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ON A CERTAIN FIRST ORDER FREDHOLM INTEGRODIFFERENTIAL EQUATION

BY

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Abstract. The aim of this paper is to study the existence, uniqueness and other properties of solutions of a certain first order Fredholm integrodifferential equation. The main tools employed in the analysis are based on the representation formula, application of the Banach fixed point theorem and a certain integral inequality which provides explicit bound on the unknown function.

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1. Introduction. Consider the first order Fredholm integrodifferential system of the form

$$x'(t) = A(t)\,x(t) + f\left(t, x(t), \int_0^a k(t,\sigma,x(\sigma))\,d\sigma\right),\ x(0) = x_0, \tag{1.1}$$

as a perturbation of the linear system

$$y'(t) = A(t)\,y(t)\,,\ y(0) = x_0, \tag{1.2}$$

for $t \in [0,a]$ where x, y, f are real n-vectors; A is $n \times n$ matrix. Let R^n denotes the real n-dimensional Euclidean space of colomn vectors. The symbol $|.|$ will denote the norm in R^n or $n \times n$ matrix norm depending on whether is applied to vector or matrix. Let $J = [0,a]$, $R_+ = [0,\infty)$ be the given subsets of R, the set of real numbers. Throughout, we assume that $f \in C(J \times R^n \times R^n, R^n)$, $k \in C\left(J^2 \times R^n, R^n\right)$ and $A(t)$ is a continuous $n \times n$ matrix on R_+.

The study of Fredholm integrodifferential and integral equations are of great interest, since they arise in a variety of applications. Many authors have studied the various types of Fredholm integrodifferential and integral equations with different view points, see [1,3-8,11,12] and the references given therein. In this paper we are concerned with the existence, uniqueness and other properties of solutions of equation (1.1) under various assumptions on the functions involved therein and the solutions of equation (1.2). The representation formula, the well known Banach fixed point theorem (see [4, p.37]) coupled with the Bielecki type norm [2] and the integral inequality with explicit estimate given in [10, p.47] are used to establish the results.

2. Existence and uniqueness. Let S be the space of those functions $z(t) \in R^n$ which are continuous for $t \in R_+$ and fulfil the condition

$$|z(t)| = O(\exp(\lambda t)), \tag{2.1}$$

where $\lambda > 0$ is a constant. In the space S we define the norm (see [2,9])

$$|z|_S = \sup_{t \in R_+} [|z(t)| \exp(-\lambda t)]. \tag{2.2}$$

It is easy to see that S with norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a constant $N \geq 0$ such that $|z(t)| \leq N \exp(\lambda t)$ for $t \in R_+$. Using this fact in (2.2) we observe that

$$|z|_S \leq N. \tag{2.3}$$

Our approach and arguments led us to make use of the variation of constants formula, namely, any solution $x(t)$ of equation (1.1) considered as a perturbation of equation (1.2) can be represented by the equivalent integral equation (see [12])

$$\begin{aligned} x(t) &= Y(t) Y^{-1}(0) x_0 \\ &\quad + \int_0^t Y(t) Y^{-1}(s) f\left(s, x(s), \int_0^a k(s, \sigma, x(\sigma)) d\sigma\right) ds, \end{aligned} \tag{2.4}$$

where $Y(t)$ is the fundamental solution matrix of equation (1.2) such that $Y(0) = I$, the identity matrix.

We need the following special version of the inequality given by PACHPATTE in [10, p.47], see also [11].

Lemma. *Let* $u(t), p(t), q(t) \in C(J, R_+)$, $c \geq 0$ *be a real constant and*

$$u(t) \leq c + \int_0^t p(s) \left[u(s) + \int_0^a q(\sigma) u(\sigma) d\sigma \right] ds, \tag{2.5}$$

for $t \in J$. *If*

$$d = \int_0^a q(\sigma) \exp\left(\int_0^\sigma p(\tau) d\tau \right) d\sigma < 1, \tag{2.6}$$

then

$$u(t) \leq \frac{c}{1-d} \exp\left(\int_0^t p(s) ds \right), \tag{2.7}$$

for $t \in J$.

Now we shall prove the following main result of this section.

Theorem 1. *Let* $Y(t)$ *be the fundamental solution matrix of equation* (1.2) *such that*

$$\left| Y(t) Y^{-1}(s) \right| \leq M, \tag{2.8}$$

for $s \leq t$; $s, t \in J$, *where* M *is a positive constant. Suppose that*

(i) *the functions* f, k *in equation* (1.1) *satisfy the conditions*

$$|f(t, x, u) - f(t, \bar{x}, \bar{u})| \leq g(t) [|x - \bar{x}| + |u - \bar{u}|], \tag{2.9}$$

$$|k(t, \sigma, x) - k(t, \sigma, \bar{x})| \leq h(t, \sigma) |x - \bar{x}|, \tag{2.10}$$

where $g \in C(J, R_+)$, $h \in C(J^2, R_+)$,

(ii) *for* λ *as in* (2.1)

(a) *there exists a nonnegative constant* α *such that* $\alpha < 1$ *and*

$$\int_0^t M g(s) \left[\exp(\lambda s) + \int_0^a h(s, \sigma) \exp(\lambda \sigma) d\sigma \right] ds \leq \alpha \exp(\lambda t), \tag{2.11}$$

(b) there exists a nonnegative constant β such that

$$M|x_0| + \int_0^t M\left|f\left(s, 0, \int_0^a k(s,\sigma,0)\,d\sigma\right)\right|ds \le \beta \exp(\lambda t), \tag{2.12}$$

for $t \in J$. Then the equation (1.1) *has a unique solution on J.*

Proof. Let $x(t) \in S$ and define the operator

$$\begin{aligned}(Tx)(t) &= Y(t)Y^{-1}(0)x_0 \\ &+ \int_0^t Y(t)Y^{-1}(s) f\left(s, x(s), \int_0^a k(s,\sigma,x(\sigma))\,d\sigma\right)ds.\end{aligned} \tag{2.13}$$

Now, we show that Tx maps S into itself. Evidently, Tx is continuous on J and $Tx \in R^n$. We verify that (2.1) is fulfilled. From (2.13) and using the hypotheses and (2.3) we have

$$\begin{aligned}|(Tx)(t)| &\le \left|Y(t)Y^{-1}(0)\right||x_0| + \int_0^t \left|Y(t)Y^{-1}(s)\right| \\ &\times\left|f\left(s, x(s), \int_0^a k(s,\sigma,x(\sigma))\,d\sigma\right) - f\left(s, 0, \int_0^a k(s,\sigma,0)\,d\sigma\right)\right|ds \\ &+ \int_0^t \left|Y(t)Y^{-1}(s)\right|\left|f\left(s, 0, \int_0^a k(s,\sigma,0)\,d\sigma\right)\right|ds \\ &\le M|x_0| + \int_0^t M\left|f\left(s, 0, \int_0^a k(s,\sigma,0)\,d\sigma\right)\right|ds \\ &+ \int_0^t Mg(s)\left[|x(s)| + \int_0^a h(s,\sigma)|x(\sigma)|\,d\sigma\right]ds \\ &\le \beta\exp(\lambda t) + |x|_S \int_0^t Mg(s)\left[\exp(\lambda s) + \int_0^a h(s,\sigma)\exp(\lambda\sigma)\right]ds \\ &\le [\beta + N\alpha]\exp(\lambda t).\end{aligned} \tag{2.14}$$

From (2.14) it follows that $Tx \in S$. This proves that T maps S into itself.

Now, we verify that the operator T is a contraction map. Let $x(t), z(t) \in S$. From (2.13) and using the hypotheses we have

$$\begin{aligned}|(Tx)(t) - (Tz)(t)| &\le \int_0^t \left|Y(t)Y^{-1}(s)\right| \\ &\times\left|f\left(s, x(s), \int_0^a k(s,\sigma,x(\sigma))\,d\sigma\right)\right.\end{aligned} \tag{2.15}$$

$$
\begin{aligned}
&-f\left(s, z(s), \int_0^a k(s, \sigma, z(\sigma))\, d\sigma\right)\Big|\, ds \\
&\leq \int_0^t Mg(s)\left[|x(s) - z(s)| + \int_0^a h(s, \sigma)\, |x(\sigma) - z(\sigma)|\, d\sigma\right] ds \\
&\leq |x - z|_S \int_0^t Mg(s)\left[\exp(\lambda s) + \int_0^a h(s, \sigma) \exp(\lambda\sigma)\right] ds \\
&\leq |x - z|_S\, \alpha \exp(\lambda t).
\end{aligned}
$$

From (2.15) we obtain

$$|Tx - Tz|_S \leq \alpha\, |x - z|_S .$$

Since $\alpha < 1$, it follows from Banach fixed point theorem (see, [4, p.37]) that T has a unique fixed point in S. The fixed point of T is however a solution of equation (1.1). The proof is complete. □

Remark 1. We note that in [2] BIELECKI first used the norm defined in (2.2) for proving global existence and uniqueness of solutions of ordinary differential equations. Bielecki's method has been used very frequently to obtain global existence and uniqueness results for wide classes of differential and integral equations, see the references given in [4].

The following theorem deals with the uniqueness of solutions of equation (1.1) in R^n without existence part.

Theorem 2. *Let the fundamental solution matrix $Y(t)$ of equation* (1.2) *be as in Theorem* 1. *Assume that the functions f and k in equation* (1.2) *satisfy the conditions* (2.9) *and* (2.10) *with $g \in C(J, R_+)$ and $h(t, \sigma) = e(t)\, r(\sigma)$; $e, r \in C(J, R_+)$ and $e(t) \geq 1$. Suppose that*

$$\bar{d} = \int_0^a r(\sigma) \exp\left(\int_0^\sigma Mg(\tau)\, e(\tau)\, d\tau\right) d\sigma < 1. \tag{2.16}$$

Then the equation (1.1) *has at most one solution on J.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of equation (1.1) and

$u(t) = |x_1(t) - x_2(t)|$. Then by hypotheses we have

$$\begin{aligned} u(t) &\leq \int_0^t \left|Y(t)Y^{-1}(s)\right| \left| f\left(s, x_1(s), \int_0^a k(s,\sigma,x_1(\sigma))\,d\sigma\right)\right. \\ &\qquad \left. -f\left(s, x_2(s), \int_0^a k(s,\sigma,x_2(\sigma))\,d\sigma\right)\right| ds \\ &\leq \int_0^t Mg(s)\left[|x_1(s) - x_2(s)|\right. \\ &\qquad \left. +\int_0^a e(s)\,r(\sigma)\,|x_1(\sigma) - x_2(\sigma)|\,d\sigma\right] ds \\ &\leq \int_0^t Mg(s)\,e(s)\left[u(s) + \int_0^a r(\sigma)\,u(\sigma)d\sigma\right] ds. \end{aligned} \tag{2.17}$$

Now a suitable application of Lemma to (2.17) yields

$$|x_1(t) - x_2(t)| \leq 0,$$

and hence $x_1(t) = x_2(t)$. Thus there is at most one solution to equation (1.1) on J. □

3. Properties of solutions. In this section we shall study some fundamental properties of solutions of equation (1.1) by using some suitable conditions on the functions involved therein.

First, we shall give the following theorem which deals with the estimate on the solution of equation (1.1).

Theorem 3. *Suppose that the functions f, k in equation* (1.1) *satisfy the conditions*

$$|f(t,x,u)| \leq \bar{g}(t)\left[|x| + |u|\right], \tag{3.1}$$

$$|k(t,\sigma,u)| \leq \bar{e}(t)\,\bar{r}(\sigma)\,|u|, \tag{3.2}$$

where $\bar{g}, \bar{e}, \bar{r} \in C(J, R_+)$ and $\bar{e}(t) \geq 1$. Assume that

$$d_0 = \int_0^a \bar{r}(\sigma)\exp\left(\int_0^\sigma M\bar{g}(\tau)\,\bar{e}(\tau)\,d\tau\right)d\sigma < 1. \tag{3.3}$$

If $x(t)$, $t \in J$ is any solution of equation (1.1), *then*

$$|x(t)| \leq \frac{M|x_0|}{1-d_0} \exp\left(\int_0^t M\bar{g}(s)\,\bar{e}(s)\,ds\right), \tag{3.4}$$

for $t \in J$.

Proof. Using the fact that $x(t)$, $t \in J$ is a solution of equation (1.1) and the hypotheses we have

$$\begin{aligned} |x(t)| &\leq \left|Y(t)\,Y^{-1}(0)\right| |x_0| + \int_0^t \left|Y(t)\,Y^{-1}(s)\right| \\ &\times \left| f\left(s, x(s), \int_0^a k(s,\sigma,x(\sigma))d\sigma\right)\right| ds \\ &\leq M|x_0| + \int_0^t M\bar{g}(s)\left[|x(s)| + \int_0^a \bar{e}(s)\,\bar{r}(\sigma)\,|x(\sigma)|\,d\sigma\right] ds \\ &\leq M|x_0| + \int_0^t M\bar{g}(s)\,\bar{e}(s)\left[|x(s)| + \int_0^a \bar{r}(\sigma)\,|x(\sigma)|\,d\sigma\right] ds. \end{aligned} \tag{3.5}$$

Now an application of Lemma to (3.5) yields (3.4). □

Remark 2. We note that, if the estimate obtained in (3.4) is bounded, then the solution $x(t)$ of equation (1.1) is bounded on J.

The next theorem deals with the dependency of solutions of equation (1.1) on given initial values.

Theorem 4. *Suppose that $Y(t), f, k$ and $\bar{d}$ be as in Theorem* 2. *Let $x(t)$ and $z(t)$ be the solutions of equation* (1.1) *with the given initial conditions*

$$x(0) = x_0, \tag{3.6}$$

and

$$z(0) = z_0. \tag{3.7}$$

Then

$$|x(t) - z(t)| \leq \frac{M|x_0 - z_0|}{1-\bar{d}} \exp\left(\int_0^t M g(s)\,e(s)\,ds\right), \tag{3.8}$$

for $t \in J$.

Proof. Since $x(t)$ and $z(t)$ are the solutions of equation (1.1) with initial values (3.6) and (3.7) we have

$$\begin{aligned} |x(t) - z(t)| &\leq \left|Y(t)Y^{-1}(0)\right| |x_0 - z_0| + \int_0^t \left|Y(t)Y^{-1}(s)\right| \\ &\times \left| f\left(s, x(s), \int_0^a k(s,\sigma,x(\sigma))\,d\sigma\right) \right. \\ &\left. - f\left(s, z(s), \int_0^a k(s,\sigma,z(\sigma))\,d\sigma\right)\right| ds \leq M|x_0 - z_0| \\ &+ \int_0^t Mg(s)\left[|x(s) - z(s)|\right. \\ &\left. + \int_0^a e(s)r(\sigma)|x(\sigma) - z(\sigma)|\,d\sigma\right] ds \leq M|x_0 - z_0| \\ &+ \int_0^t Mg(s)e(s)\left[|x(s) - z(s)|\right. \\ &\left. + \int_0^a r(\sigma)|x(\sigma) - z(\sigma)|\,d\sigma\right] ds. \end{aligned} \tag{3.9}$$

Now an application of Lemma to (3.9) yields the estimate (3.8), which shows the dependency of solutions of equation (1.1) on given initial values.

We now consider the following first order Fredholm integrodifferential systems of the forms

$$x'(t) = A(t)x(t) + F\left(t, x(t), \int_0^a K(t,\sigma,x(\sigma))\,d\sigma, \mu\right), x(0) = x_0, \tag{3.10}$$

$$x'(t) = A(t)x(t) + F\left(t, x(t), \int_0^a K(t,\sigma,x(\sigma))\,d\sigma, \mu_0\right), x(0) = x_0, \tag{3.11}$$

for $t \in J$, considered as a perturbation of the linear system (1.2), where $F \in C(J \times R^n \times R^n \times R, R^n)$, $K \in C\left(J^2 \times R^n, R^n\right)$ and μ, μ_0 are real parameters. □

The following theorem shows the dependency of solutions of equations (3.10) and (3.11) on parameters.

Theorem 5. *Let the fundamental solution matrix* $Y(t)$ *of equation* (1.2) *be as in Theorem* 1. *Assume that the functions* F, K *in* (3.10), (3.11) *satisfy the conditions*

$$|F(t,x,u,\mu) - F(t,\bar{x},\bar{u},\mu)| \leq G(t)\left[|x-\bar{x}| + |u-\bar{u}|\right], \tag{3.12}$$

$$|F(t,x,u,\mu) - F(t,x,u,\mu_0)| \leq H(t)\,|\mu-\mu_0|, \tag{3.13}$$

$$|K(t,\sigma,x) - K(t,\sigma,\bar{x})| \leq E(t)\,Q(\sigma)\,|x-\bar{x}|, \tag{3.14}$$

where $G, H, E, Q \in C(J, R_+)$, $E(t) \geq 1$ *and*

$$\int_0^t H(s)\,ds \leq \varepsilon, \tag{3.15}$$

for $t \in J$, *where* $\varepsilon > 0$ *is an arbitrary small constant. Assume that*

$$d_1 = \int_0^a Q(\sigma)\exp\left(\int_0^\sigma MG(\tau)\,E(\tau)\,d\tau\right)d\sigma < 1. \tag{3.16}$$

Let $x_1(t)$ *and* $x_2(t)$ *be the solutions of equations* (3.10) *and* (3.11) *respectively, then*

$$|x_1(t) - x_2(t)| \leq \frac{M\varepsilon\,|\mu-\mu_0|}{1-d_1}\exp\left(\int_0^t MG(s)\,E(s)\,ds\right), \tag{3.17}$$

for $t \in J$.

Proof. *Let* $x(t) = x_1(t) - x_2(t)$, *where* $x_1(t)$ *and* $x_2(t)$ *are the solutions of equations* (3.10) *and* (3.11). *From the hypotheses we have*

$$\begin{aligned} |x(t)| \leq & \int_0^t \left|Y(t)\,Y^{-1}(s)\right| \Big| F\left(s, x_1(s), \int_0^a K(s,\sigma,x_1(\sigma))\,d\sigma, \mu\right) \\ & - F\left(s, x_2(s), \int_0^a K(s,\sigma,x_2(\sigma))\,d\sigma, \mu\right) \\ & + F\left(s, x_2(s), \int_0^a K(s,\sigma,x_2(\sigma))\,d\sigma, \mu\right) \\ & - F\left(s, x_2(s), \int_0^a K(s,\sigma,x_2(\sigma))\,d\sigma, \mu_0\right)\Big|\,ds \end{aligned} \tag{3.18}$$

$$\leq \int_0^t MG(s)\left[|x_1(s)-x_2(s)|+\int_0^a E(s)\,Q(\sigma)\,|x_1(\sigma)-x_2(\sigma)|d\sigma\right]ds$$
$$+\int_0^t MH(s)\,|\mu-\mu_0|\,ds$$
$$\leq M\varepsilon\,|\mu-\mu_0|+\int_0^t MG(s)\,E(s)\left[|x(s)|+\int_0^a Q(\sigma)\,|x(\sigma)|d\sigma\right]ds.$$

Now an application of Lemma to (3.18) *yields* (3.17)*, which shows the dependency of solutions of equations* (3.10) *and* (3.11) *on parameters.* □

Remark 3. We note that the idea employed in this paper can be extended to study the following more general Volterra-Fredholm integrodifferential equation

$$\begin{aligned} x'(t) &= A(t)\,x(t) \\ &\quad +f\left(t,x(t),\int_0^t h(t,\sigma,x(\sigma))\,d\sigma,\int_0^a k(t,\sigma,x(\sigma))\,d\sigma\right), \\ x(0) &= x_0, \end{aligned} \tag{3.19}$$

considered as a perturbation of the linear system (1.2), under some suitable conditions on the functions involved in (3.19) and by using the integral inequality given in [10, p. 47] (see also [11]). Here, we omit the details.

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