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PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR EVOLUTION EQUATIONS IN BANACH SPACES

BY

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Abstract. In the this paper we prove an existence result for 2π-periodic solutions for a class of nonlinear evolution equations of the form

$$u'(t) + Au(t) \ni f(t, u(t)), \quad t \in \mathbb{R},$$

where $A : D(A) \subseteq X \rightsquigarrow X$ is an $m-$accretive operator acting on the general Banach space X and $f : \mathbb{R} \times \overline{D(A)} \to X$ a continuous function which is $2\pi-$periodic with respect to its first argument and satisfies an appropriate "sign" condition.

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1. Introduction. Let X be a Banach space with norm $\|\cdot\|$, $A : D(A) \subseteq X \to X$ an $m-$accretive operator and let $f : \mathbb{R} \times \overline{D(A)} \to X$ be a continuous function which is $2\pi-$periodic with respect to its first argument.

Let us consider the periodic problem

$$\begin{cases} u'(t) + Au(t) \ni f(t, u(t)) \\ u(t) = u(t + 2\pi) \end{cases} \tag{1.1}$$

which represents the abstract formulation of many nonlinear problems of parabolic type.

Due to its practical interest, this problem has been intensive studied by many authors via an interplay of compactness arguments and monotonicity methods. See Vrabie [18], Caşcaval–Vrabie [3], Hirano [7],

Hokkanen [9], and Hirano–Shioji [8] and Shioji [15], to cite only a few. The multi-valued case, i.e., the case in which instead of a single valued function f we have a multi-function F, was studied by Lakshmikantham–Papageorgiou [11], Shushuan Hu–Papageorgiou [17], Papageorgiou [14], Castaing–Monteiro Marques [4] and Paicu [13], among others.

In the present paper we prove a general Banach space setting variant of the main result established in a Hilbert space frame by Caşcaval–Vrabie [3]. Our main result extends also the one in Vrabie [18], in specific case in which f is continuous (not only Carathéodory as assumed in Vrabie *loc. cit.*). We emphasize however that all the results in this paper can be easily reformulated to cover also the Carathéodory case.

The main difficulty in our case is due to the fact that, unlike the Hilbert space case, here we cannot approximate the integral solutions by absolutely continuous ones. Instead, to overcome this difficulty, we use a slightly different method than in both Caşcaval–Vrabie *loc. cit.* and Vrabie *loc. cit.*, method consisting in two successive fixed point arguments combined with an approximation argument. Moreover, the one sided tangential condition on f here assumed is strictly weaker than the one used in the multi-valued case by Paicu [13] where, instead of (H_3) — see Section 3 —, the perturbation $F : \mathbb{R} \times \overline{D(A)} \rightsquigarrow X$ is assumed to satisfy

(H_3') There exists $r > 0$ such that, for each $t \in \mathbb{R}_+$, each $u \in \overline{D(A)}$ with $\|u\| \geq r$ and for each $y \in F(t,u)$, we have

$$[u,y]_+ \leq 0.$$

The paper is divided into four sections, Section 2 being mainly concerned with basic prerequisites. In Section 3 we state and prove our main result, while in the last Section 4 we analyze an example.

2. Preliminaries. We assume familiarity with the basic concepts and results concerning evolution equations governed by m-accretive operators in Banach spaces. For details, we refer the reader to Barbu [2], Lakshmikantham–Leela [10] and Vrabie [18]. We begin by recalling for easy references some basic concepts and results in the field which we will use in the sequel.

Let X be a real Banach space with norm $\|\cdot\|$, let $x, y \in X$ and $h \in \mathbb{R}\backslash\{0\}$.

We denote by

$$[x,y]_h := \frac{1}{h}(\|x+hy\| - \|x\|) \text{ and by } \ (x,y)_h := \frac{1}{2h}(\|x+hy\|^2 - \|x\|^2),$$

and we recall that there exist both limits

$$[x,y]_+ = \lim_{h\downarrow 0}[x,y]_h \quad \text{and} \quad (x,y)_+ = \lim_{h\downarrow 0}(x,y)_h.$$

Remark 2.1. Both $[\,\cdot,\cdot\,]_+$ and $(\cdot,\cdot)_+$ are u.s.c. on $X\times X$ and

(i) $(x,y)_+ = \|x\|[x,y]_+$;

(ii) $|[x,y]_+| \le \|y\|$;

(iii) $[x,y+z]_+ \le [x,y]_+ + [x,z]_+$;

(iv) $[x,y+ax]_+ = [x,y]_+ + a\|x\|$ for $a\in\mathbb{R}$;

(v) $[ax,y]_+ = [x,y]_+$ for $a>0$.

For further details see Lakshmikantham–Leela [10].

An operator $A: D(A)\subseteq X\to X$ is called *accretive* if for each $x_i\in D(A)$ and $y_i\in Ax_i$, $i=1,2$, we have

$$[\,x_1-x_2, y_1, -y_2\,]_+ \le 0.$$

It is called $m-$*accretive* if it is accretive and for each (or equivalently for some) $\lambda>0$ we have $R(I+\lambda A)=X$.

Let $f\in L^1(a,b\,;X)$ and let us consider the evolution equation

$$u'(t) + Au(t) \ni f(t). \tag{2.1}$$

A function $u:[\,a,b\,]\to X$ is called *an integral solution* of (2.1) on $[\,a,b\,]$ if $u\in C([\,a,b\,];X)$, $u(t)\in\overline{D(A)}$ for each $t\in[\,a,b\,]$ and u satisfies:

$$\|u(t)-x\| \le \|u(s)-x\| + \int_s^t [u(\tau)-x, f(\tau)-y]_+ d\tau \tag{2.2}$$

for each $x\in D(A)$, $y\in Ax$ and $a\le s\le t\le b$.

Theorem 2.1. *Let X be a Banach space and let $A: D(A)\subseteq X\to X$ be $m-$accretive. Then, for each $x\in\overline{D(A)}$ and $f\in L^1(a,b\,;X)$, there exists a unique integral solution of* (2.1) *on $[\,a,b\,]$ which satisfies $u(a)=x$.*

See Lakshmikantham–Leela [10], Theorem 3.5.1, p. 104 and Theorem 3.6.1, p. 116.

Theorem 2.2. *Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be $m-$accretive. If $f, g \in L^1(a, b\,; X)$ and u, v are two integral solutions of* (2.1) *corresponding to f and g, respectively, then:*

$$\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t [u(\tau) - v(\tau), f(\tau) - g(\tau)]_+ d\tau \tag{2.3}$$

for each $a \leq s \leq t \leq b$.

See Lakshmikantham–Leela [10], Theorem 3.5.2, p. 104

Let $x \in \overline{D(A)}$, $\tau \in [\,a, b)$ and $f \in L^1(a, b\,; X)$. We denote by $u(\cdot, \tau, x, f)$ the unique integral solution $v : [\,\tau, b\,] \to \overline{D(A)}$, of the problem (2.1) which satisfies $v(\tau) = x$. We denote by $\{S(t) : \overline{D(A)} \to \overline{D(A)},\ t \geq 0\}$ the semigroup generated by $-A$ on $\overline{D(A)}$, i.e., $S(t)x = u(t, 0, x, 0)$ for each $x \in \overline{D(A)}$ and $t \geq 0$. We say that the semigroup generated by $-A$ on $\overline{D(A)}$ is *compact* if, for each $t > 0$, $S(t)$ is a compact operator.

A subset $\mathcal{F}$ in $L^1(a, b\,; X)$ is *uniformly integrable* if, for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for each measurable subset E in $[\,a, b\,]$ whose Lebesgue measure $\mu(E) < \delta(\varepsilon)$, we have

$$\int_E \|f(s)\|\, ds \leq \varepsilon,$$

uniformly for $f \in \mathcal{F}$. It is easy to see that, whenever $\mathcal{F}$ is bounded in $L^p(a, b\,; X)$ for some $p > 1$, then it is uniformly integrable.

Theorem 2.3. *Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be m-accretive and such that $-A$ generates a compact semigroup. Let $\mathcal{B} \subseteq \overline{D(A)}$ be bounded and let $\mathcal{F}$ be uniformly integrable in $L^1(a, b; X)$. Then, for each $c \in (a, b)$, the integral solutions set*

$$\mathcal{I}(\mathcal{B}, \mathcal{F}) = \{u(\cdot, a, x, f)\,;\ x \in \mathcal{B},\ f \in \mathcal{F}\}$$

is relatively compact in $C([\,c, b\,]; X)$. If, in addition, $\mathcal{B}$ is relatively compact then the integral solutions set, $\mathcal{I}(\mathcal{B}, \mathcal{F})$, is relatively compact even in $C([\,a, b\,]; X)$.

See Baras [1] or Vrabie [19], Theorems 2.3.3, p. 47.

3. The main result. The assumptions we need are listed below.

(H_1) $A : D(A) \subseteq X \rightsquigarrow X$ is an $m-$accretive operator with $\overline{D(A)}$ convex, $0 \in A0$ and such that the semigroup generated by $-A$ on $\overline{D(A)}$ is compact.

(H_2) $f : \mathbb{R} \times \overline{D(A)} \to X$ is continuous and 2π-periodic with respect to its first argument.

(H_3) There exists $r > 0$ such that, for each $t \in \mathbb{R}_+$, each $u \in \overline{D(A)}$ with $\|u\| = r$, we have

$$[u, f(t,u)]_+ \leq 0.$$

(H_4) The function f maps bounded subsets in $\mathbb{R} \times \overline{D(A)}$ into bounded subsets in X. (is bounded on $\mathbb{R} \times B(0,r) \cap \overline{D(A)}$[1], where $r > 0$ is given by (H_3).)

Now we may proceed to the statement of our main result.

Theorem 3.1. *Let X be a Banach. If $(H_1) \sim (H_4)$ are satisfied, then the problem* (1.1) *has at least one integral solution.*

The proof of Theorem 3.1 is divided into several steps. Firstly, we show that, whenever, in addition to $(H_1) \sim (H_4)$, f is locally Lipschitz with respect to its second argument and satisfies a slightly stronger condition than (H_3), the problem (1.1) has at least one integral solution. To this aim, let us first consider the Cauchy problem

$$\begin{cases} u'(t) + Au(t) \ni f(t, u(t)) \\ u(0) = \xi, \end{cases} \tag{3.1}$$

with $\xi \in B(0,r) \cap \overline{D(A)}$. The following condition will prove useful in the sequel.

(H_3'') There exist $r > 0$ and $\rho > 0$ such that, for each $t \in \mathbb{R}_+$, each $u \in D(A)$ with $\|u\| = r$, we have

$$[u, f(t,u)]_+ \leq -\rho.$$

[1]Here and thereafter $B(\xi, r)$ denotes the closed ball with center $\xi \in X$ and radius $r > 0$.

A simple successive approximation argument shows that, if f is continuous on $\mathbb{R}\times\overline{D(A)}$ and locally Lipschitz with respect to its second argument then, for each $\xi\in\overline{D(A)}$, the problem (3.1) has a unique noncontinuable integral solution $u:[\,0,T)\to\overline{D(A)}$. Moreover, if $\xi\in B(0,r)\cap\overline{D(A)}$, we have

Lemma 3.1. *Let us assume that $A:D(A)\subseteq X\rightsquigarrow X$ is an $m-$accretive operator with $0\in A0$ and $f:\mathbb{R}\times\overline{D(A)}\to X$ is continuous and 2π-periodic with respect to its first argument and locally Lipschitz with respect to the second argument. Let us assume, in addition, that (H_3'') and (H_4) are satisfied. Then, for each $\xi\in B(0,r)\cap\overline{D(A)}$, the problem (3.1) has a unique noncontinuable integral solution $u:[\,0,+\infty)\to B(0,r)\cap\overline{D(A)}$.*

Proof. Let $\xi\in B(0,r)\cap\overline{D(A)}$ and let $u:[\,0,T)\to\overline{D(A)}$ be the unique noncontinuable integral solution of (3.1). We will first show that $\|u(t)\|\le r$ for each $t\in[\,0,T)$. We proceed by contradiction. So, let us assume that

$$\sup_{t\in[\,0,T)}\|u(t)\|>r.$$

This means that there exists $t_0\in[\,0,T)$ and $\delta>0$ with $t_0+\delta<T$ and such that $\|u(t_0)\|=r$ and $\|u(t)\|>r$ for each $t\in(t_0,t_0+\delta)$. Using (2.2), the fact that $0\in A0$ and Remark 2.1, we deduce

$$r<\|u(t_0+h)\|\le\|u(t_0)\|+\int_{t_0}^{t_0+h}[\,u(\tau),f(\tau,u(\tau))\,]_+\,d\tau$$

for each $h\in(0,\delta)$. Since $[\,\cdot,\cdot\,]_+$ is u.s.c. and $[\,u(t_0),f(t_0,u(t_0))\,]\le-\rho$, we conclude that for $h\in(0,\delta)$, small enough, $r<r-h\rho+(h\rho)/2<r$. This contradiction can be eliminated only if $\|u(t)\|\le r$ for each $t\in[\,0,T)$.

We will show next that $T=+\infty$. To this aim, let us assume by contradiction that $T<+\infty$. Since f is bounded on $B(0,r)\cap\overline{D(A)}$, it follows that there exists the limit $\lim_{t\uparrow T}u(t)=u^*\in B(0,r)\cap\overline{D(A)}$. But this shows that u can be continued to the right of T which is contradiction. This contradiction come from the supposition that $T<+\infty$. Thus $T=+\infty$ and $u(t)\in B(0,r)\cap\overline{D(A)}$ for each $t\in[\,0,+\infty)$ and this completes the proof. □

Under the hypotheses of Lemma 3.1, we can define the Poincaré map $P:B(0,r)\cap\overline{D(A)}\to B(0,r)\cap\overline{D(A)}$ by

$$P(\xi)=u(2\pi,\xi)$$

where $u(\cdot,\xi):[\,0,+\infty)\to B(0,r)\cap\overline{D(A)}$ is the unique noncontinuable integral solution of the problem (3.1).

Lemma 3.2. *Let us assume that* (H_1), (H_2), (H_3'') *and* (H_4) *are satisfied and* $f : \mathbb{R} \times \overline{D(A)} \to X$ *is locally Lipschitz with respect to the second argument. Then, the Poincaré map* $P : B(0,r) \cap \overline{D(A)} \to B(0,r) \cap \overline{D(A)}$ *is continuous and has compact range.*

Proof. To prove that P is continuous, let $\xi \in K$ and let $(\xi_n)_n$ in K with $\lim_n \xi_n = \xi$. From Theorem 2.3, we deduce that $\{P(\xi_n);\ n = 1, 2, \dots\}$ is relatively compact. So, to complete the proof, it suffices to show that each convergent subsequence of $(P(\xi_n))_n$ converges to $P(\xi)$. To this aim, let us assume for simplicity that $(P(\xi_n))_n$ is itself convergent, and let us denote by u_n the unique C^0-solution of (3.1) corresponding to the initial datum ξ_n. By (2.3) and (i) in Remark 2.1, we deduce

$$\|u_n(t) - u(t)\| \le \|\xi_n - \xi\| + \int_0^t \|f(s, u_n(s)) - f(s, u(s))\|\, ds \tag{3.2}$$

for $n = 1, 2, \dots$ and $t \in [\,0, 2\pi\,]$.

Since f is locally Lipschitz, for each $\tau \in [\,0, 2\pi\,]$, there exist $\rho_\tau > 0$ and $L_\tau > 0$ such that

$$\|f(t,x) - f(t,y)\| \le L_\tau \|x - y\|$$

for each $(t,x), (t,y) \in [\,\tau - \rho_\tau, \tau + \rho_\tau\,] \times D(\xi, \rho_\tau)$. But $[\,0, 2\pi\,]$ is compact, and therefore there exists a finite family, $\{\tau_i;\ i = 1, 2, \dots, n\}$ in $[\,0, 2\pi\,]$, such that

$$[\,0, 2\pi\,] \subseteq \cup_{i=1}^n [\,\tau_i - \rho_{\tau_i}, \tau_i + \rho_{\tau_i}\,].$$

Let us denote by $\rho = \min\limits_{1 \le i \le n} \rho_{\tau_i}$ and by $L = \max\limits_{1 \le i \le n} L_{\tau_i}$. Then

$$\|f(t,x) - f(t,y)\| \le L \|x - y\|$$

for each $(t,x), (t,y) \in [\,0, 2\pi\,] \times D(\xi, \rho)$. As $\lim_n \xi_n = \xi$, we may assume with no loss of generality that $\|\xi_n - \xi\| \le \rho e^{-2\pi L}$ for $n = 1, 2, \dots$. Fix $n = 1, 2, \dots$ and let us remark that, on a right neighborhood $[\,0, a\,]$ of 0 we have $\|u_n(s) - u(s)\| \le \rho$. Let a be the greatest number in $(0, 2\pi\,]$ with the property above. We will show that $a = 2\pi$. Indeed, if we assume by contradiction that $a < 2\pi$, from (3.2) and the Lipschitz condition, we get

$$\|u_n(t) - u(t)\| \le \|\xi_n - \xi\| + \int_0^t L \|u_n(s) - u(s)\|\, ds$$

for $n = 1, 2, \dots$ and $t \in [\,0, a\,]$. By Gronwall Lemma, we have

$$\|u_n(t) - u(t)\| \le \|\xi_n - \xi\| e^{aL} \le \rho e^{-(2\pi - a)L}$$

for $n = 1, 2, \dots$ and $t \in [\,0, a\,]$. Since both u_n and u are continuous, this inequality contradicts the maximality of a. This contradiction can be eliminated only if $a = 2\pi$. Thus we have

$$\|u_n(t) - u(t)\| \le \|\xi_n - \xi\| e^{2\pi L}$$

for $n = 1, 2, \dots$ and $t \in [\,0, 2\pi\,]$. Hence each convergent subsequence of $(P(\xi_n))_n$ converges to $P(\xi)$. As $\{P(\xi_n);\ n = 1, 2, \dots\}$ is relatively compact, this shows that even $\lim_n P(\xi_n) = P(\xi)$, and thus P is continuous. Finally, again by Theorem 2.3, we conclude that $P(K)$ is relatively compact. □

Lemma 3.3. *Let us assume that* (H_1), (H_2), (H_3'') *and* (H_4) *are satisfied and* $f : \mathbb{R} \times \overline{D(A)} \to X$ *is locally Lipschitz with respect to the second argument. Then, the problem* (1.1) *has at least one integral solution.*

Proof. In view of Lemma 3.2 the Poincaré map satisfies the hypotheses of Schauder Fixed Point Theorem and thus is has at least one fixed point $\xi \in B(0, r) \cap \overline{D(A)}$. Clearly the integral solution of the Cauchy problem (3.1), corresponding to that fixed point ξ is an integral solution of the periodic problem (1.1) and this completes the proof. □

We can now pass to the proof of Theorem 3.1.

Proof. Let $\varepsilon > 0$ be arbitrary. By a well-known continuation theorem due to Dugundji [6], $f : \mathbb{R} \times B(0, r) \cap \overline{D(A)} \to X$ can be extended by continuity to a function $\widetilde{f}$ defined on $\mathbb{R} \times X$. Next, by an approximation result due to Lasota–Yorke [12], there exists a locally Lipschitz function $\widetilde{f}_\varepsilon : \mathbb{R} \times X \to X$ such that

$$\|\widetilde{f}(t, x) - \varepsilon x - \widetilde{f}_\varepsilon(t, x)\| \le \frac{\varepsilon}{2}$$

for each $(t, x) \in \mathbb{R} \times X$. Forgetting about $\widetilde{f}$, we deduce that there exists a locally Lipschitz function $f_\varepsilon : \mathbb{R} \times \overline{D(A)} \to X$ such that

$$\|f(t, x) - \varepsilon x - f_\varepsilon(t, x)\| \le \frac{\varepsilon}{2}$$

for each $(t, x) \in \mathbb{R} \times \overline{D(A)}$.

By the inequality above and (iv) in Remark 2.1, we get

$$[\, x, f_\varepsilon(t,x)\,]_+ \le -\frac{\varepsilon}{2}$$

for each $(t,x) \in \mathbb{R} \times \overline{D(A)}$, with $\|x\| = r$. In view of the first part of the proof, we know that the periodic problem

$$\begin{cases} u'_\varepsilon(t) + Au_\varepsilon(t) \ni f_\varepsilon(t, u_\varepsilon(t)) \\ u_\varepsilon(0) = u_\varepsilon(2\pi) \end{cases}$$

has at last one 2π-periodic solution $u_\varepsilon : \mathbb{R} \to K$. Let $(\varepsilon_n)_n$ be a sequence in $(0,1)$ with $\varepsilon_n \downarrow 0$ and let us denote by $u_n = u_{\varepsilon_n}$. By Theorem 2.3, we deduce that $\{u_n;\ n = 1, 2, \dots\}$ is relatively compact in $C([\, a, 2\pi\,]; X)$ for each $a \in (0, 2\pi)$. Then, $\{u_n(2\pi);\ n = 1, 2, \dots\}$ is relatively compact in X. Taking into account that $u_n(0) = u_n(2\pi)$, again by Theorem 2.3, we conclude that $\{u_n;\ n = 1, 2, \dots\}$ is relatively compact even in $C([\, 0, 2\pi\,]; X)$. So, we may assume with no loss of generality that there exists a continuous function $u : [\, 0, 2\pi\,] \to K$ such that $\lim_n u_n(t) = u(t)$ uniformly for $t \in [\, 0, 2\pi\,]$. Let us denote by $f_n = f_{\varepsilon_n}$. Since $\lim_n f_n(t, u_n(t)) = f(t, u(t))$ uniformly for $t \in [\, 0, 2\pi\,]$, we deduce that u is a 2π-periodic solution of (1.1). The proof is complete. □

Remark 3.1. An inspection of the arguments above shows that the conclusion of Theorem 3.1 remains valid if, instead of (H_2), we assume

(H'_2) $f : \mathbb{R} \times \overline{D(A)} \to X$ is continuous and, for each $u \in B(0,r) \cap \overline{D(A)}$, $f(\cdot, u)$ is 2π-periodic.

More than this, we can assume, in addition, that f is defined merely on $\mathbb{R} \times (B(0,r) \cap \overline{D(A)})$, with $r > 0$ given by (H_3). Indeed, if this is the case, by a Tietze-type result due to Dugundji [6], f can be extended to a function defined on the whole $\mathbb{R} \times \overline{D(A)}$ and satisfying (H'_2).

4. An application. Let Ω be a nonempty, open and bounded subset in $\mathbb{R}^n$ with C^2-boundary Γ and let us consider the nonlinear diffusion equation

$$\text{(4.1)} \qquad \begin{cases} u_t(t,x) - \Delta\varphi(u(t,x)) = f(t, u(t,\cdot))(x) & \text{in } \mathbb{R} \times \Omega \\ u(t,x) = 0 & \text{on } \mathbb{R} \times \Gamma \\ u(t,x) = u(t + 2\pi, x) & \text{on } \mathbb{R} \times \Omega, \end{cases}$$

where $f : \mathbb{R} \times L^1(\Omega) \to L^1(\Omega)$ is continuous and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing.

Theorem 4.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, C^1 on $\mathbb{R} \setminus \{0\}$ and for which there exist two constants $C > 0$ and $a > 0$ if $n \leq 2$ and $a > (n-2)/n$ if $n \geq 3$ such that*

$$\varphi'(r) \geq C|r|^{a-1}$$

for each $r \in \mathbb{R} \setminus \{0\}$. Let $f : \mathbb{R} \times L^1(\Omega) \to L^1(\Omega)$ be a continuous function which is 2π-periodic with respect to its first argument. Let us assume that there exists $r > 0$ such that

$$\int_{\Omega^+_{t,u}} f(t, u(t,\cdot))(x)\,dx - \int_{\Omega^-_{t,u}} f(t, u(t,\cdot))(x)\,dx + \int_{\Omega^0_{t,u}} |f(t, u(t,\cdot))(x)|\,dx \leq 0$$

for each $t \in \mathbb{R}$ and each $u \in L^1(\Omega)$ with $\|u\|_{L^1(\Omega)} = r$, where

$$\begin{cases} \Omega^+_{t,u} = \{x \in \Omega;\ u(t,x) > 0\} \\ \Omega^-_{t,u} = \{x \in \Omega;\ u(t,x) < 0\} \\ \Omega^0_{t,u} = \{x \in \Omega;\ u(t,x) = 0\}. \end{cases}$$

Further, let us assume that there exists $k > 0$ such that $\|f(t,u)\|_{L^1(\Omega)} \leq k$ for each $t \in \mathbb{R}$, each $u \in L^1(\Omega)$ with $\|u\|_{L^1(\Omega)} \leq r$. Then (4.1) has at least one 2π-periodic solution $u \in C(\mathbb{R}; L^1(\Omega))$.

Proof. Let $X = L^1(\Omega)$ and let $A : D(A) \subseteq X \to X$ be defined by

$$\begin{cases} D(A) = \{u \in W^{1,1}_0(\Omega);\ \Delta\varphi(u) \in L^1(\Omega)\} \\ Au = \Delta\varphi(u) \text{ for } u \in D(A). \end{cases}$$

Clearly $0 \in D(A)$ and $0 \in A0$. In view of Showalter [16], Example 9.B, p. 234, A is m-accretive and, by Vrabie [19], Lemma 2.7.2, p. 71, $-A$ generates a compact semigroup on $\overline{D(A)} = L^1(\Omega)$. Thus A satisfies (H_1). We recall that, in $L^1(\Omega)$, the right directional derivative of the norm, i.e., $[\,\cdot,\cdot\,]_+$ is given by

$$[u,v]_+ = \int_{\Omega^+_u} v(x)\,dx - \int_{\Omega^-_u} v(x)\,dx + \int_{\Omega^0_u} |v(x)|\,dx$$

for all $u, v \in L^1(\Omega)$, where

$$\begin{cases} \Omega^+_u = \{x \in \Omega;\ u(x) > 0\} \\ \Omega^-_u = \{x \in \Omega;\ u(x) < 0\} \\ \Omega^0_u = \{x \in \Omega;\ u(x) = 0\}. \end{cases}$$

See VRABIE [19], Example 1.4.3, p. 12. So, (H_2), (H_3) and (H_4) are ensured by hypotheses. From Theorem 3.1, we deduce that (4.1) has at least one integral solution u and the proof is complete. □

We conclude with an example, inspired from DEIMLING [5], Exercise 6, p. 84, of a function f satisfying the hypotheses (H_2'), (H_3) and (H_4) and for which (4.1) has no trivial solution, i.e., which is constant with respect to t.

Example 4.1. Let $r > 0$ and $f : \mathbb{R} \times B(0,r) \to L^1(\Omega)$ be defined by

$$f(t,u)(x) = \alpha(t,x)u(x) + \left(r - \|u\|_{L^1(\Omega)}\right)\beta(t,x)$$

for each $t \in \mathbb{R}$, each $u \in B(0,r)$ and a.e. for $x \in \Omega$, where $\alpha, \beta : \mathbb{R} \times \Omega \to \mathbb{R}$ satisfy:

(i) $\alpha(t,x) < 0$ for each $(t,x) \in \mathbb{R} \times \Omega$;

(ii) $\alpha(t,\cdot), \beta(t,\cdot) \in L^\infty(\Omega)$ for each $t \in \mathbb{R}$;

(iii) $\beta(t,\cdot)/\alpha(t,\cdot) \notin L^1(\Omega)$ for each $t \in \mathbb{R}$;

(iv) both α and β are 2π-periodic with respect to the first argument.

One may easily verify that f satisfies (H_2') and both (H_3) and (H_4). Furthermore, for each fixed $t \in \mathbb{R}$, the equation $f(t,u) = 0$ has no solution $u \in B(0,r)$. Thus the 2π-periodic solutions of (4.1) given by Theorem 4.1 in conjunction with Remark 3.1, are nontrivial.

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