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STABILITY AND BOUNDEDNESS OF SOLUTIONS OF SOME NON-AUTONOMOUS DELAY DIFFERENTIAL EQUATION OF THE THIRD ORDER

BY

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Abstract. This paper gives sufficient conditions for the asymptotic stability and uniform boundedness of some non-autonomous nonlinear third order delay differential equations. By constructing Lyapunov functionals, we obtained results in a very simple form.

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Key words: stability, boundedness, Lyapunov functional, delay differential equations, third-order differential equations.

1. Introduction and statement of results. In this paper, we consider the third order non-autonomous nonlinear differential equations with delays:

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}) + c(t)h(x(t-r)) = 0 \tag{1.1}$$

and

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}) + c(t)h(x(t-r)) = p(t) \tag{1.2}$$

where r is a positive constant, $a(t), b(t), c(t), g(\dot{x}), h(x)$ are real valued functions continuous in their respective arguments; $g(0) = h(0) = 0$. The dots indicate differential with respect to t and all solutions are assumed real.

Equations of the forms (1.1) and (1.2) in which $a(t), b(t)$ and $c(t)$ are constants have been studied by several authors namely, Sadek [3] and Zhu [7], to mention a few. They obtained criteria which ensure the stability, uniform boundedness and uniform ultimate boundedness of solutions.

Recently, in [4], SADEK establishes conditions under which all solutions of the non-autonomous equation

$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)h(x(t-r)) = 0$$

tend to the zero solution as $t \to \infty$. This result is now extended to equation (1.1) by considering the semi-invariant set of a related non-autonomous system. Using the same technique, boundedness conditions are then obtained for (1.2). The main objective of this paper is to prove the following:

Theorem 1. *Suppose that* $a(t), b(t), c(t) \in C'(I), h \in C'(\mathbb{R})$ *and* $g \in C(\mathbb{R})$ *and that these functions satisfy the following conditions:*

(*i*) $h(0) = 0, \frac{h(x)}{x} \geq \delta_0 > 0 \;\; x \neq 0,$

(*ii*) $h'(x) \leq c$

(*iii*) $g(0) = 0, \frac{g(y)}{y} \geq b > 0 \;\;\; y \neq 0,$

(*iv*) $0 < \delta_1 \leq c(t) \leq b(t), -L \leq b'(t) \leq c'(t) \leq 0 \quad t \in I,$

(*v*) $0 < a \leq a(t) \leq L, \quad t \in I$. *Then every solution* $x \equiv x(t)$ *of* (1.1) *is uniform-bounded and satisfies* $x(t) \to 0, \dot{x}(t) \to 0, \ddot{x}(t) \to 0$ *as* $t \to \infty$ *provided there exists* α *satisfying* $\frac{b}{c} > \alpha > \frac{1}{a}$ *such that*

(*vi*) $\frac{1}{2}a'(t) \leq \delta_2 < \delta_1[b - \alpha c] \;\; t \in I$ *and* $r < \min[\frac{2\delta_5}{3LC}, \frac{\delta_6}{L\alpha c}]$.

Remark. If (1.1) is the constant coefficient delay differential equation $\dddot{x} + a\ddot{x} + b\dot{x} + cx(t-r) = 0$, then conditions (i) - (vi) reduce to the Routh-Hurwitz conditions $a > 0, c > 0$ and $ab > c$. To show this we set $a(t) \equiv a, b(t) \equiv c(t) \equiv 1, \;\; g(\dot{x}) = b\dot{x}$ and $h(x(t-r)) = cx(t-r)$.

Theorem 2. *If all the conditions of Theorem* 1 *are satisfied then all solutions of the perturbed equation* (1.2) *are bounded provided*

$$\int_0^t |p(s)|ds < \infty,$$

for all $t \geq 0$.

Now, we will give the stability criteria for the general non-autonomous delay differential system. We consider:

$$\dot{x} = f(t, x_t), x_t = x(t+\theta) \quad -r \leq \theta \leq 0, t \geq 0, \tag{1.3}$$

where $f : I \times C_H \to I\!R^n$ is a continuous mapping,

$$f(t,0) = 0, C_H := \{\phi \in (C[-r,0], I\!R^n) : \|\phi\| \leq H\}$$

and for $H_1 \leq H$, there exists $L(H_1) > 0$, with

$$|f(\phi)| \leq L(H_1) \text{ when } \|\phi\| \leq H_1.$$

Definition 1 ([3]). *An element $\psi \in C$ is in the ω-limit set of ϕ, say, $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where*

$$x_{t_n}(\phi) = x(t_n + \theta, 0, \phi) \text{ for } -r \leq \theta \leq 0.$$

Definition 2 ([3,7]). *A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of* (1.3)*, $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.*

Lemma 1 ([3,7]). *If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of* (1.3) *with $x_0(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \text{ as } t \to \infty.$$

Lemma 2 ([2,5]). *Let $V(t,\phi)) : I \times C_H \to I\!R$ be a continuous functional satisfying a local Lipschitz condition. $V(t,0) = 0$, and such that:*

(*i*) $W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(\|\phi\|)$ *where $W_1(r), W_2(r)$ are wedges*

(*ii*) $\dot{V}_{(1.3)}(t,\phi) \leq 0$, *for $\phi \leq C_H$.*

Then the zero solution of (1.3) is uniformly stable. If we define $Z = \{\phi \in C_H : V_{(1.3)}(t,\phi) = 0\}$, then the zero solution of (1.3) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

2. Proof of Theorem 1. We write the Eq. (1.1) as the following equivalent system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -a(t)z - b(t)g(y) - c(t)h(x) + c(t)\int_{t-r}^{t} h'(x(s))y(s)ds \end{aligned} \tag{2.1}$$

Define its Lyapunov functional as:

$$\begin{aligned} 2V(t, x_t, y_t, z_t) &= 2c(t)H(x) + 2\alpha b(t)G(y) + 2\alpha c(t)h(x)y \\ &\quad + a(t)y^2 + \alpha z^2 + 2yz + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \end{aligned} \tag{2.2}$$

where $H(x) \equiv \int_0^x h(s)ds$, $G(y) \equiv \int_0^y g(s)ds$ and λ is a positive constant which will be determined later. This Lyapunov functional derives from Lyapunov functions used by Omeike [2] and Swick [4] when they studied certain autonomous and non-uatonomous differential systems, respectively.

From (iv) it follows that $b(t)$ and $c(t)$ are non-decreasing functions on $[0, \infty)$. Thus, since they are continuous on this interval and bounded below by $\delta_1 > 0$, they are bounded on $[0, \infty)$ and the limit of each exists as $t \to \infty$. Since L in (iv) and (v) is an arbitrary selected bound, we can also assume that:

$$\begin{aligned} &0 < \delta_1 \le c(t) \le b(t) \le L, \\ &\lim_{t\to\infty} c(t) = c_0, \lim_{t\to\infty} b(t) = b_0, \\ &\delta_1 \le c_0 \le b_0 \le L. \end{aligned} \tag{2.3}$$

Due to (2.1), we rewrite V as

(2.4)

$$\begin{aligned} V &= c(t)\left[H(x) + \alpha\frac{b(t)}{c(t)}G(y) + \alpha h(x)y\right] + \frac{1}{2}\left[a(t)y^2 + yz + \alpha z^2\right] \\ &+ \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \equiv V_1 + \frac{1}{2}V_2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds. \end{aligned}$$

First, consider

$$\begin{aligned} V_2 &= a(t)y^2 + yz + \alpha z^2 \\ &= a(t)\left(y + \frac{z}{2a(t)}\right)^2 + \frac{1}{4a(t)}(4\alpha a(t) - 1)z^2. \end{aligned}$$

By (v), $\alpha a(t) \geq \alpha a > 1$ since $\alpha > \frac{1}{a}$. Clearly, $4\alpha a(t) - 1$ is positive. Thus, there is a $\delta_3 > 0$ such that

$$V_2 \geq \frac{1}{2}\delta_3 y^2 + \frac{1}{2}\delta_3 z^2. \tag{2.5}$$

$$V_1 = c(t)\left[H(x) + \alpha\frac{b(t)}{c(t)}G(y) + \alpha h(x)y\right] \geq \delta_1\left[H(x) + \frac{\alpha}{2}by^2 + \alpha h(x)y\right],$$

since $\frac{b(t)}{c(t)} \geq 1, c(t) \geq \delta_1 > 0$ and $\frac{g(y)}{y} \geq b > 0$ implies that $G(y) \geq \frac{1}{2}by^2$. The quantity in the brackets above can be written thus,

(2.6)

$$\begin{aligned}
&\frac{1}{2}\left[2H(x) + \alpha by^2 + 2\alpha h(x)y\right] = \frac{1}{2}\left[\frac{\alpha}{b}(by + h(x))^2 + 2H(x) - \frac{\alpha}{b}h^2(x)\right] \\
&= \frac{1}{2}\frac{\alpha}{b}(by + h(x))^2 + \int_0^x \left[1 - \frac{\alpha h'(s)}{b}\right] h(s)ds \\
&\geq \int_0^x \left[1 - \frac{\alpha c}{b}\right] h(s)ds \geq \delta_4 \int_0^x h(s)ds = \delta_4 H(x)
\end{aligned}$$

where $\delta_4 \equiv 1 - \frac{\alpha c}{b} > 1 - \frac{(b/c)\cdot c}{b} = 0$. Thus, since $\frac{h(x)}{x} \geq \delta_0 > 0$, we have

$$V_1 \geq \frac{\delta_0\delta_1\delta_4}{2}\, x^2. \tag{2.7}$$

Combining (2.4), (2.5) and (2.7), we have

$$V(t, x_t, y_t, z_t) \geq \frac{\delta_0\delta_1\delta_4}{2}x^2 + \frac{\delta_3}{4}y^2 + \frac{\delta_3}{4}z^2 + \lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)d\theta ds$$

Hence we can easily check that $V(t, x_t, y_t, z_t)$ satisfies condition (i) of Lemma 2.

From (2.2) and (2.1), we obtain

$$\begin{aligned}
&\frac{d}{dt}V(t, x_t, y_t, z_t) = c'(t)H(x) + \alpha b'(t)G(y) + \alpha c'(t)h(x)y \\
&+\frac{1}{2}a'(t)y^2 - [b(t)g(y)y - \alpha c(t)h'(x)y^2 - \lambda ry^2] - [\alpha a(t) - 1]z^2 \\
&+c(t)y\int_{t-r}^t h'(x(s))y(s)ds \\
&+\alpha c(t)z\int_{t-r}^t h'(x(s))y(s)ds - \lambda\int_{t-r}^t y^2(s)ds.
\end{aligned} \tag{2.8}$$

Since $h'(x) \leq c$ and using $2uv \leq u^2 + v^2$, we have

$$\begin{aligned}\alpha c(t)z\int_{t-r}^{t} h'(x(s))y(s)ds &\leq \frac{1}{2}\alpha c(t)crz^2 + \frac{1}{2}cc(t)\int_{t-r}^{t} y^2(s)ds \\ &\leq \frac{1}{2}L\alpha crz^2 + \frac{1}{2}Lc\int_{t-r}^{t} y^2(s)ds\end{aligned}$$

and

$$c(t)y\int_{t-r}^{t} h'(x(s))y(s)ds \leq cc(t)\int_{t-r}^{t} y^2(s)ds \leq Lc\int_{t-r}^{t} y^2(s)ds.$$

Thus,

$$\begin{aligned}&\frac{d}{dt}V(t, x_t, y_t, z_t) \leq c'(t)H(x) + \alpha b'(t)G(y) + \alpha c'(t)h(x)y \\ &+\frac{1}{2}a'(t)y^2 - [b(t)g(y)y - \alpha c(t)h'(x)y^2 - \lambda ry^2] \\ &-\frac{1}{2}[2(\alpha a(t) - 1 - L\alpha rc]z^2 + \left[\frac{3}{2}Lc - \lambda\right]\int_{t-r}^{t} y^2(s)ds.\end{aligned} \tag{2.9}$$

If $y = 0$, then $b(t)g(y)y - \alpha c(t)h'(x)y^2 - \lambda ry^2 = 0$. If $y \neq 0$, we can rewrite the term as

$$\begin{aligned}[b(t)g(y)y - \alpha c(t)h'(x)y^2 - \lambda ry^2] &= \left[b(t)\frac{g(y)}{y} - \alpha c(t)h'(x) - \lambda r\right]y^2 \\ &\geq [bb(t) - \alpha cc(t) - \lambda r]y^2 \\ &= c(t)\left[\frac{b(t)}{c(t)}b - \alpha c\right]y^2 - \lambda ry^2 \\ &\geq \delta_1\left[b - \alpha c - \delta_1^{-1}\lambda r\right]y^2.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}a'(t)y^2 &- [b(t)g(y)y - \alpha c(t)h'(x)y^2 - \lambda ry^2] \\ &\leq \left[\frac{1}{2}a'(t) - \delta_1(b - \alpha c - \delta_1^{-1}\lambda r)\right]y^2 \\ &\leq \{\delta_2 - \delta_1(b - \alpha c) + \lambda r\}y^2 \\ &\leq -\{\delta_5 - \lambda r\}y^2\end{aligned} \tag{2.10}$$

where $\delta_5 \equiv \delta_1[b - \alpha c] - \delta_2 > 0$ by (vi).

According to (v), $\alpha a(t) \geq \alpha a > 1$, thus

$$[2(\alpha a(t) - 1) - L\alpha rc]z^2 \geq (\delta_6 - L\alpha rc)z^2, \tag{2.11}$$

where $\delta_6 \equiv \alpha a - 1 > 0$.

Substituting (2.10) and (2.11) into (2.9) and also choose $\lambda = \frac{3}{2}Lc$, we have that

$$\begin{aligned} \frac{d}{dt}V(t, x_t, y_t, z_t) \quad &\leq \quad c'(t)H(x) + \alpha b'(t)G(y) + \alpha c'(t)h(x)y \\ &\quad -(\delta_5 - \frac{3}{2}Lcr)y^2 - (\delta_6 - L\alpha rc)z^2. \end{aligned} \tag{2.12}$$

Next, we show that

$$c'(t)H(x) + \alpha b'(t)G(y) + \alpha c'(t)h(x)y \leq 0$$

for all x, y and $t \geq 0$.

From (iv), $-L \leq b'(t) \leq c'(t) \leq 0$ for $t \geq 0$. If $c'(t) = 0$ then

$$c'(t)H(x) + \alpha b'(t)G(y) + \alpha c'(t)h(x)y = \alpha b'(t)G(y) \leq 0$$

since $b'(t) \leq 0$ and $G(y) \geq 0$. For those $t's$ such that $c'(t) < 0$, we have

$$\begin{aligned} &c'(t)H(x) + \alpha b'(t)G(y) + \alpha c'(t)h(x)y \\ &= c'(t)\left[H(x) + \alpha\frac{b'(t)}{c'(t)}G(y) + \alpha h(x)y\right] \\ &\leq c'(t)[H(x) + \alpha G(y) + \alpha h(x)y] \\ &\leq c'(t)\{\delta_4 H(x)\} \\ &\leq 0 \end{aligned}$$

for all x, y, z and t, since, in (2.6) it was shown that

$$H(x) + \alpha G(y) + \alpha h(x)y \geq \delta_4 H(x) \geq 0.$$

Thus,

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq -(\delta_5 - \frac{3}{2}Lcr)y^2 - (\delta_6 - L\alpha cr)z^2.$$

Therefore, if

$$r < \min\left[\frac{2\delta_5}{3Lc}, \frac{\delta_6}{L\alpha c}\right],$$

we have

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \le -\beta(y^2 + z^2), \quad \text{for some } \beta > 0.$$

By $\frac{d}{dt}V(t, x_t, y_t, z_t) = 0$ and system (2.1), we can easily obtain: $x = y = z = 0$. Thus, the conditions of Lemma 2 are satisfied. Therefore the proof of Theorem 1 is now complete. □

3. Proof of Theorem 2. The proof of Theorem 2 will depend upon the same scalar valued function $V(t, x_t, y_t, z_t)$ as used in the proof of Theorem 1. It was shown in the proof of Theorem 1 that there is a positive constant δ_7 such that $\delta_7\|X\|^2 \le V(t, x_t, y_t, z_t)$, where $X = (x, y, z)$ and thus that $V(t, x_t, y_t, z_t) \to \infty$ as $x^2 + y^2 + z^2 \to \infty$.

The proof that all solutions of (1.2) are bounded is based on the method in ([1], [6]). If it can be shown that there exists a constant $K > 0$ such that $V \le K$ for all t, x, y and z their since $V(t, x_t, y_t, z_t) \to \infty$ as $x^2 + y^2 + z^2 \to \infty$, thus there exists a $D > 0$ such that if $x \equiv x(t)$ is a solution of (1.2) then $|x(t)| \le D$, $|\dot{x}(t)| \le D$, $|\ddot{x}(t)| \le D$ for all $t \ge 0$. Equation (1.2) is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -a(t)z - b(t)g(y) - c(t)h(x) \\ &\quad +c(t)\int_{t-1}^{t} h'(x(s)y(s)ds + p(t) \end{aligned} \tag{3.1}$$

Along any solution $(x(t), y(t), z(t))$ we have

$$\dot{V}_{(3.1)}(t, x_t, y_t, z_t) = \dot{V}_{(2.1)}(t, x_t, y_t, z_t) + (y + \alpha z)p(t).$$

Since $\dot{V}_{(2.1)} \le 0$ for all t, x, y, z, thus

$$\dot{V}_{(3.1)} \le (y + \alpha z)p(t) \le (|y| + \alpha|z|)|p(t)| \le \delta_8(|y| + |z|)|p(t)|$$

where $\delta_8 \equiv \max\{1, \alpha\}$. Noting that $|x| < 1 + x^2$ we get

$$\begin{aligned} \dot{V}_{(3.1)} &\le \delta_8(2 + y^2 + z^2)|p(t)| \\ &\le 2\delta_8|p(t)| + \delta_8\|X\|^2|p(t)| \\ &\le 2\delta_8|p(t)| + (\delta_8/\delta_7)V(t, x_t, y_t, z_t)|p(t)| \end{aligned}$$

recalling that $\delta_7\|X\|^2 \le V(t, x_t, y_t, z_t)$.

Let $\eta = \max\left(2\delta_8, \frac{\delta_8}{\delta_7}\right)$, then

$$\dot{V}_{(3.1)} \leq \eta|p(t)| + \eta V|p(t)|$$

or

$$\dot{V}_{(3.1)} - \eta V|p(t)| \leq \eta|p(t)|.$$

Multiplying each side of this inequality by the integrating factor $\exp\left(-\eta\int_0^t |p(s)|ds\right)$, we get

$$\dot{V}_{(3.1)} \exp\left(-\eta\int_0^t |p(s)|ds\right) - \eta V|p(t)|\exp\left(-\eta\int_0^t |p(s)|ds\right),$$

we get

$$\dot{V}_{(3.1)} \exp\left(-\eta\int_0^t |p(s)|ds\right) - \eta V|p(t)|\exp\left(-\eta\int_0^t |p(s)|ds\right) \leq \eta|p(t)|\exp\left(-\eta\int_0^t |p(s)|ds\right).$$

Integrating each side of this inequality from 0 to t, we get, where $0 = (x(0), y(0), z(0))$,

$$V\exp\left(-\eta\int_0^t |p(s)|ds\right) - V(0) \leq 1 - \exp\left(-\eta\int_0^t |p(s)|ds\right)$$

or

$$V \leq V(0)\exp\left(\eta\int_0^t |p(s)|ds\right) + \exp\left(\eta\int_0^t |p(s)|ds\right) - 1.$$

Since $\int_0^t |p(s)|ds \leq A$ for all t, this implies

$$V(t, x_t, y_t, z_t) \leq V(0)e^{\eta A} + [e^{\eta A} - 1] \text{ for } t \geq 0.$$

Now, since the right-hand side is a constant, and since $V(t, x_t, y_t, z_t) \to \infty$ as $x^2 + y^2 + z^2 \to \infty$, it follows that there exists a $D > 0$ such that

$$|x(t)| \leq D, |y(t)| \leq D, |z(t)| \leq D \text{ for } t \geq 0.$$

From the system (3.1) this implies that

$$|x(t)| \leq D, |\dot{x}(t)| \leq D, |\ddot{x}(t)| \leq D \text{ for } t \geq 0.$$

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