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HEIGHTS OF A TRIANGLE IN A LINEAR NORMED SPACE AND HILBERTIAN NORMS

BY

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Abstract. In this paper we study some geometric properties using the norm derivatives. First, we prove that for the norm to be hilbertian it is sufficient to show the symmetry of norm derivatives only for the pairs of elements where the norm is differentiable. Also, we discuss the problem of heights of a triangle with respect to Birkhoff's orthogonality. Finally we remark that the triangles in a smooth linear space have orthocenters only in the case of hilbertian norms.

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1. Introduction. Let X be a smooth real linear normed space, i.e. the norm is *Gateaux differentiable* ([4], [10]). Let us denote:

$$n'(x;y) = \lim_{t\to 0} \frac{\|x+ty\|^2 - \|x\|^2}{2t}, \quad \text{for all } x, y \in X. \tag{1}$$

If the norm on X is generated by an inner product, and in this case the norm is usually called *hilbertian*, we have:

$$n'(x;y) = \langle x, y\rangle, \quad \text{for all } x, y \in X.$$

Obviously, in this case the norm's differential, which in this paper is defined by (1), is symmetric. In [12], LEDUC has established that the symmetry property of differential characterizes the hilbertian norms, i.e.

$n'(x;y) = n'(y;x)$, for all $x, y \in X$
if and only if the norm of X is generated by an inner product.

If $\dim X \geq 3$ it is sufficient the next property to be satisfied:

$$n'(x;y) = 0 \Rightarrow n'(y;x) = 0. \tag{2}$$

(see [12], Theorem 3).

On the other hand, in a general real linear normed space we may consider the following orthogonality relation:

$$x \perp y \Leftrightarrow \|x + ty\| \leq \|x\|, \quad \text{for all } t \in \mathbb{R}.$$

Taking into account the property of minimum elements of a convex function (see, for example, [4], [9], [10]) if the space is smooth, this type of ortogonality (usually known as *Birkhoff ortogonality* [5]) can be characterized using the differential. Precisely,

$$x \perp y \Leftrightarrow n'(x;y) = 0. \tag{3}$$

So, when $\dim X \geq 3$ *the symmetry of Gateaux differential vanishing is the same with the symmetry of Birkhoff orthogonality.*

In the case of norms which are not Gateaux differentiable, the orthogonality condition can be characterized by directional derivatives of the norm, $n'_-(x;y)$, $n'_+(x;y)$, which are obtained taking in the right side of (1) the limit for $t \to 0_-$, respectively $t \to 0_+$, limits which always exist ([4]).

The following properties are well known:

(i) $n'_+(-x;y) = -n'_-(x;y), \forall x, y \in X.$

(ii) $|n'_+(x;y)| \leq \|x\|\|y\|, \forall x, y \in X,$

(iii) $n'_+(\alpha x; \beta x + \gamma y) = \alpha\gamma n'_+(x;y) + \alpha\beta\|x\|^2$, $\forall x, y \in X, \forall \alpha, \beta \in \mathbb{R}, \gamma > 0.$

(iv) $n'_+(x; y_1 + y_2) \leq n'_+(x;y_1) + n'_+(x;y_2), \forall x, y_1, y_2 \in X.$

(v) $x \perp y \Leftrightarrow n'_+(x;y) \cdot n'_-(x;y) \leq 0, x, y \in X.$

(vi) $n'_+(x;y) = \inf_{t>0} \frac{\|x+ty\|^2 - \|x\|^2}{2t}, \forall x, y \in X,$

because the function:

$$t \to \frac{\|x + ty\|^2 - \|x\|^2}{2t}, t \in \mathbb{R}\backslash\{0\}, \text{ is non-decreasing.} \tag{4}$$

(See [4], [9], [10].)

Some geometric properties used in Euclidian spaces can be extended to general linear normed spaces using Birkhoff's orthogonality. If $\dim X \geq 3$, the symmetry of Birkhoff's orthogonality can be used only in the case of norms generated by an inner product (see, for instance, [3], pp. 143 or [12]). In this way we can obtain a few characterizations of hilbertian norms considering some special geometric properties generated by the orthogonality in Euclidian spaces, but which in linear normed spaces X, lead to the orthogonality symmetry. On the other side, most of the characterizations which use the directional derivatives lead us frequently to the symmetry of directional derivatives, which allows us to use the well-known characterization:

(5) $n'_+(x;y) = n'_+(y;x)$, *for all* $x, y \in X$
if and only if the norms is generated by an inner product.

(See [3], [6], [7], [11].)

In this paper we first show that the symmetry property (5) may be satisfied only at the pairs of points in which the norm is quite differentiable, taking into account that for an arbitrary norm, the set of differentiable points is dense in the space ([4], [9]). Consequently, a series of characterizations of hilbertian norms established by different authors can be easily extended asking for those properties be satisfied only in the norm's differntiabilty points of the space. In the 3rd section we will study the problem of the heights of a triangle with respect to Birkhoff's orthogonality. We prove that the orthocenter's existence property for any triangle characterizes the hilbertian norm. Moreover, this special property can be equivalently formulated by an equality concerning the norm derivatives.

A similar result was established by Alsina and Thomas [2], but they use a different form for the heights of a triangle in a linear normed space.

2. The differentiability's symmetry and orthogonality. Let X be a real linear normed space, and $D \subset X$ the set of differentiability points of the norm. As we have mentioned before, a well known result of convex analysis shows that D is dense set in X ([4], [9]). Moreover, the directional derivatives in any point can be approximated by directional derivatives in points of differentiability. First of all, because $n'_+(x;\bullet)$ is a continuous function for all $x \in X$, taking a sequence of points of differentiability $(y_n) \subset D$ which converges to y, we get:

$$n'_+(x;y) = \lim_{n\to\infty} n'_+(x;y_n). \tag{6}$$

A similar property cannot be obtained in first variable. In fact, as Giles [8] had shown, continuity in first variable leads to differentiability.

Yet, the function $n'_+(\bullet; y)$ is upper-semicontinuous for each $y \in X$, being a lower bound of continuous functions as we can see from property (vi). So, if we consider a sequence of points of differentiability $(x_n) \subset D$ which converges to $x \in X$, then:

$$\lim_{n\to\infty} \sup n'(x_n; y) \leq n'_+(x; y). \tag{7}$$

If the space is smooth the inequality (7) becomes equality since the duality mapping, i.e. the differential of norm, is demicontinuous [4].

Now we can get an easy extension of a characterization given by the property (5) of the norms generated by an inner product with the help of directional derivatives. In this sense, let us recall a minimax result which will be essential in what follows (see [14] or [4], Corollary 36, pp. 160).

Theorem 1. *If $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence of real functions, upper-semicontinuous on a compact set A, then:*

$$\lim_{n\to\infty} \max_{u\in A} f_n(u) = \max_{u\in A} \lim_{n\to\infty} f_n(u).$$

Theorem 2. *The following properties are equivalent:*

(i) $n'(x; y) = n'(y; x)$, *for all* $x, y \in D$;

(ii) $n'_+(x; y) = n'_+(y; x)$, *for all* $x, y \in X$;

(iii) the norm on X is generated by an inner product.

Proof. According to property (5) we have the implication (ii)⇒(iii). Also, it is obvious that (iii)⇒(i).

Let us now show that (i)⇒(ii).

Let us consider two arbitrary elements $x, y \in X$ and two sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset D$ which converge to x, respectively y.

We must consider first the case when one of the points x, y in D. Let's suppose, for example, that $x \in D$. We define the sequence of functions (f_n) by:

$$f_n(u) = \frac{\|u + t_n x\|^2 - \|u\|^2}{2t_n}, \quad n \in \mathbb{N}^*, u \in X,$$

where (t_n) is a decreasing sequence which converges to zero and so, $\lim_{n\to\infty} f_n(u) = n'_+(u;x)$. Considering $A_k = \{y_i; i \geq k\} \cup \{y\}, k \in \mathbb{N}^*, y_i = y + t_i x, i \in \mathbb{N}$, we observe that this set is a compact in X for each $k \in \mathbb{N}^*$. Using Theorem 1 and the fact that f_n are continuous functions, we get:

$$\inf_{n\in\mathbb{N}} \sup_{i\geq k} \frac{\|y_i + t_n x\|^2 - \|y_i\|^2}{2t_n}$$
$$= \sup_{i\geq k} \inf_{n\in\mathbb{N}} \frac{\|y_i + t_n x\|^2 - \|y_i\|^2}{2t_n} = \sup_{i\geq k} n'_+(y_i;x) = \sup_{i\geq k} n'_+(x;y_i).$$

And so we get:

$$n'_+(y;x) = \inf_{n\in\mathbb{N}} \frac{\|y + t_n x\|^2 - \|y\|^2}{2t_n} = \inf_{n\in\mathbb{N}} \lim_{i\to\infty} \frac{\|y_i + t_n x\|^2 - \|y_i\|^2}{2t_n}$$
$$\inf_{n\in\mathbb{N}} \inf_{k\in\mathbb{N}} \sup_{i\geq k} \frac{\|y_i + t_n x\|^2 - \|y_i\|^2}{2t_n} = \inf_{k\in\mathbb{N}} \sup_{i\geq k} n'_+(x;y_i) = n'_+(x;y),$$

meaning

$$n'_+(y;x) = n'_+(x;y). \tag{8}$$

Therefore, this equality holds whenever $x \in D$ or $y \in D$. Now, using (6), (7) and (8) we get:

$$n'_+(x;y) = \lim_{n\to\infty} n'_+(x;y_n) = \lim_{n\to\infty} n'_+(y_n;x) \leq n'_+(y;x), \text{ for all } x, y \in X.$$

Interchanging x with y we obtain the reverse inequality, so (ii) is true what finishes the proof. □

3. Heights in a triangle. Let us suppose for the beginning that X is a real smooth linear normed space endowed with Birkhoff's orthogonality. Therefore, the orthogonality can be equivalently defined by (3). Let us consider $x, y \in X$ and denote by $D(x,y)$ the straight line determined by these points, meaning:

$$D(x,y) = \{x_t; x_t = (1-t)x + ty, t \in \mathbb{R}\}. \tag{9}$$

Because of the homogeneity of Birkhoff's orthogonality in both variables it can be extended to the orthogonality of two straight lines. We define:

$$D(x_1,y_1) \perp D(x_2,y_2) \text{ if } x_1 - x_2 \perp y_1 - y_2. \tag{10}$$

If we consider a third point $z \notin D(x,y)$ we can discuss about the existence of the perpendicular line from z to $D(x,y)$.

As the orthogonality is not usually symmetric, we must consider the existence of a $t \in \mathbb{R}$ such that $x_t - z \perp x - y$ or the existence of a $t \in \mathbb{R}$ such that $x - y \perp x_t - z$. In each of this cases the straight line $D(x_t, z)$ can be considered as the as *a perpendicular line from z to the straight line $D(x,y)$*. We can call the right perpendicular line, respectively left perpendicular line from z to $D(x,y)$. Analytically, this leads to the existence of a real number t with:

$$n'(x_t - z; x - y) = 0, \tag{11}$$

respectively

$$n'(x - y; x_t - z) = 0. \tag{12}$$

In the first case of the right perpendicular line from z to $D(x,y)$ we get the next equation:

$$n'(x - z + t(y - x); y - x) = 0, \tag{13}$$

For this equation the existence and uniqueness of solutions is quite complicated taking into account that properties of the norm's differential in first variable are slightly poor. A special case is that one that $x - z \perp y - x$, when $D(z,y)$ is the right perpendicular line from z to $D(x,y)$. But it is possible to exist other perpendicular lines from z to $D(x,y)$, which means that equation (13) might have some other solutions $t \in \mathbb{R}$, where $t \neq 0$.

In the case of the left perpendicular line from z to $D(x,y)$ we will obtain a linear equation:

$$n'(y - x; x - z + t(y - x)) = 0, \tag{14}$$

which has the unique solution:

$$t = \frac{n'(x - y; x - z)}{\|x - y\|^2},\ x \neq y. \tag{15}$$

In fact, the case $x = y$ must be trivial because $D(x,x) = \{x\}$.

So, for any $x, y \in X, x \neq y$, and $z \notin D(x,y)$ there exists a unique left perpendicular line from z to $D(x,y)$ having the following form:

$$z + \sigma[x - z + \frac{n'(x - y; x - z)}{\|x - y\|^2} \cdot (y - x)],\ \ \sigma \in \mathbb{R} \tag{16}$$

or

$$z+\sigma[y-z+\frac{n'(y-x;y-z)}{\|x-y\|^2}(x-y)],\ \ \sigma\in\mathbb{R}, \tag{17}$$

according to the symmetry between x and y.

So, if we consider $y=0$ and x,z two orthogonal linearly independent elements, $x\perp z$, then the left perpendicular line from z to $D(0,x)$ will be the straight $\alpha z,\alpha\in\mathbb{R}$, while the right perpendicular from z to $D(0,x)$ will be the straight determined by z and a point tx, such that $n'(z-tx;x)=0$.

In the next we will consider *only the case of the left perpendicular line* from a point to a straight, meaning the relation (16) or (17).

Moreover, this type of orthogonality can be extended to spaces with the norm not necessarily differentiable using the directional derivatives and the property (v). For example, if the norm is not differentiable in $x-y$ then any straight line like:

$$z+\sigma[x-z+\frac{a(x-y,x-z)}{\|x-y\|^2}(y-z)],\ \ \alpha\in\mathbb{R}, \tag{18}$$

where $a(x-y,x-z)\in\mathbb{R}$ has the following property:

$$n'(x-y;x-z)\leq a(x-y,x-z)\leq n'_+(x-y;x-z), \tag{19}$$

can be considered the perpendicular line from z to $D(x,y)$ since $x-y\perp x-z+t(y-x)$, where $t=\frac{a(x-y,x-z)}{\|x-y\|}$.

Therefore, we obtain a symmetric cone with the vertex z of perpendiculars having two limit positions for $a(x-y,x-z)=n'_-(x-y;x-z)$, respectively $a(x-y,x-z)=n'_+(x-y;x-z)$. So, if X is a space with a *semi-inner product* in Lumer's sense [13], it is natural that the perpendicular line from z to $D(x,y)$ to be considered as the next straight line:

$$z+\sigma[x-z+\frac{[x-y,x-z]}{\|x-y\|^2}]\cdot(y-x)],\ \ \sigma\in\mathbb{R}, \tag{20}$$

where $[.,.]$ is the semi-inner product in X.

Equivalently, we will consider a certain selection $f:X\to X^*$, of the *duality application* and we will choose $a(u,v)=(f(u)(v)),u,v\in X$, where X^* is the dual space of X. We recall that *the duality mapping coincides with the subdifferential of* $\frac{1}{2}\|\cdot\|^2$ (see [4]).

If the norm is differentiable, it is well known that $[x,y]=n'(x;y)$ for all $x,y\in X$. We must return now to the case of smooth linear normed spaces and analyse the problem of existence of common points between the heights of a triangle. Let us consider a triangle with vertices $x,y,z\in X$, which are not all on the same straight. Let us denote $h_{z;x,y}$ the height from z to $D(x,y)$, meaning the (left) perpendicular line from z to $D(x,y)$. Its elements are given by (16) or (17). Similarly, we obtain the heights $h_{x;y,z}$ and $h_{y;z,x}$. It can be easily verified that *the heights of a triangle's translate are the translated heights of the triangle*. Therefore, for simplicity, we can suppose that one of the vertices is quite the origin. So, if $x,z\in X$ are linearly independent and we take the triangle with the third vertex in origin, we obtain:

$$
\begin{aligned}
&h_{z;x,0}: z+\sigma\left[-z+\frac{n'(x;z)}{\|x\|^2}\cdot x\right],\ \ \sigma\in\mathbb{R},\\
&h_{x;0,z}: x+\sigma\left[-x+\frac{n'(z;z)}{\|z\|^2}\cdot z\right],\ \ \sigma\in\mathbb{R},\\
&h_{0;z,x}: \sigma\left[z+\frac{n'(z-x;z)}{\|x-z\|^2}\cdot(x-z)\right],\ \ \sigma\in\mathbb{R},
\end{aligned}
\tag{21}
$$

because n' is linear in second variable.

If the triangle is a rectangular one, for example if $x\perp z$, then the height from x is the straight $D(0,z)$, while the height from z is not going to be generally $D(0,z)$, but a straight determined by z and a point of $D(0,z)$ which can be other than the origin. It can be noted once again that the right, respectively left height from a vertex can be different.

Let us obtain now the intersection point of two heights. Let us denote:

$$
\cap_{y;z,x}=h_{x;y,z}\cap h_{z;x,y}. \tag{22}
$$

By direct calculations we get:

$$
\cap_{0;z,x}=x+\frac{n'(x;z)-\|x\|^2}{n'(x;z)\cdot n'(z;x)-\|x\|^2\|z\|^2}\cdot[n'(z;x)\cdot z-\|z\|^2\cdot x], \tag{23}
$$

whenever

$$
n'(x;z)n'(z;x)\neq\|x\|^2\|z\|^2. \tag{24}
$$

Taking into account the translation behavior we obtain:

$$
\cap_{y;z,x}=y+\cap_{0;z-y,x-y}, \tag{25}
$$

which means

$$\cap_{y;z,x} : x + \frac{n'(x-y;z-y) - \|x-y\|^2}{n'(x-y;z-y)n'(z-y;x-y) - \|x-y\|^2\|z-y\|^2} \cdot[n'(z-y;x-y)(z-y) - \|z-y\|^2(x-y)], \tag{26}$$

whenever $n'(x-y;z-y)n'(z-y;x-y) \neq \|x-y\|^2\|z-y\|^2$..

Remark 3. Let us specify now the case of (24) being not satisfied. It can be easily observed that if

$$n'(x;z)n'(z;x) = \|x\|^2\|z\|^2, \tag{27}$$

then necessarily

$$n'(x;z) = n'(z;x) = \|x\| \cdot \|z\| \text{ (or } -\|x\|\|z\|), \tag{27'}$$

because both inequalities of the norm's differential given by (ii) becomes equalities. Therefore, *in this case the norm's differential is symmetric in those two linearly independent elements*, x and z. Moreover, if the equality (27) holds, then the heights $h_{z;0,x}$ and $h_{x;z,0}$ are parallel if and only if $n'(x;z) \neq \|x\|^2$. If $n'(x;z) = \|x\|^2$, these heights coincide with the straight $D(x,z)$ and $n'(x;z) = n'(z;x) = \|x\|^2 = \|z\|^2$. On the other hand, if X is a strictly convex space then the equality (27) holds if and only if the elements x, z are linearly independent.

Similarly we can proceed with the relation (26), too.

We shall study now the problem of existence of the *orthocenter* in a triangle. In this case we have the following result:

Theorem 4. *Let X be a smooth linear normed space with $\dim X \geq 3$. Then the following properties are equivalent:*

(*i*) *the norm is hilbertian;*

(*ii*) *the heights of any triangle are meeting in a unique point;*

(*iii*) $n'(y;x)[n'(x;y)n'(x-y;x) - \|x\|^2 n'(x-y;y)] = n'(x;y)[\|y\|^2 n'(x-y;x) - n'(y;x)n'(x-y;y)]$, *for all* $x, y \in X$.

(*iv*) $n'(x+y;x)n'(x;y)[n'(y;x)+\|y\|^2] = n'(x+y;y)n'(y;x)[\|x\|^2+n'(x;y)]$, *for all* $x, y \in X$;

(v) $n'(x;y)n'(x+ty;x)n'(y;x+ty) = n'(y;x)n'(x+ty;y)n'(x;x+ty)$, *for any* $t \in \mathbb{R}$ *and* $x, y \in X$.

Proof. Obviously, we have implication (i)→(ii). Next, we prove (ii)⇒(iii). Assume that (ii) holds.

Obviously, the equality (iii) is verified whenever the elements x and y are linearly dependent. Also, (iv) and (v) are equivalent with the equality (iii).

Let $x, y \in X$ be two linearly independents elements and let us consider the triangle with vertices $0, x, y$. Taking into account the relations (23) and (26) we get

$$\cap_{0;x,y} : y + \frac{n'(y;x) - \|y\|^2}{n'(x;y)n'(y;x) - \|x\|^2\|y\|^2}[n'(x;y)x - y],$$
$$\text{if } n'(x;y)n'(y;x) \neq \|x\|^2\|y\|^2,$$

and

$$\bigcap_{x;y,0} : \frac{n'(x;y)}{n'(x-y;z)n'(x;x-y) - \|x-y\|^2\|x\|^2}[n'(x-y;y)x - n'(x-y;x)y],$$
$$\text{if } n'(x-y;x)n'(x;x-y) \neq \|x-y\|^2\|x\|^2.$$

Therefore, if $\cap_{0;x,y} = \cap_{x;y,0}$ we must have

$$1 - \frac{n'(y;x) - \|y\|^2}{n'(x;y)n'(y;x) - \|x\|^2\|y\|^2} = -\frac{n'(x;y)n'(x-y;x)}{n'(x-y;x)n'(x;x-y) - \|x-y\|^2\|x\|^2},$$

whenever $n'(x;y)n'(y;x) \neq \|x\|^2\|y\|^2$ and $n'(x-y;x)n'(x;x-y) \neq \|x-y\|^2\|x\|^2$. Now, using the properties of differential, the previous equality becomes just equality (iii). Moreover, according to Remark 3 if $n'(x;y)n'(y;x) = \|x\|^2\|y\|^2$ it follows $n'(x;y) = n'(y;x) = \|x\|^2 = \|y\|^2$ because by hypothesis (ii) the heights $h_{x;0,y}$ and $h_{y;x,0}$ are not parallel. Thus, the equality (*iii*) is also verified. Similarly, if $n'(x-y;x)n'(x;x-y) = \|x-y\|^2\|x\|^2$ we must have $n'(x;y) = 0$ and $n'(y(x)) = 0$, because $h_{y;x,0}$ and $h_{0;y,x}$, respectively $h_{x;y,0}$ and $h_{0;x,y}$ are not parallel. Therefore, also in this case the equality (iii) is verified. Thus, we proved (iii) for any $x, y \in X$ as claimed.

Finally, the implication (iii)→(i) follows by Leduc's characterization (2). Indeed, if $n'(x;y) = 0$, by (iii) it follows that $n'(y;x) = 0$ or $n'(x-y;y) = 0$. Now, if $n'(y;x) \neq 0$ and $n'(x;y) = 0$ we also have $n'(tx;y) = 0$ for any $t \in \mathbb{R}$,

and so $n'(tx-y;y)=0$ for any $t\in\mathbb{R}$. For $t\searrow 0$ we obtain $y=0$, which is absurd since x,y are linearly independent elements. Hence, it is necessary $n'(y;x)=0$ and the proof is finished. □

Remark 5. In the last part of the above proof we can interchange x with $tx, t\in\mathbb{R}$, because of the differential's homogeneity in both variables. In the same manner, the property that $\lim_{t\to 0}(y-tx;y)=\|y\|^2$ can be obtained from equality (iii). In fact, this property holds whenever the norm is differentiable. Moreover, Giles ([8]) proved that in a space endowed with a semi-inner product $\lim_{t\to 0} n'(x+ty;y)=[x,y]$ if and only if the norm is differentiable. On the other hand, if the norm is differentiable then there exists only a unique inner-product, namely $[x,y]=n'(x;y)$, for all $x,y\in X$. According to property (iii) of differential we get that $\lim_{t\to\infty} n'(tx+y;y)=\|y\|^2$.

A result similar to Theorem 4 concerning the characterization of hilbertian norms with the aid of the property which says that in an arbitrary triangle of a linear normed space the heights are meeting in a unique point has been established by Alsina and Tomas in [2]. But in their paper they used another expression for a triangle's height. For example, the height from z in a triangle x,y,z is defined in [2] by:

$$z-\sigma[z-y+\frac{\|z-y\|^2-n'_+(z-x;z-y)}{\|x-y\|^2}(y-x)],\quad \sigma\in\mathbb{R}. \tag{28}$$

Also, in [2] (see also [1]) the authors had shown that the heights (28) of a triangle have the property of orthogonality if and only if the norm is hilbertian. In fact, the heights of (16) and (28) are the same in a smooth linear normed space for any triangle only in the case of hilbertian norms.

Proposition 6. *In a smooth linear normed space the heights of* (16) *and* (28) *are the same for any triangle if and only if the norm is hilbertian.*

Proof. If for any two linearly independent elements $x,y\in X$ the heights (16) and (30) corresponding the origin in triangle $0,x,y$ are the same, then necessary we have:

$$n'(y-x;y)=\|y\|^2-n'(x;y),\ \text{for all } x,y\in X.$$

Interchanging x with y we also obtain:

$$n'(x-y;x)=\|x\|^2-n'(y;x).$$

Summing these two relations we get:

$$n'(y-x;y)+n'(y-x;-x)=\|x\|^2+\|y\|^2-(n'(x;y)+n'(y;x))$$

which means

$$\|y-x\|^2=\|x\|^2+\|y\|^2-(n'(x;y)+n'(y;x)),\ \text{for all } x,y\in X.$$

Interchanging x with $-x$ we also have:

$$\|y+x\|^2=\|x\|^2+\|y\|^2+(n'(x;y)+n'(y;x)).$$

Finally, summing these two last relations we obtain that *the parallelogram identity* is satisfied ([3], [7]), and so the norm is hilbertian.

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