ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LV, 2009, f.1

A GENERALIZED GOULD TYPE INTEGRAL WITH RESPECT TO A SUBMEASURE II

BY

ALINA EMILIA PETCU

Abstract. In [4] written jointly with A. Gavriluţ we have introduced and studied a Gould type integral for a bounded real valued function with respect to a submeasure of finite variation. In [7] we have defined a Gould type integral in a more general case for a real function with respect to a submeasure which may be not of finite variation. Also, we have pointed out several basic properties of this integral and we gave a characterization of the integrability in generalized sense. In the present paper we establish some other properties of the generalized Gould integral and a measure change type theorem.

Mathematics Subject Classification 2000: 28B20, 28C15, 49J53.
Key words: Gould type integral, submeasure, multisubmeasure.

1. Introduction. In [6], Gould has introduced and studied a special type of integral for a real valued function with respect to a finite additive set function taking values in a Banach space. In [8] and [9], Precupanu and Croitoru have introduced a multivalued Gould type integral for real functions with respect to a multimeasure with values in the family of all nonvoid, compact, convex sets of a Banach space. Taking as starting point their above mentioned papers, Gavriluţ has defined and studied in [2] and [3] a Gould type integral for a bounded, real valued function with respect to a multisubmeasure of finite variation. In [4] Gavriluţ and Petcu have studied a Gould type integral for a bounded, real valued function with respect to a real submeasure of finite variation. Petcu has treated in [7] a Gould type integral in a more general case for a real function with respect to a submeasure which may be not of finite variation.

In this paper we obtain some other properties of the generalized Gould integral and we give a measure change type theorem.

In Section 2, we recall from [7] the notion of the Gould $m-$integrability in general case and a characterization of it.

In Section 3, we prove some other properties of the generalized Gould integral concerning its behavior with respect to the submeasure.

We also establish a measure change type theorem.

Let T be an abstract, nonvoid set, $\mathcal{A}$ an algebra of subsets of T and $f : T \to \mathbb{R}$ a function.

All over this paper we assume that $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ is a submeasure in Drewnowski's sense [1].

Let us also consider the following set functions associated to m:

$$\overline{m}(A) = \sup\{\sum_{i=1}^{n} m(A_i)\}, \text{ for every } A \subset T, \tag{1}$$

where the supremum is extended over all finite partitions $\{A_i\}_{i=\overline{1,n}}$ of A, called *the variation* of m; and

$$\widetilde{m}(A) = \inf\{\overline{m}(B); A \subseteq B, B \in \mathcal{A}\}, \text{ for every } A \subseteq T. \tag{2}$$

We note that $\widetilde{m}(A) = \overline{m}(A)$, for every $A \in \mathcal{A}$ and $\overline{m}$ is a finitely additive set function on $\mathcal{A}$.

Definition 1.1.

I) A set $A \in \mathcal{A}$ is said to be $m-$*bounded* if $\{m(B); B \subseteq A, B \in \mathcal{A}\}$ is a bounded set in $\mathbb{R}$.

II) The space $(T, \mathcal{A}, m)$ is said to be *non-degenerate* if for any m-unbounded set $A \in \mathcal{A}$, there exists $\alpha > 0$ such that every m-unbounded set $B \in \mathcal{A}, B \subseteq A$, contains a $m-$bounded set $C \in \mathcal{A}$, $C \subseteq B$ such that $m(C) > \alpha$.

Definition 1.2. I) A function $f : T \to \mathbb{R}$ is said to be $\overline{m}$-*totally-measurable on* $(T, \mathcal{A}, m)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

i) $\overline{m}(A_0) < \varepsilon$ and

ii) $\sup_{t,s \in A_i} |f(t) - f(s)| = osc(f, A_i) < \varepsilon$, for every $i = \overline{1, n}$.

II) f is said to be $\overline{m}$-*totally-measurable on* $B \in \mathcal{A}$ if its restriction $f|_B$ of f to B is $\overline{m}$-totally measurable on $(B, \mathcal{A}_B, m_B)$, where $m_B = m|_{\mathcal{A}_B}$ and $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$.

2. Definition of the g-m-integral. In this section we recall from [7] some results concerning the generalized Gould integral with respect to a submeasure.

Without any special assumptions, $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ is a submeasure with total variation $\overline{m}$, $(T, \mathcal{A}, \overline{m})$ is a non-degenerate space and $f : T \to \mathbb{R}$ is a real valued function.

We define:

$$M(f) = \{K \in \mathcal{A};\ \overline{m}(K) < \infty \text{ and } f \text{ is bounded over } K\} \tag{3}$$

and

$$M_B(f) = M(f) \cap \mathcal{A}_B, \forall B \in \mathcal{A}, \text{ where } \mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}. \tag{4}$$

We recall now the notion of the $g - m-$integrability.

Definition 2.1. I) We say that f is *generalized Gould $m-$integrable* (briefly, *$g-m-$integrable*) *on* T if the following conditions are satisfied:

i) f is $\overline{m}$-totally-measurable;

ii) there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for any $K \in M(f)$ with $K \supseteq K_\varepsilon$, we have:

$$\left| \int_K^* f dm - I \right| < \varepsilon,$$

where $\int_K^* f dm$ is the integral in the sense of Definition 2.3 [4].

The real number I is called the *generalized Gould integral of* f (briefly, *$g-m-$integral*). We denote it by $\int_T f dm$.

II) f is said to be *$g - m$-integrable on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is *$g - m$-integrable* on $(B, \mathcal{A}_B, m_B)$.

Remark 2.2. I) If $f : T \to \mathbb{R}$ is $g - m-$integrable, then $\int_T f dm$ is uniquely determined.

II) If $f : T \to \mathbb{R}$ is a bounded $g - m-$integrable function on T and $\overline{m}(T) < \infty$, then $\int_T^* f dm = \int_T f dm$.

Thus we can use the same symbol $\int_T f dm$ for the Gould integral in both situations "bounded" or "unbounded".

In the following, we recall from [7] a characterization of the $g-m-$integrability:

Theorem 2.3. *A $\overline{m}$-totally-measurable function $f : T \to \mathbb{R}$ is $g - m-$integrable on T if and only if : for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for any $K \in M(f)$ with $K \cap K_\varepsilon = \emptyset$, we have*:

$$\left| \int_K f dm \right| < \varepsilon.$$

3. Properties of the g-m-integral with respect to a submeasure. In this section we study some properties of the $g - m-$integral with respect to a submeasure.

Theorem 3.1. *If f is a $g - m$-integrable function on T and $\alpha \in \mathbb{R}_+$, then f is $g - \alpha m$-integrable on T and*

$$\int_T f d(\alpha m) = \alpha \int_T f dm.$$

Proof. The case $\alpha = 0$ is trivial. Let $\alpha \neq 0$.

We easily observe that the real valued set function $\alpha m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$, defined by $(\alpha m)(A) := \alpha m(A)$, for every $A \in \mathcal{A}$ is a submeasure.

We consider $M_{\alpha m}(f) = \{K \in \mathcal{A}; \overline{\alpha m}(K) < \infty$ and f is bounded over $K\}$.

Because f is $g - m$-integrable on T for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ such that for any $K \in M(f)$ with $K \supseteq K_\varepsilon$ we have:

$$\left| \int_K f dm - \int_T f dm \right| < \frac{\varepsilon}{|\alpha|}. \tag{5}$$

Let $K \in M_{\alpha m}(f)$ such that $K \supseteq K_\varepsilon$. We observe that $K \in M(f)$. By virtue of Theorem 2.7-ii) [4] and from (5) we obtain:

$$\begin{aligned} \left| \int_K f d(\alpha m) - \alpha \int_T f dm \right| &= \left| \alpha \int_K f dm - \alpha \int_T f dm \right| \\ &= |\alpha| \left| \int_K f dm - \int_T f dm \right| < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon, \end{aligned}$$

which means that f is g-αm-integrable on T and $\int_T f d(\alpha m) = \alpha \int_T f dm$.

Theorem 3.2. *Let $m_1, m_2 : \mathcal{A} \mapsto \mathbb{R}_+$ be two submeasures with $\overline{m_1}(T) < \infty$, $\overline{m_2}(T) < \infty$ and $f : T \mapsto \mathbb{R}$ a $g - m_1$-integrable and $g - m_2$-integrable*

function on T. If $m : \mathcal{A} \mapsto \mathbb{R}_+$ is the submeasure defined by $m(A) = m_1(A) + m_2(A)$, for every $A \in \mathcal{A}$, then f is g-m-integrable on T.

Proof. First, we shall prove that f is $\overline{m_1 + m_2}-$totally measurable.

Since f is g-m_1-integrable and g-m_2-integrable f is $\overline{m_1}-$totally measurable and $\overline{m_2}-$totally measurable on T. Hence, for every $\varepsilon > 0$ there exists $P_\varepsilon^1 = \{A_i\}_{i=\overline{0,n}}$ and $P_\varepsilon^2 = \{B_j\}_{j=\overline{0,p}}$ partitions of T such that $\overline{m_1}(A_0) < \frac{\varepsilon}{2}$, $\overline{m_2}(B_0) < \frac{\varepsilon}{2}$ and $osc(f, A_i) < \varepsilon$, $i = \overline{1,n}; osc(f, B_j) < \varepsilon$, $j = \overline{1,p}$.

Let us consider $P_\varepsilon = \{A_i \cap B_j\}_{\substack{i=\overline{0,n} \\ j=\overline{0,p}}}$.

We have $\overline{m_1 + m_2}(A_0 \cap B_0) \leq \overline{m_1}(A_0 \cap B_0) + \overline{m_2}(A_0 \cap B_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and $osc(f, A_i \cap B_j) < \varepsilon, i = \overline{1,n}$, $j = \overline{1,p}$.

Hence, f is $\overline{m_1 + m_2}-$totally measurable on T.

To prove the $g - m-$integrability of f we shall use Theorem 2.3. So, for every $\varepsilon > 0$ we have to find a set $K_\varepsilon \in \mathcal{A}$ with f bounded on K_ε such that for every $K \in \mathcal{A}$ with f bounded on K and $K \cap K_\varepsilon = \emptyset$ holds

$$\left| \int_K f d(m_1 + m_2) < \varepsilon \right|.$$

According to Corollary 3.3-ii) [7] we get that for every $\varepsilon > 0$, there exist δ_1, $\delta_2 > 0$ such that for every $A \in \mathcal{A}$ with $\overline{m_1}(A) < \delta_1$, $\overline{m_2}(A) < \delta_2$

$$\left| \int_A f dm_1 \right| < \frac{\varepsilon}{4} \text{ and } \left| \int_A f dm_2 \right| < \frac{\varepsilon}{4}. \tag{6}$$

Because f is $\overline{m_1}-$totally measurable and $\overline{m_2}-$totally measurable on T we can apply Lemma 2.2 from [7]. Then, for δ_1, $\delta_2 > 0$ from above, there exist $N_1 > 0$, $N_2 > 0$ and there exist $A_{N_1}, B_{N_2} \in \mathcal{A}$ such that

$$\overline{m_1}(T \backslash A_{N_1}) < \delta_1, \overline{m_1}(T \backslash B_{N_2}) < \delta_2 \text{ and } \sup_{t \in A_{N_1}} |f(t)| < N_1, \sup_{t \in B_{N_2}} |f(t)| < N_2.$$

We observe that for every $A \in \mathcal{A}$ with $A \cap (A_{N_1} \cup B_{N_2}) = \emptyset$ we have $\overline{m_1}(A) < \delta_1$, $\overline{m_2}(A) < \delta_2$ and holds (6).

Since f is g-m_1-integrable and g-m_2-integrable on T, according to Theorem 2.3, there exist $K_\varepsilon^1 \in \mathcal{A}$ with f bounded on K_ε^1 such that for every $K \in \mathcal{A}$ with f bounded on K and $K \cap K_\varepsilon^1 = \emptyset$ holds

$$\left| \int_K f dm_1 \right| < \frac{\varepsilon}{4} \tag{7}$$

and $K_\varepsilon^2 \in \mathcal{A}$ with f bounded on K_ε^2 such that for every $K \in \mathcal{A}$ with f bounded on K with $K \cap K_\varepsilon^2 = \emptyset$ we have

$$\left| \int_K f dm_2 \right| < \frac{\varepsilon}{4}. \tag{8}$$

Now, we prove that for every $C \in \mathcal{A}$ with $C \cap K_\varepsilon^1 = \emptyset$ we have $\left| \int_C f dm_1 \right| < \varepsilon$.

Indeed, let $C \in \mathcal{A}$ be such that $C \cap K_\varepsilon = \emptyset$. According to Theorem 3.2-i) [7] it follows that f is $g - m$-integrable on C. Thus, there exists $\widetilde{K_\varepsilon^1} \subset C$ with f bounded on $\widetilde{K_\varepsilon^1}$ such that for every $K \subset C$ with f bounded on K and $K \supseteq \widetilde{K_\varepsilon^1}$,

$$\left| \int_K f dm_1 - \int_C f dm_1 \right| < \frac{\varepsilon}{2}. \tag{9}$$

Since $C \cap K_\varepsilon^1 = \emptyset$ it results that $K \cap K_\varepsilon^1 = \emptyset$.

From (9) and (7), we obtain

$$\left| \int_C f dm_1 \right| \leq \left| \int_K f dm_1 - \int_C f dm_1 \right| + \left| \int_K f dm_1 \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon. \tag{10}$$

Analogously, we can prove that for every $C \in \mathcal{A}$ with $C \cap K_\varepsilon^2 = \emptyset$ we have $\left| \int_C f dm_2 \right| < \varepsilon$.

Thus, for every $C \in \mathcal{A}$ with $C \cap K_\varepsilon^1 = \emptyset$ and $C \cap K_\varepsilon^2 = \emptyset$ we have

$$\left| \int_C f dm_1 \right| < \frac{\varepsilon}{4} \text{ and } \left| \int_C f dm_2 \right| < \frac{\varepsilon}{4}. \tag{11}$$

Let us take $K_\varepsilon = (K_\varepsilon^1 \cup K_\varepsilon^2) \cap (A_{N_1} \cup B_{N_2})$. We see that $\overline{m_1 + m_2}(K_\varepsilon) < \infty$ and f is bounded on K_ε.

We shall prove that for every $\varepsilon > 0$ and for every $K \in \mathcal{A}$ with f bounded on K and $K \cap K_\varepsilon = \emptyset$ holds

$$\left| \int_K f d(m_1 + m_2) \right| < \varepsilon.$$

Let us consider $K \in \mathcal{A}$ such that f is bounded on K and $K \cap K_\varepsilon = \emptyset$. We can write $K = K_1 \cup K_2$, $K_1 \cap K_2 = \emptyset$ and $K_1 \cap (A_{N_1} \cup B_{N_2}) = \emptyset, K_2 \cap (K_\varepsilon^1 \cup K_\varepsilon^2) = \emptyset$.

By virtue of Theorem 2.12 [4], we have

$$\int_K fd(m_1+m_2)=\int_{K_1} fd(m_1+m_2)+\int_{K_2} fd(m_1+m_2).$$

Now, using Theorem 2.3, we prove that for every $\varepsilon>0$

$$\left|\int_{K_1} fd(m_1+m_2)\right|<\frac{\varepsilon}{2}.$$

Indeed, because $K_1\cap(A_{N_1}\cup B_{N_2})=\emptyset$, using Theorem 2.9-i) [4] and (11), we obtain

$$\begin{aligned}\left|\int_{K_1} fd(m_1+m_2)\right| &= \left|\int_{K_1} fdm_1+\int_{K_1} fdm_2\right| \\ &\leq \left|\int_{K_1} fdm_1\right|+\left|\int_{K_1} fdm_2\right|<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}.\end{aligned} \tag{12}$$

Analogously, we can prove that

$$\left|\int_{K_2} fd(m_1+m_2)\right|<\frac{\varepsilon}{2}. \tag{13}$$

Consequently, from (12) and (13), we have

$$\left|\int_{K} fd(m_1+m_2)\right|<\varepsilon.$$

Thus, f is $g-m-$integrable.

Remark 3.3. If $f:T\to\mathbb{R}$ is a bounded function and $m_1,m_2:\mathcal{A}\mapsto\mathbb{R}_+$ are submeasures of finite variation, then, according to Theorem 2.9-i) [4], the equality of integrals holds.

Theorem 3.4. *Let m_1, $m_2:A\mapsto\mathbb{R}_+$ be two submeasures such that $m_1(A)\leq m_2(A)$, for every $A\in A$ and $\overline{m_2}(T)<\infty$.*

i) If $f:T\mapsto\mathbb{R}_+$ is a g-m_2-integrable function on T, then f is g-m_1-integrable on T and

ii)

$$\int_T fdm_1\leq\int_T fdm_2.$$

Proof. i) To prove that f is $g-m_1$-integrable on T we shall use Theorem 2.3.

It is easy to see that f is $\overline{m_1}-$totally measurable.

Because f is $g-m_2$-integrable on T, applying Theorem 2.3, we get that for every $\varepsilon > 0$, there is $K_\varepsilon \in \mathcal{A}$ with f bounded on K_ε such that for every $K \in \mathcal{A}$ with f bounded on K and $K \cap K_\varepsilon = \emptyset$

$$\int_K f dm_2 < \varepsilon.$$

Since $m_1(A) \leq m_2(A)$, for every $A \in \mathcal{A}$ and f is bounded on K, by virtue of Theorem 2.9-ii) from [4], we have

$$\int_K f dm_1 \leq \int_K f dm_2 < \varepsilon.$$

which assures that f is $g - m_1$-integrable on T

ii) Because f is m_1-g-integrable and m_2-g-integrable on T, reasoning as in the proof of Theorem 3.3 for every $\varepsilon > 0$ and for every $C \in \mathcal{A}$ with $C \cap K_\varepsilon^1 = \emptyset$ and respectively $C \cap K_\varepsilon^2 = \emptyset$ we have

$$\left| \int_C f dm_1 \right| < \frac{\varepsilon}{2} \tag{14}$$

and

$$\left| \int_C f dm_2 \right| < \frac{\varepsilon}{2}. \tag{15}$$

Let us take $K_\varepsilon = K_\varepsilon^1 \cup K_\varepsilon^2$. We observe that f is bounded on K_ε and $(T \backslash K_\varepsilon) \cap K_\varepsilon^1 = \emptyset$, respectively $(T \backslash K_\varepsilon) \cap K_\varepsilon^2 = \emptyset$. Then, taking $C = T \backslash K_\varepsilon$ in (14) and (15), we obtain

$$\left| \int_{T \backslash K_\varepsilon} f dm_1 \right| < \frac{\varepsilon}{2} \text{ and } \left| \int_{T \backslash K_\varepsilon} f dm_2 \right| < \frac{\varepsilon}{2}. \tag{16}$$

Now, applying Theorem 3.2-ii) [7], (16) and then Theorem 2.9-ii) [4], we

have

$$\begin{aligned}
&\int_T f dm_1 - \int_T f dm_2 = \int_{T\setminus K_\varepsilon} f dm_1 \\
&+ \int_{K_\varepsilon} f dm_1 - \int_{T\setminus K_\varepsilon} f dm_2 - \int_{K_\varepsilon} f dm_2 \\
&\leq \left|\int_{T\setminus K_\varepsilon} f dm_1\right| + \left|\int_{T\setminus K_\varepsilon} f dm_2\right| + \int_{K_\varepsilon} f dm_1 - \int_{K_\varepsilon} f dm_2 \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The proof is completed. □

In the sequel, we establish a measure change type theorem.

Let T,$T^{'}$ be two nonvoid sets.

Let $G : T \to T^{'}$be a bijective function and $\mathcal{A}$ an algebra of subsets of T.

Let us denote by $\mathcal{A}^{'} = \{A^{'} \subseteq T^{'}, G^{-1}(A^{'}) \in \mathcal{A}\}$. We observe that $\mathcal{A}^{'}$ is an algebra of subsets of $T^{'}$.

We consider $m : \mathcal{A} \to \overline{\mathbb{R}}_+$ a submeasure and m^G: $\mathcal{A}^{'} \to \overline{\mathbb{R}}_+$ defined by

$$m^G(A^{'}) = m(G^{-1}(A^{'})), \text{ for every } A^{'} \in \mathcal{A}^{'}.$$

It is easy to prove that m^G is a submeasure on $T^{'}$.

Theorem 3.5. *Let $f : T^{'} \to \mathbb{R}$ be a $g-m^G-$integrable function on $T^{'}$. Then*

i) $f \circ G$ is g-m−integrable on T and

ii)

$$\int_{T'} f \ dm^G = \int_T f \circ G dm. \tag{17}$$

Proof. i) To prove that $f \circ G$ is g-m-integrable on T we shall use Theorem 2.3.

First, we prove that $f \circ G$ is $\overline{m}-$totally measurable on T.

Using $\overline{m^G}$-totally measurability of f we find a partition $P_\varepsilon^{'} = \{A_i^{'}\}_{i=\overline{0,n}}$ of $T^{'}$ such that $\overline{m^G}(A_0^{'}) < \varepsilon$ and $osc(f, A_i^{'}) < \varepsilon$.

Let us consider $P_\varepsilon = \{G^{-1}(A_i^{'})\}_{i=\overline{0,n}}$.

We observe that P_ε is a partition of T and $\overline{m}(G^{-1}(A_0^{'})) = \overline{m^G}(A_0^{'}) < \varepsilon$, $osc(f \circ G, G^{-1}(A_i^{'})) = osc(f, A_i^{'}) < \varepsilon$, which assures that $f \circ G$ is $\overline{m}-$totally measurable on T.

Since f is g-m^G- integrable on $T^{'}$, applying Theorem 2.3 we have for every $\varepsilon > 0$ there is $K_\varepsilon^{'} \in M^{m^G}(f)$ such that for every $K^{'} \in M^{m^G}(f)$ with $K^{'} \cap K_\varepsilon^{'} = \emptyset$

$$\left| \int_{K^{'}} f \ dm^G \right| < \varepsilon. \tag{18}$$

Let us take $K_\varepsilon = G^{-1}(K_\varepsilon^{'}) \in \mathcal{A}$. Because $\overline{m^G}(K_\varepsilon^{'}) = \overline{m}(G^{-1}(K_\varepsilon^{'})) < \infty$ and $\sup_{t^{'} \in K_\varepsilon^{'}} \left| f(t^{'}) \right| = \sup_{t^{'} \in G(K_\varepsilon)} \left| f(t^{'}) \right| = \sup_{t \in K_\varepsilon} |(f \circ G)(t)|$ it results that $K_\varepsilon = G^{-1}(K_\varepsilon^{'}) \in M(f \circ G)$.

Let $K \in M(f \circ G)$ such that $K \cap K_\varepsilon = \emptyset$.

Since $G : T \to T^{'}$ is a bijective function we have $K = G^{-1}(K^{'})$ and $G(K \cap K_\varepsilon) = G(K) \cap G(K_\varepsilon) = K^{'} \cap K_\varepsilon^{'} = \emptyset$.

Consequently, according to Theorem 2.13 [5] and from (18) there exists $K_\varepsilon = G^{-1}(K_\varepsilon^{'}) \in M(f \circ G)$ such that for every $K \in M(f \circ G)$ with $K \cap K_\varepsilon = \emptyset$,

$$\left| \int_K f \circ G dm \right| = \left| \int_{K^{'}} f \ dm^G \right| < \varepsilon.$$

Hence, $f \circ G$ is $g - m-$integrable on T.

ii) We prove now the equality (17).

By g-m^G- integrability of f on $T^{'}$ we have that for every $\varepsilon > 0$ there is $K_\varepsilon^1 \in M^{m^G}(f)$ such that for every $K^{'} \in M^{m^G}(f)$ with $K^{'} \supseteq K_\varepsilon^1$

$$\left| \int_{K^{'}} f \ dm^G - \int_{T^{'}} f \ dm^G \right| < \frac{\varepsilon}{2}. \tag{19}$$

Analogously, because $f \circ G$ is $g - m-$integrable on T we get that for every $\varepsilon > 0$ there is $K_\varepsilon \in M(f \circ G)$ such that for every $K \in M(f \circ G)$ with $K \supseteq K_\varepsilon$

$$\left| \int_K f \circ G dm - \int_T f \circ G dm \right| < \frac{\varepsilon}{2}. \tag{20}$$

Taking $K_\varepsilon^0 = K_\varepsilon \cup G^{-1}(K_\varepsilon^1)$ we see that $\overline{m}(K_\varepsilon^0) = \overline{m}(K_\varepsilon \cup G^{-1}(K_\varepsilon^1) \leq \overline{m}(K_\varepsilon) + \overline{m}(G^{-1}(K_\varepsilon^1) = \overline{m}(K_\varepsilon) + \overline{m^G}(K_\varepsilon^1) < \infty$ and $f \circ G$ is bounded on K_ε^0.

Also, for every $K \in M(f \circ G), K \supseteq K_\varepsilon^0$ because $G : T \to T^{'}$is a bijective function we can write $K = G^{-1}(K^{'})$.

Then $G^{-1}(K^{'}) \supseteq K_\varepsilon \cup G^{-1}(K_\varepsilon^1)$, which is equivalent to $K^{'} \supseteq G(K_\varepsilon) \cup K_\varepsilon^1$.

Consequently, from (19), (20) and using Theorem 2.13 [5] we obtain

$$\begin{aligned}\left|\int_{T'} f\ dm^G - \int_T f \circ G dm\right| &\leq \left|\int_{K'} f\ dm^G - \int_{T'} f\ dm^G\right| \\ &\quad + \left|\int_{K'} f\ dm^G - \int_T f \circ G dm\right| \\ &\leq \left|\int_{K'} f\ dm^G - \int_{T'} f\ dm^G\right| \\ &\quad + \left|\int_K f \circ G dm - \int_T f \circ G dm\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus the proof is completed. □

Acknowledgement. The author is indebted to Prof. Dr. A. M. Precupanu for the careful reading of the paper and for the useful suggestions.

REFERENCES

1. Drewnowski, L. – *Topological rings of sets, continuous set functions, Integration, I, II,* Bull. Acad. Polon. Sci. Ser. Math. Astron. Phys., 20(1972), 269-276, 277–286.
2. Gavriluţ, A. – *A Gould type integral with respect to a multisubmeasure,* Math. Slovaca, 58(2008), 43–62.
3. Gavriluţ, A. – *On some properties of the Gould type integral with respect to a multisubmeasure,* An. Stiint. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.), 52(2006), 177–194.
4. Gavriluţ, A.; Petcu, A. – *A Gould type integral with respect to a submeasure,* An. Stiint. Univ. "Al. I. Cuza" Iasi. Mat. (N.S.), 53(2007), 351–368.
5. Gavriluţ, A.; Petcu, A. – *Some properties of the Gould type integral with respect to a submeasure,* Bul. Inst. Polit. Iasi, LIII (LVII), Sec. Matematică, Mecanică Teoretică, Fizică, 2007, 121–130.

6. Gould, G.G. – *Integration over vector-valued measures*, Proc. London. Math. Soc., 15(1965), 193–205.

7. Petcu, A. – *A generalized Gould type integral with respect to a submeasure I,* An. Stiint. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.) (in print).

8. Precupanu, A.; Croitoru, A. – *A Gould type integral with respect to a multimeasure, I*, An. Stiint. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.), 48(2002), 165–200.

9. Precupanu, A.; Croitoru, A. – *A Gould type integral with respect to a multimeasure, II*, An. Stiint. Univ. "Al. I. Cuza" Iaşi. Mat. (N.S.), 49(2003), 183–207.

Received: 10.IV.2008

Petroleum-Gas University of Ploieşti,
Bd. Bucureşti, no 39, Ploieşti,
ROMANIA
pet_aly@yahoo.com