)\Phi(y/Q) = 0/Q$. Then $F\Phi y \subseteq Q$, and hence $y \in Q$ i.e. $y/Q = 0/Q$. Thus S/Q is a right strongly prime Γ-semiring. Hence Q is a right strongly prime ideal of S.

Definition 2.17. [7] *An ideal I of a semiring S is called a right strongly prime ideal if for all $a \notin I$ there is a finite set $F \subseteq < a >$ such that $Fb \subseteq I$ implies that $b \in I$.*

Proposition 2.18. [7] *A k-ideal I of a semiring S is a right strongly prime ideal if and only if S/I is a right strongly prime semiring.*

Definition 2.19. [7] *Let A be a nonempty subset of a semiring S. The right annihilator of A, denoted by $ann_R(A)$, is defined by $ann_R(A) = \{s \in S : As = 0\}$.*

Theorem 2.20. [7] *A semiring S is right strongly prime if and only if every nonzero ideal of S contains a finite subset G such that $ann_R(G) = (0)$.*

Proposition 2.21. *Let Q be a k-ideal of a semiring S. Then Q is a right strongly prime ideal of S if and only if for every ideal I of S not*

contained in Q, there exist finite subset F of I such that $Fy \subseteq Q$ implies that $y \in Q$ for all $y \in S$.

Proof. The proof is similar to that of Proposition 2.16. □

We can prove the following Proposition as G. L. Booth did in the case of Γ-rings in [1](Lemma 5.1).

Proposition 2.22. *Let S be a semiring and I be an ideal of the Γ-semiring S, where $\Gamma = S$. Let $\bar{I}$ be the ideal of the semiring S generated by the set I. Then $(\bar{I})^5 \subseteq I$.*

Proposition 2.23. *Let S be a semiring. Then a proper subset P of the semiring S is a right strongly prime k-ideal of the semiring S if and only if P is a right strongly prime k-ideal of the Γ-semiring S where $\Gamma = S$.*

Proof. Let P be a right strongly prime k-ideal of the semiring S. Then P is a k-ideal of the Γ-semiring S where $\Gamma = S$. Let I be an ideal of the Γ-semiring S not contained in P. Then by Proposition 2.22 $(\bar{I})^5 \subseteq I$. Suppose $(\bar{I})^5 \subseteq P$. Since P is a prime ideal of the semiring S, this implies that $(\bar{I}) \subseteq P$ i.e. $I \subseteq P$ which contradicts the fact that I is not contained in P. Hence $(\bar{I})^5 \not\subseteq P$. Since P is a right strongly prime ideal of the semiring S, then by Proposition 2.19 there exists a finite subset F of $(\bar{I})^5$ such that $Fx \subseteq P$ implies that $x \in P$ for all $x \in S$. Let $FFx \subseteq P$. Then $F(fx) \subseteq P$ for all $f \in F$ implies that $fx \in P$ for all $f \in F$ i.e. $Fx \subseteq P$. Again this implies that $x \in P$ for all $x \in S$. Since $F \subseteq (\bar{I})^5 \subseteq I$, by Proposition 2.16, P is a right strongly prime ideal of the Γ-semiring S where $\Gamma = S$.

Conversely, suppose that Q is a right strongly prime ideal of the Γ-semiring S where $\Gamma = S$. Then by Proposition 2.22, $(\bar{Q})^5 \subseteq Q$. Since $\bar{Q}S \subseteq \bar{Q}$, it follows that $(\bar{Q}S)^4\bar{Q} \subseteq Q$. Since every right strongly prime ideal of a Γ-semiring S is prime, so we have $\bar{Q} \subseteq Q$. Hence $Q = \bar{Q}$ i.e. Q is an ideal of the semiring S. Let J be an ideal of the semiring S which is not contained in Q. Then J is an ideal of the Γ-semiring S which is not contained in Q. By Proposition 2.15 there exist finite subsets F and G of J and S respectively such that $FGx \subseteq Q$ implies $x \in Q$ for all $x \in S$. Let $H = \{fg : f \in F,\ g \in G\}$. Clearly H is a finite subset of J, and $Hy \subseteq Q$ implies $y \in Q$ for all $y \in S$. Hence by Proposition 2.21, Q is a right strongly prime ideal of the semiring S. □

Definition 2.24. [4] *Let S be a Γ-semiring and F be the free additive commutative semigroup generated by $S \times \Gamma$. Then the relation ρ on F defined by $\sum_{i=1}^{m}(x_i, \alpha_i)\rho \sum_{j=1}^{n}(y_j, \beta_j)$ if and only if $\sum_{i=1}^{m} x_i\alpha_i a = \sum_{j=1}^{n} y_j\beta_j a$*

for all $a \in S (m, n \in Z^+$, the set of all positive integer), is a congruence on F. We denote the congruence class containing $\sum_{i=1}^{m}(x_i, \alpha_i)$ by $\sum_{i=1}^{m}[x_i, \alpha_i]$. Then F/ρ is an additive commutative semigroup. Now F/ρ forms a semiring with the multiplication defined by $(\sum_{i=1}^{m}[x_i, \alpha_i])(\sum_{j=1}^{n}[y_j, \beta_j]) = \sum_{i,j}[x_i\alpha_i y_j, \beta_j]$. We denotes this semiring by L and call it the left operator semiring of the Γ-semiring S.

Dually We define the right operator semiring R of the Γ-semiring S where $R = \{\sum_{i=1}^{m}[\alpha_i, x_i] : \alpha_i \in \Gamma, x_i \in S, i = 1, 2, ...m; m \in Z^+\}$ and the multiplication on R is defined as $(\sum_{i=1}^{m}[\alpha_i, x_i])(\sum_{j=1}^{n}[\beta_j, y_j]) = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$.

Proposition 2.25. [4] *Let S be a Γ-semirig with 0 and L (respectively R) be its left (respectively right) operator semiring. Then for any $\alpha \in \Gamma$, $[0, \alpha]$ (respectively $[\alpha, 0]$) is the zero of L (respectively R).*

Definition 2.26. [4] *Let S be a Γ-semiring and L (respectively R) be its left (respectively right) operator semiring. Then for $A \subseteq L, B \subseteq R, C \subseteq S$ we define $A^+ = \{x \in S : [x, \alpha] \in A$ for all $\alpha \in \Gamma\}$,$B^* = \{x \in S : [\alpha, x] \in B$ for all $\alpha \in \Gamma\}$, $C^{+'} = \{\sum_{i=1}^{n}[x_i, \alpha_i] \in L : \sum_{i=1}^{n} x_i\alpha_i s \in C$ for all $s \in S\}$ and $C^{*'} = \{\sum_{j=1}^{m}[\beta_j, y_j] \in R : \sum_{j=1}^{m} s\beta_j y_j \in C$ for all $s \in S\}$.*

Proposition 2.27. [4] *Let S be a Γ-semiring and L be its left operator semiring. Then:*
(i) if Q is an ideal of S then $Q^{+'}$ is an ideal of L;
(ii) if P is an ideal of L then P^+ is an ideal of S;
(iii) $(Q^{+'})^+ = Q$ and $(P^+)^{+'} = P$.

Theorem 2.28. *Let S be a Γ-semiring and L be its left operator semiring. Then S is a right strongly prime Γ-semiring if only if L is a right strongly prime semiring and $x\Gamma S = \{0\}$ implies that $x = 0$ for all $x \in S$.*

Proof. Let S be a right strongly prime Γ-semiring. Let I be a non zero ideal of the operator semiring L of S. Then by Proposition 2.27 I^+ is an ideal of S. Since I is a non zero ideal, there exits $\sum_{i=1}^{n}[x_i, \alpha_i] \in I$ such that $\sum_{i=1}^{n}[x_i, \alpha_i] \neq [0, \alpha]$ i.e. $\sum_{i=1}^{n} x_i\alpha_i s \neq 0$ for some $s \in S$. Now $\sum_{i=1}^{n}[x_i, \alpha_i][s, \gamma] \in I$ for all $\gamma \in \Gamma$ i.e. $[\sum_{i=1}^{n} x_i\alpha_i s, \gamma] \in I$ for all $\gamma \in \Gamma$ i.e.$0 \neq \sum_{i=1}^{n} x_i\alpha_i s \in I^+$. Therefore $I^+ \neq (0)$. Since S is a right strongly prime Γ-semiring, there exist finite subsets F and Φ of I^+ and Γ respectively such that $F\Phi x = (0)$ implies that $x = 0$ for all $x \in S$. Let $G = \{[x, \gamma] :$

$x \in F,\ \gamma \in \Phi\}$. Clearly, G is a finite subset of I as F and Φ are finite. Let $l = \sum_{i=1}^{n}[x_i, \alpha_i] \in L$ be such that $Gl = 0$ i.e. $[x,\ \gamma]\sum_{i=1}^{n}[x_i,\ \alpha_i] = [0,\ \alpha]$ for all $x \in F,\ \gamma \in \Phi$ and for some $\alpha \in \Gamma$ i.e. $x\gamma(\sum_{i=1}^{n} x_i\alpha_i s) = 0\alpha s = 0$ for all $x \in F,\ \gamma \in \Phi$, for all $s \in S$ and for some $\alpha \in \Gamma$ i.e. $F\Phi(\sum_{i=1}^{n} x_i\alpha_i s) = 0$ for all $s \in S$. Then by the above argument $\sum_{i=1}^{n} x_i\alpha_i s = 0$ for all $s \in S$ i.e. $l = \sum_{i=1}^{n}[x_i,\ \alpha_i] = [0,\ \alpha]$. Hence L is a right strongly prime semiring. Since S is a right strongly prime Γ-semiring, it is a prime Γ-semiring. Hence $x\Gamma S = 0$ implies that $x = 0$ for all $x \in S$.

Conversely, let L be a right strongly prime semiring and $x\Gamma S = 0$ implies that $x = 0$ for all $x \in S$. Let I be a non zero ideal of S. Then by Proposition 2.27, $I^{+'}$ is an ideal of L. Since I is non zero, there exists $x \in I$ such that $x \neq 0$. By hypothesis $x\Gamma S \neq 0$ i.e. $x\gamma s \neq 0$ for some $\gamma \in \Gamma$ and $s \in S$ i.e. $[x,\ \gamma] \neq [0,\ \alpha]$. Since I is an ideal and $x \in I$, $x\gamma s \in I$ for all $s \in S$. Hence $[x,\ \gamma] \in I^{+'}$ with $[x,\ \gamma] \neq [0,\ \alpha]$ for some $\alpha \in \Gamma$. Therefore $I^{+'} \neq ([0,\ \alpha])$. Then by Theorem 2.18 there exists a finite subset F of $I^{+'}$ such that $Fl = [0,\ \alpha]$ implies that $l = [0,\ \alpha]$ for all $l \in L$. Let $F = \{f_1, f_2, f_3, ..., f_m\}$ where each f_j is of the form $\sum_{i=1}^{n}[x_i,\ \alpha_i]$. Set $G = \{f_i f_j : f_i,\ f_j \in F\}$. Then G is a finite subset of $I^{+'}$. Let $Gl = 0$. Then $F(fl) = \{0\}$ for all $f \in F$ which implies that $fl = 0$ for all $f \in F$ and for all $l \in L$ i.e. $Fl = 0$. Again this implies that $l = 0$ for all $l \in L$. Also if $f_k = \sum_{i=1}^{n}[x_i,\ \alpha_i]$ and $f_t = \sum_{j=1}^{p}[x_j,\ \alpha_j] \in F$, then $f_k f_t = \sum_{i=1}^{n}[x_i,\ \alpha_i]\sum_{j=1}^{p}[x_j,\ \alpha_j] = \sum_{j=1}^{p}[\sum_{i=1}^{n} x_i\alpha_i x_j,\ \alpha_j]$. Since $\sum_{i=1}^{n}[x_i,\ \alpha_i] \in F \subseteq I^{+'}$, $\sum_{i=1}^{n} x_i\alpha_i x_j \in I$. It follows that $G \subseteq [I,\ \Gamma]$. Suppose that $G = \{g_1, g_2, ..., g_n\}$ and $g_j = \sum_{i=1}^{k}[x_{ij},\ \gamma_{ij}]$ for all $j = 1, 2, ..., n$ where all $x_{ij} \in I$. Let H and Φ denote the sets of all the x_{ij}'s and γ_{ij}'s, respectively. Then H and Φ both are finite subsets of I and Γ respectively. Let $y \in S$ be such that $H\Phi y = 0$. Then $x_{ij}\gamma_{ij}y = 0$ for all $x_{ij} \in H,\ \gamma_{ij} \in \Phi$ i.e. $x_{ij}\gamma_{ij}y\gamma s = 0\alpha s$, for all $x_{ij} \in H;\ \gamma_{ij} \in \Phi,\ \alpha,\ \gamma \in \Gamma$ and for all $s \in S$ i.e. $\sum_{i=1}^{n}[x_{ij},\ \gamma_{ij}][y,\ \gamma] = [0,\ \alpha]$ for all j and $\alpha,\ \gamma \in \Gamma$ i.e. $G[y,\ \gamma] = [0,\ \alpha]$, for all $\alpha,\ \gamma \in \Gamma$. It follows from above $[y,\ \gamma] = [0,\ \alpha]$ for all $\gamma,\ \alpha \in \Gamma$ i.e. $y\gamma s = 0\alpha s = 0$ for all $\gamma \in \Gamma$ and for all $s \in S$ i.e. $y\Gamma S = 0$. Then by hypothesis $y = 0$, and hence S is right strongly prime. □

Definition 2.29. [4] *Let S be a Γ-semiring and L be the left operator semiring and R be the right operator semiring. If there exists an element $\sum_{i=1}^{m}[e_i, \delta_i] \in L$ (respectively $\sum_{j=1}^{n}[\nu_j, f_j] \in R$) such that $\sum_{i=1}^{m} e_i\delta_i a = a$(respectively $\sum_{j=1}^{n} a\nu_j f_j = a$) for all $a \in S$ then S is said to have the left*

unity $\sum_{i=1}^{m}[e_i, \delta_i]$ (*respectively the right unity* $\sum_{j=1}^{n}[\nu_j, f_j]$).

Corollary 2.30. *Let* S *be a* Γ*-semiring with left unity* $\sum_{i=1}^{m}[e_i, \delta_i]$ *and right unity* $\sum_{j=1}^{n}[\nu_j, f_j]$ *and* L *be its left operator semiring. Then* S *is right strongly prime if and only if* L *is right strongly prime.*

Proof. The Proof follows from Theorem 2.28. □

Proposition 2.31. [4] *Let* S *be a* Γ*-semiring and* R *be its right operator semiring. Then:*

(*i*) *if* Q *is an ideal of* S *then* $Q^{*'}$ *is an ideal of* R*;*

(*ii*) *if* P *is an ideal of* R *then* P^{*} *is an ideal of* S*;*

(*iii*) $(Q^{*'})^{*} = Q$ *and* $(P^{*})^{*'} = P$.

Theorem 2.32. *Let* S *be a* Γ*-semiring with left and right unities and* R *be its right operator semiring. Then* S *is right strongly prime if and only if* R *is right strongly prime.*

Proof. Let S be a Γ-semiring with left unity $\sum_{i=1}^{m}[e_i, \delta_i]$ and right unity $\sum_{j=1}^{n}[\nu_j, f_j]$. Suppose that S is a right strongly prime Γ-semiring. Let I be a non zero ideal of the right operator semiring R. Then by Proposition 2.31, I^{*} is an ideal of S. Also we have $I^{*} \neq 0$. Then there exist finite subsets F and Φ of I^{*} and Γ respectively such that $F\Phi x = 0$ implies that $x = 0$ for all $x \in S$. Let $G = \{[\delta_i, x\gamma f_j] : 1 \leq i \leq m,\ 1 \leq j \leq n,\ x \in F,\ \gamma \in \Phi\}$. Then G is finite. Since I^{*} is an ideal of S, $x\gamma f_j \in I^{*}$ for all j and for all $x \in F,\ \gamma \in \Phi$. Hence $s\delta_i x\gamma f_j \in I^{*}$ for all $i,\ j$ and for all $x \in F,\ \gamma \in \Phi,\ s \in S$. It follows that $[\delta_i, x\gamma f_j] \in I$, for all $i,\ j$ and for all $x \in F,\ \gamma \in \Phi$. Hence G is a finite subset of I. Let $r = \sum_{k=1}^{p}[\beta_k, y_k] \in R$ be such that $Gr = 0$. Then $[\delta_i,\ x\gamma f_j]\sum_{k=1}^{p}[\beta_k, y_k] = [\alpha,\ 0]$ for all $i,\ j$ and for all $x \in F$, for all $\gamma \in \Phi$ i.e. $\sum_k s\delta_i x\gamma f_j\beta_k y_k = s\alpha 0 = 0$ for all $i,\ j$ and for all $x \in F$, for all $s \in S$; $\gamma \in \Phi$ i.e. $(\sum_{i=1}^{m} e_i\delta_i x)\ \gamma\ (\sum_{k=1}^{p} f_j\beta_k y_k) = 0$ for all j and for all $x \in F,\ \gamma \in \Phi$ i.e. $x\gamma(\sum_{k=1}^{p} f_j\beta_k y_k) = 0$ for all j and for all $x \in F,\ \gamma \in \Phi$ as S has left unity $\sum_{i=1}^{m}[e_i,\ \delta_i]$ i.e. $F\Phi(\sum_{k=1}^{p} f_j\beta_k y_k) = 0$ for all j. Since S is right strongly prime, $\sum_{k=1}^{p} f_j\beta_k y_k = 0$ for all j i.e. $\sum_{j,k} s\nu_j f_j\beta_k y_k = 0$ for all $s \in S$. Since $\sum_{j=1}^{n}[\nu_j,\ f_j]$ is right unity of S, we get $\sum_{k=1}^{p} s\beta_k y_k = 0$ for all $s \in S$ i.e. $r = \sum_{k=1}^{p}[\beta_k,\ y_k] = [\alpha,\ 0]$. Hence R is right strongly prime semiring.

Conversely, suppose R is a right strongly prime semiring. Let J be a non zero ideal of S. Then by Proposition 2.31, $J^{*'}$ is an ideal of R, which is non zero. Then there exists a finite subset H of $J^{*'}$ such that $Hr = 0$ implies that $r = 0$ for all $r \in R$. Let $H = \{h_1, h_2, ..., h_r\}$. Since S has right unity

$\sum_{j=1}^{n}[\nu_j, f_j]$, $h_t = \sum_{j=1}^{n}[\nu_j, f_j]\sum_{k=1}^{p}[\beta_k, y_k] = \sum_{j=1}^{n}[\nu_j,\ \sum_{k=1}^{p} f_j\beta_k y_k]$. Let $x_{jk} = \sum_{k=1}^{p} f_j\beta_k y_k$. Then $x_{jk} \in J$ as $h_t = \sum_{k=1}^{p}[\beta_k, y_k] \in J^{*'}$for all $j,\ k$. Hence every h_t can be expressed as $\sum_{k=1}^{p}[\nu_j,\ x_{jk}]$. Let G be the set of all x_{jk}' s and $\Phi = \{\delta_1, \delta_2, ..., \delta_m\}$. Then G and Φ are finite subsets of J and Γ respectively. Let $x \in S$ be such that $G\Phi x = 0$. Then $x_{jk}\delta_i x = 0$ for all $i,\ j,\ k$. Hence $\sum_{j=1}^{n}[\nu_j,\ x_{jk}][\delta_i,\ x] = \sum_{j=1}^{n}[\nu_j,\ x_{jk}\delta_i x] = [\alpha,\ 0]$ for all $i,\ k$. It follows that $[\delta_i,\ x] = [\alpha,\ 0]$ for all i i.e. $s\delta_i x = 0$ for all i and for all $s \in S$ i.e. $x = \sum_{i=1}^{m} e_i\delta_i x = 0$ as $\sum_{i=1}^{m}[e_i,\ \delta_i]$ is the left unity of S. Hence, S is a right strongly prime Γ-semiring. □

Proposition 2.33. [9] *Let S be a Γ-semiring with left and right unities and L be its left operator semiring and Q be a proper ideal of S. Then $L_{S/Q}$ and $L/Q^{+'}$ are isomorphic via the mapping $\sum_{i=1}^{m}[x_i/Q, \alpha_i] \longrightarrow \sum_{i=1}^{m}[x_i, \alpha_i]/Q^{+'}$, where $L_{S/Q}$ is the left operator semiring of the Bourne factor Γ- semiring S/Q.*

Theorem 2.34. *Let S be a Γ-semiring with left and right unities and L be its left operator semiring. Then the mapping $P \to P^{+}$ defines a one-to-one correspondence between the sets of right strongly prime k-ideals of L and S.*

Proof. Suppose P is a right strongly prime k-ideal of the semiring L. Then by Proposition 2.27, P is an ideal of L and $P = (P^{+})^{+'}$. It is also easy to proof P is a k-ideal of the Γ-semiring S. Let $L^{'}$ denote the left operator semiring of the factor Γ-semiring S/P^{+}. By Proposition 2.33, $L^{'}$ and $L/(P^{+})^{+'}$ i.e. $L^{'}$ and L/P are isomorphic. Again by Proposition 2.18 L/P is right strongly prime semiring and hence so is $L^{'}$. Thus by Corollary 2.30 the factor Γ-semiring S/P^{+} is strongly prime. Hence P^{+} is a right strongly prime ideal of S.

Conversely, suppose Q is a right strongly prime k-ideal of Γ-semiring S. Then the factor Γ-semiring S/Q is right strongly prime. By Corollary 2.30, the left operator semiring of S/Q i.e. $L_{S/Q}$ is right strongly prime. Again by Proposition 2.33, $L_{S/Q}$ and $L/Q^{+'}$ are isomorphic. It follows that $L/Q^{+'}$ is a right strongly prime semiring. Hence, by Proposition 2.18, $Q^{+'}$ is a right strongly prime k-ideal of L. □

Proposition 2.35. *Let S be a Γ-semiring with left and right unities and R be its right operator semiring. Then the mapping $Q \to Q^{*}$ defines a one-to-one correspondence between the sets of strongly prime k-ideals of R and S.*

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Received: 6.XI.2007

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