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ON WEAKLY CONCIRCULAR SYMMETRIC MANIFOLDS

BY

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Abstract. The object of the present paper is to introduce a type of non-flat Riemannian manifold called *weakly concircular symmetric manifold* and study its geometric properties as well as its decomposibility. Among others it is shown that every weakly concircular symmetric manifold of vanishing scalar curvature is a weakly symmetric manifold. Also it is proved that in a decomposable weakly concircular symmetric manifold both the decompositions are concircularly recurrent. The existence of such a manifold is ensured by several interesting examples.

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1. Introduction. The notions of weakly symmetric and weakly projective symmetric manifolds were introduced by TAMÁSSY and BINH [5] and later BINH [1] studied decomposable weakly symmetric manifolds. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called a weakly symmetric manifold if its curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned}(1.1)\quad (\nabla_X R)(Y,Z,U,V) &= A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V)\\ &+C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X)\end{aligned}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, C, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. Moreover, it is to be noted that in a

$(WS)_n$, $B = C$ and $D = E$ [2] and hence the defining condition (1.1) reduces to

$$(1.2)\quad (\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) + B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) + D(V)R(Y,Z,U,X),$$

where A, B, D are 1-forms (not simultaneously zero). The present paper deals with a type of non-flat Riemannian manifold $(M^n, g)(n > 2)$ whose concircular curvature tensor $\tilde{C}$ of type (0,4) is not identically zero and satisfies the condition

$$(1.3)\quad (\nabla_X \tilde{C})(Y,Z,U,V) = A(X)\tilde{C}(Y,Z,U,V) + B(Y)\tilde{C}(X,Z,U,V) + C(Z)\tilde{C}(Y,X,U,V) + D(U)\tilde{C}(Y,Z,X,V) + E(V)\tilde{C}(Y,Z,U,X)$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where A, B, C, D and E are 1-forms (not simultaneously zero). Such a manifold will be called a *weakly concircular symmetric manifold* and denoted by $(W\tilde{C}S)_n$.

Section 2 is concerned with preliminaries. It is shown that in a $(W\tilde{C}S)_n$ the associated 1-forms $B = C$ and $D = E$ and hence the defining condition (1.3) of a $(W\tilde{C}S)_n$ reduces to the following form:

$$(1.4)\quad (\nabla_X \tilde{C})(Y,Z,U,V) = A(X)\tilde{C}(Y,Z,U,V) + B(Y)\tilde{C}(X,Z,U,V) + B(Z)\tilde{C}(Y,X,U,V) + D(U)\tilde{C}(Y,Z,X,V) + D(V)\tilde{C}(Y,Z,U,X),$$

where A, B and D are 1-forms (not simultaneously zero).

In section 3, some basic results of $(W\tilde{C}S)_n$ are investigated. It is proved that every $(W\tilde{C}S)_n$ of vanishing scalar curvature is a $(WS)_n$. It is proved that in a $(W\tilde{C}S)_n$, if the Ricci tensor is of Codazzi type, then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X, L) = \alpha(X)$. Also it is shown that in a $(W\tilde{C}S)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_2 defined by $g(X, \rho_2) = T(X) = B(X) + D(X)$ for all X.

Section 4 is devoted to the study of decomposable $(W\tilde{C}S)_n$ which is generally called the product $(W\tilde{C}S)_n$ and it is shown that in such a manifold one of the decomposition is Ricci symmetric with constant scalar curvature and the other is concircularly flat. It is proved that if a Riemannian manifold (M^n, g) is a decomposable $(W\tilde{C}S)_n$ with non-vanishing scalar curvature such that $M = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$, then both M_1 and M_2 are concircularly recurrent. The last section deals with several non-trivial examples of $(W\tilde{C}S)_n$ and also of decomposable $(W\tilde{C}S)_n$.

2. Preliminaries. In this section, some formulas are derived, which will be useful to the study of $(W\tilde{C}S)_n$. It is known [6] that in a Riemannian manifold the concircular curvature tensor $\tilde{C}$ of type (0,4) is given by

(2.1)
$$\tilde{C}(Y,Z,U,V) = R(Y,Z,U,V) - \frac{r}{n(n-1)}[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)],$$

where r is the scalar curvature of the manifold. Let $\{e_i : i = 1,2,\ldots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then the Ricci tensor S of type (0,2) and the scalar curvature r are given by the following:

$$S(X,Y) = \sum_{i=1}^{n} R(e_i,X,Y,e_i) \text{ and } r = \sum_{i=1}^{n} S(e_i,e_i) = \sum_{i=1}^{n} g(Qe_i,e_i),$$

where Q is the Ricci-operator i.e., $g(QX,Y) = S(X,Y)$.

Now from (2.1), we have the following:

$$\sum_{i=1}^{n} \tilde{C}(Y,Z,e_i,e_i) = 0 = \sum_{i=1}^{n} \tilde{C}(e_i,e_i,U,V) \tag{2.2}$$

and

$$\sum_{i=1}^{n} \tilde{C}(e_i,Z,U,e_i) = S(Z,U) - \frac{r}{n}g(Z,U). \tag{2.3}$$

Also from (2.1) it follows that

$$\begin{array}{ll} (i) & \tilde{C}(X,Y,Z,U) = -\tilde{C}(Y,X,Z,U), \\ (ii) & \tilde{C}(X,Y,Z,U) = -\tilde{C}(X,Y,U,Z), \\ (iii) & \tilde{C}(X,Y,Z,U) = \tilde{C}(Z,U,X,Y), \\ (iv) & \tilde{C}(X,Y,Z,U) + \tilde{C}(Y,Z,X,U) + \tilde{C}(Z,X,Y,U) = 0. \end{array} \tag{2.4}$$

Proposition 2.1. *The defining condition of a* $(W\tilde{C}S)_n$ *can always be expressed in the form of* (1.4).

Proof. Interchanging U and V in (1.3) we get

$$\begin{aligned}(\nabla_X\tilde{C})(Y,Z,V,U) &= A(X)\tilde{C}(Y,Z,V,U) + B(Y)\tilde{C}(X,Z,V,U) \\ &+C(Z)\tilde{C}(Y,X,V,U) + D(V)\tilde{C}(Y,Z,X,U) + E(U)\tilde{C}(Y,Z,V,X).\end{aligned} \tag{2.5}$$

Now, adding (1.3) and (2.5), we obtain by virtue of (2.4)(ii) that

$$\gamma(U)\tilde{C}(Y,Z,X,V) - \gamma(V)\tilde{C}(Y,Z,U,X) = 0, \tag{2.6}$$

where $\gamma(X) = D(X) - E(X)$ for all X.

If we choose a particular vector field ρ such that $\gamma(\rho) \neq 0$, then putting $U = V = \rho$ in (2.6) we get $\tilde{C}(Y,Z,X,\rho) = 0$. Again setting $V = \rho$ in (2.6) we obtain $\tilde{C}(Y,Z,U,X) = 0$ for all vector fields Y, Z, U, X which contradicts to our assumption that the concircular curvature tensor is not identically zero. Hence we must have $\gamma(X) = 0$ for all X and consequently $D(X) = E(X)$, for all X.

Also in view of (2.4)(i), in a similar manner it can be shown that $B = C$. Hence the defining condition of a $(W\tilde{C}S)_n$ can be written as (1.4). This proves the proposition.

We now consider a $(W\tilde{C}S)_n$ whose Ricci tensor is of Codazzi type. Then we have([3], [4])

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z) \quad \text{for all } X, Y, Z,$$

which yields

$$dr(X) = 0 \quad \text{for all } X. \tag{2.7}$$

Now from (2.1) it follows by virtue of Bianchi identity that

$$\begin{aligned} &(\nabla_X \tilde{C})(Y,Z,U,V) + (\nabla_Y \tilde{C})(Z,X,U,V) + (\nabla_Z \tilde{C})(X,Y,U,V) \\ &= -\frac{1}{n(n-1)}[dr(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} \\ &+dr(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\} \\ &+dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}]. \end{aligned} \tag{2.8}$$

By virtue of (2.7), (2.8) yields

$$(\nabla_X \tilde{C})(Y,Z,U,V) + (\nabla_Y \tilde{C})(Z,X,U,V) + (\nabla_Z \tilde{C})(X,Y,U,V) = 0. \tag{2.9}$$

The above results will be needed in the later sections.

3. Some basic results of $(W\tilde{C}S)_n$. This section deals with some basic results of $(W\tilde{C}S)_n$. If the scalar curvature vanishes then from (2.1) we have $\tilde{C}(Y,Z,U,V) = R(Y,Z,U,V)$ and hence $(\nabla_X \tilde{C})(Y,Z,U,V) = (\nabla_X R)(Y,Z,U,V)$. This leads to the following:

Theorem 3.1. *Every $(W\tilde{C}S)_n$ of vanishing scalar curvature is a $(WS)_n$.*

In view of (1.4), the relation (2.9) reduces to the following:

$$\alpha(X)\tilde{C}(Y,Z,U,V) + \alpha(Y)\tilde{C}(Z,X,U,V) + \alpha(Z)\tilde{C}(X,Y,U,V) = 0, \tag{3.1}$$

where $\alpha(X) = A(X) - 2B(X)$ for all X.

Setting $Y = V = e_i$ in (3.1) and taking summation over $i, 1 \le i \le n$, we get

$$\begin{aligned} &\alpha(X)[S(Z,U) - \frac{r}{n}g(Z,U)] + \alpha(\tilde{C}(Z,X)U) \\ &-\alpha(Z)[S(X,U) - \frac{r}{n}g(X,U)] = 0. \end{aligned} \tag{3.2}$$

Again putting $Z = U = e_i$ in (3.2) and taking summation over $i, 1 \le i \le n$, we get

$$\alpha(QX) = \frac{r}{n}\alpha(X), \tag{3.3}$$

that is, $S(X,L) = \frac{r}{n}g(X,L)$. This leads to the following:

Theorem 3.2. *If the Ricci tensor of a $(W\tilde{C}S)_n$ is of Codazzi type then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X,L) = \alpha(X)$ for all X.*

Next by virtue of (1.4), the relation (2.8) takes the form

$$\begin{aligned} &\alpha(X)\tilde{C}(Y,Z,U,V) + \alpha(Y)\tilde{C}(Z,X,U,V) + \alpha(Z)\tilde{C}(X,Y,U,V) \\ &= -\frac{1}{n(n-1)}[dr(X)\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\} \\ &+dr(Y)\{g(X,U)g(Z,V) - g(Z,U)g(X,V)\} \\ &+dr(Z)\{g(Y,U)g(X,V) - g(X,U)g(Y,V)\}]. \end{aligned} \tag{3.4}$$

Setting $Y = V = e_i$ in (3.4) and taking summation over $i, 1 \le i \le n$, we get

$$\begin{aligned} &\alpha(X)[S(Z,U) - \frac{r}{n}g(Z,U)] - \alpha(Z)[S(X,U) - \frac{r}{n}g(X,U)] \\ &+\alpha(\tilde{C}(Z,X)U) = -\frac{n-2}{n(n-1)}[dr(X)g(Z,U) - dr(Z)g(X,U)]. \end{aligned} \tag{3.5}$$

Putting $Z = U = e_i$ in (3.5) and taking summation over $i, 1 \leq i \leq n$, we obtain

$$dr(X) = \frac{2n}{n-2}[\alpha(QX) - \frac{r}{n}\alpha(X)]. \tag{3.6}$$

If the manifold is of constant scalar curvature then (3.6) reduces to (3.3) and hence we can state the following:

Theorem 3.3. *If a $(W\tilde{C}S)_n$ is of constant scalar curvature, then $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector L defined by $g(X, L) = \alpha(X)$ for all X.*

Again using (2.1) the equation (1.4) becomes

$$\begin{aligned}
&(\nabla_X R)(Y, Z, U, V) - \frac{dr(X)}{n(n-1)}[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)] \\
&= A(X)[R(Y, Z, U, V) - \frac{r}{n(n-1)}\{g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\}] \\
&+B(Y)[R(X, Z, U, V) - \frac{r}{n(n-1)}\{g(Z, U)g(X, V) - g(X, U)g(Z, V)\}] \\
&+B(Z)[R(Y, X, U, V) - \frac{r}{n(n-1)}\{g(X, U)g(Y, V) - g(Y, U)g(X, V)\}] \\
&+D(U)[R(Y, Z, X, V) - \frac{r}{n(n-1)}\{g(Z, X)g(Y, V) - g(Y, X)g(Z, V)\}] \\
&+D(V)[R(Y, Z, U, X) - \frac{r}{n(n-1)}\{g(Z, U)g(Y, X) - g(Y, U)g(Z, X)\}].
\end{aligned} \tag{3.7}$$

Setting $Y = V = e_i$ in (3.7) and taking summation over $i, 1 \leq i \leq n$, we have

$$\begin{aligned}
&(\nabla_X S)(Z, U) - \frac{dr(X)}{n}g(Z, U) = A(X)[S(Z, U) - \frac{r}{n}g(Z, U)] \\
&+B(R(X, Z)U) - \frac{r}{n(n-1)}\{B(X)g(Z, U) - B(Z)g(X, U)\} \\
&+B(Z)[S(X, U) - \frac{r}{n}g(X, U)] + D(U)[S(Z, X) - \frac{r}{n}g(Z, X)] \\
&+D(R(X, U)Z) - \frac{r}{n(n-1)}\{D(X)g(Z, U) - D(U)g(Z, X)\}.
\end{aligned} \tag{3.8}$$

Also putting $Z = U = e_i$ in (3.8) and taking summation over $i, 1 \leq i \leq n$, we obtain

$$T(QX) = \frac{r}{n}T(X), \tag{3.9}$$

which can be written as

$$S(X,\rho_2) = \frac{r}{n} g(X,\rho_2), \tag{3.10}$$

where $g(X,\rho_2) = T(X) = B(X) + D(X)$ for all X.

This leads to the following:

Theorem 3.4. *In a $(W\tilde{C}S)_n$, $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_2 defined by $g(X,\rho_2) = T(X)$ for all X.*

4. Decomposable $(W\tilde{C}S)_n$. A Riemannian manifold (M^n, g) is said to be decomposable [7] if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \le p \le n-2$, that is, in some coordinate neighbourhood of the Riemannian manifold (M^n, g), the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + \overset{*}{g}_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{4.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, \cdots, x^p (p < n)$ denoted by $\bar{x}$ and $\overset{*}{g}_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, \cdots, x^n$ denoted by $\overset{*}{x}$; $a, b, c, \cdots$ run from 1 to p and $\alpha, \beta, \gamma, \cdots$ run from $p+1$ to n. The two parts of (4.1) are the metrics of $M_1^p (p \ge 2)$ and $M_2^{n-p} (n-p \ge 2)$ which are called the decomposition of the manifold $M^n = M_1^p \times M_2^{n-p} (2 \le p \le n-2)$.

Let (M^n, g) be a Riemannian manifold such that $M_1^p \times M_2^{n-p}$ for $2 \le p \le n-2$. Here throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by a 'star' is assumed to be from M_2.

Let $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{U}$, $\bar{V} \in \chi(M_1)$ and $\overset{*}{X}$, $\overset{*}{Y}$, $\overset{*}{Z}$, $\overset{*}{U}$, $\overset{*}{V} \in \chi(M_2)$, then we have the following relations:

$$\begin{aligned}
&R(\overset{*}{X}, \bar{Y}, \bar{Z}, \bar{U}) = 0 = R(\bar{X}, \overset{*}{Y}, \bar{Z}, \overset{*}{U}) = R(\bar{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),\\
&(\nabla_{\overset{*}{X}} R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}} R)(\bar{Y}, \overset{*}{Z}, \bar{U}, \overset{*}{V}) = (\nabla_{\overset{*}{X}} R)(\bar{Y}, \overset{*}{Z}, \bar{U}, \overset{*}{V}),\\
&R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}); R(\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}) = \overset{*}{R} (\overset{*}{X}, \overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}),\\
&S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}); S(\overset{*}{X}, \overset{*}{Y}) = \overset{*}{S} (\overset{*}{X}, \overset{*}{Y}),\\
&(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}); (\nabla_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}) = (\overset{*}{\nabla}_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{Z}),\\
&\text{and } r = \bar{r} + \overset{*}{r},
\end{aligned}$$

where r, $\bar{r}$, and $\overset{*}{r}$ are the scalar curvature of M, M_1, M_2 respectively. Let us consider a Riemannian manifold (M^n, g) which is decomposable $(W\tilde{C}S)_n$. Then $M^n = M_1^p \times M_2^{n-p}$, $(2 \leq p \leq n-2)$.

Now from (2.1), we have

$$\tilde{C}(\overset{*}{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = \tilde{C}(\bar{Y}, \overset{*}{Z}, \overset{*}{U}, \overset{*}{V}), \tag{4.2}$$

$$\tilde{C}(\overset{*}{Y}, \bar{Z}, \bar{U}, \overset{*}{V}) = -\frac{r}{n(n-1)} g(\bar{Z}, \bar{U}) g(\overset{*}{Y}, \overset{*}{V}), \tag{4.3}$$

$$\tilde{C}(\overset{*}{Y}, \overset{*}{Z}, \bar{U}, \bar{V}) = 0 = \tilde{C}(\bar{Y}, \bar{Z}, \overset{*}{U}, \overset{*}{V}) \tag{4.4}$$

and

$$\tilde{C}(\overset{*}{Y}, \bar{Z}, \overset{*}{U}, \bar{V}) = \frac{r}{n(n-1)} g(\overset{*}{Y}, \overset{*}{U}) g(\bar{Z}, \bar{V}). \tag{4.5}$$

Again from (1.4), we have

$$\begin{aligned}(\nabla_{\bar{X}} \tilde{C})(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) &= A(\bar{X}) \tilde{C}(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) + B(\bar{Y}) \tilde{C}(\bar{X}, \bar{Z}, \bar{U}, \bar{V}) \\ &+ B(\bar{Z}) \tilde{C}(\bar{Y}, \bar{X}, \bar{U}, \bar{V}) + D(\bar{U}) \tilde{C}(\bar{Y}, \bar{Z}, \bar{X}, \bar{V}) + D(\bar{V}) \tilde{C}(\bar{Y}, \bar{Z}, \bar{U}, \bar{X}).\end{aligned} \tag{4.6}$$

Changing $\bar{X}$ by $\overset{*}{X}$ in (4.6) we get

$$A(\overset{*}{X}) \tilde{C}(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0. \tag{4.7}$$

Similarly we have

$$B(\overset{*}{Y}) \tilde{C}(\bar{X}, \bar{Z}, \bar{U}, \bar{V}) = 0, \tag{4.8}$$

$$D(\overset{*}{U}) \tilde{C}(\bar{Y}, \bar{Z}, \bar{X}, \bar{V}) = 0. \tag{4.9}$$

Now putting $\bar{X} = \overset{*}{X}$, $\bar{Y} = \overset{*}{Y}$ in (4.6) we get

$$r[D(\bar{U}) g(\bar{Z}, \bar{V}) - D(\bar{V}) g(\bar{Z}, \bar{U})] = 0. \tag{4.10}$$

Similarly putting $\bar{X} = \overset{*}{X}$, $\bar{U} = \overset{*}{U}$ in (4.6) we obtain

$$r[B(\bar{Y})g(\bar{Z},\bar{V}) - B(\bar{Z})g(\bar{Y},\bar{V})] = 0. \tag{4.11}$$

Also putting $\bar{Y} = \overset{*}{Y}$, $\bar{Z} = \overset{*}{Z}$, $\bar{U} = \overset{*}{U}$ in (4.6) we have

$$r[B(\overset{*}{Z})g(\overset{*}{Y},\overset{*}{U}) - B(\overset{*}{Y})g(\overset{*}{Z},\overset{*}{U})] = 0. \tag{4.12}$$

In the similar way, from (4.6), we have the following:

$$r[D(\overset{*}{U})g(\overset{*}{Z},\overset{*}{V}) - D(\overset{*}{V})g(\overset{*}{Z},\overset{*}{U})] = 0. \tag{4.13}$$

Also from (1.4), we get

$$\begin{aligned}(\nabla_{\overset{*}{X}}\tilde{C})(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) &= A(\overset{*}{X})\tilde{C}(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) + B(\overset{*}{Y})\tilde{C}(\overset{*}{X},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) \\ &+B(\overset{*}{Z})\tilde{C}(\overset{*}{Y},\overset{*}{X},\overset{*}{U},\overset{*}{V}) + D(\overset{*}{U})\tilde{C}(\overset{*}{Y},\overset{*}{Z},\overset{*}{X},\overset{*}{V}) + D(\overset{*}{V})\tilde{C}(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{X}).\end{aligned} \tag{4.14}$$

From (4.14), it follows that

$$A(\bar{X})\tilde{C}(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0, \tag{4.15}$$

$$B(\bar{Y})\tilde{C}(\overset{*}{X},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0, \tag{4.16}$$

$$D(\bar{U})\tilde{C}(\overset{*}{Y},\overset{*}{Z},\overset{*}{X},\overset{*}{V}) = 0. \tag{4.17}$$

From (4.7) — (4.9) we have two cases, namely,

(I) $A = B = D = 0$ on M_2,

(II) M_1 is concircularly flat.

At first we consider the case (I). Then from (4.14) we have

$$(\nabla_{\overset{*}{X}}\tilde{C})(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) = 0,$$

that is,

$$(\nabla_{\overset{*}{X}}R)(\overset{*}{Y},\overset{*}{Z},\overset{*}{U},\overset{*}{V}) - \frac{d\,\overset{*}{r}}{n(n-1)}[g(\overset{*}{Z},\overset{*}{U})g(\overset{*}{Y},\overset{*}{V}) - g(\overset{*}{Y},\overset{*}{U})g(\overset{*}{Z},\overset{*}{V})] = 0. \tag{4.18}$$

Setting $\overset{*}{Z}=\overset{*}{U}=\overset{*}{e}_\alpha$ in (4.18) and taking summation over α, $p+1 \leq \alpha \leq n$, we obtain

$$(4.19) \qquad (\nabla_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{V}) = \frac{n-p-1}{n(n-1)} d \overset{*}{r} (\overset{*}{X}) g(\overset{*}{Y}, \overset{*}{V}),$$

since $r = \bar{r} + \overset{*}{r}$ and $d\bar{r}(\overset{*}{X}) = 0$. Hence from (4.19) we have

$$(\nabla_{\overset{*}{X}} S)(\overset{*}{Y}, \overset{*}{V}) = 0,$$

provided that M_2 is of constant scalar curvature. This implies that M_2 is Ricci symmetric manifold if it is of constant scalar curvature.

Secondly, we discuss the case of (II). Since M_1 is concircularly flat. Therefore, it is a manifold of constant curvature. Hence we can state the following:

Theorem 4.1. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(W\tilde{C}S)_n$ then the following holds:*
(I) In the case of $A = B = D = 0$ on M_2, the manifold M_2 is Ricci symmetric, provided that M_2 is of constant scalar curvature.
(II) When M_1 is concircularly flat, it is a manifold of constant curvature.

Similarly we have from (4.15) — (4.17) that

Theorem 4.2. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(W\tilde{C}S)_n$ then the following holds:*
(I) In the case of $A = B = D = 0$ on M_1, the manifold M_1 is Ricci symmetric, provided that M_1 is of constant scalar curvature.
(II) When M_2 is concircularly flat, it is a manifold of constant curvature.

Next we consider the contraction with respect to $\bar{Z}$ and $\bar{V}$ in (4.10) and obtain

$$(4.20) \qquad rD(\bar{U}) = 0, \text{ since } p \geq 2.$$

If r is non-vanishing then (4.20) yields

$$(4.21) \qquad D(\bar{U}) = 0 \text{ for all } \bar{U} \in \chi(M_1).$$

Similarly from (4.11) we have

$$B(\bar{Y}) = 0 \text{ for all } \bar{Y} \in \chi(M_1), \tag{4.22}$$

provided r is non-vanishing. Thus if $r \neq 0$ then from (4.21) and (4.22) we have $B = 0$ and $D = 0$ on M_1 and hence from (4.6) we get

$$(\nabla_{\bar{X}}\tilde{C})(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = A(\bar{X})\tilde{C}(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}). \tag{4.23}$$

Again if we consider the contraction with respect to $\overset{*}{Y}$ and $\overset{*}{U}$ in (4.12) then obtain

$$rB(\overset{*}{Z}) = 0, \text{ since } n - p \geq 2. \tag{4.24}$$

If r is non-zero then from (4.24) we have

$$B(\overset{*}{Z}) = 0 \text{ for all } \overset{*}{Z} \in \chi(M_2). \tag{4.25}$$

Similarly if $r \neq 0$ from (4.13) we get

$$D(\overset{*}{U}) = 0 \text{ for all } \overset{*}{U} \in \chi(M_2) \tag{4.26}$$

and hence from (4.14) we get

$$(\nabla_{\overset{*}{X}}\tilde{C})(\overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}, \overset{*}{V}) = A(\overset{*}{X})\tilde{C}(\overset{*}{Y}, \overset{*}{Z}, \overset{*}{U}, \overset{*}{V}). \tag{4.27}$$

Thus from (4.23) and (4.27) we can state the following:

Theorem 4.3. *Let (M^n, g) be a Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(W\tilde{C}S)_n$ with non-vanishing scalar curvature, then both the decompositions are concircularly recurrent.*

5. Some examples of $(W\tilde{C}S)_n$

Example 5.1. Let M^4 be an open subset of R^4 endowed with the metric

$$\begin{aligned} ds^2 = g_{ij}dx^i dx^j &= f(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2, \\ &(i, j = 1, 2, 3, 4), \end{aligned} \tag{5.1}$$

where $f = a_0 + a_1x^3 + e^{x^1}\{\frac{(x^3)^2}{2} - \frac{(x^3)^3}{6} + \frac{(x^3)^4}{12} - \cdots + \frac{(-1)^{n-1}(x^3)^{n+1}}{n(n+1)} + \cdots\}$, a_0, a_1 are non-constant functions of x^1 only and $-1 < x^3 \leq 1$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, scalar curvature, concircular curvature tensor and its covariant derivatives are given by

$$\Gamma^2_{11} = \frac{1}{2}f_{.1}, \Gamma^2_{13} = \frac{1}{2}f_{.3} = -\Gamma^3_{11},$$
$$R_{1331} = \frac{1}{2}f_{.33} = \frac{e^{x^1}}{2(1+x^3)} \neq 0,$$
$$r = 0, \tilde{C}_{1331} = \frac{1}{2}f_{.33} = \frac{e^{x^1}}{2(1+x^3)} \neq 0,$$

$$\tilde{C}_{1331,1} = \frac{1}{2}f_{.331} = \frac{e^{x^1}}{2(1+x^3)} \neq 0, \tag{5.2}$$

$$\tilde{C}_{1331,3} = \frac{1}{2}f_{.333} = -\frac{e^{x^1}}{2(1+x^3)^2} \neq 0 \tag{5.3}$$

and the components that can be obtained from these by the symmetry properties, where '.' denotes the partial differentiation with respect to the coordinates, ',' denotes the covariant differentiation with respect to the metric tensor g and r is the scalar curvature of the manifold whose value is zero here. Therefore, our M^4 with the considered metric is a Riemannian manifold of vanishing scalar curvature.

We shall now show that this M^4 is a $(W\tilde{C}S)_4$, that is, it satisfies (1.4). In terms of local coordinate system, let us consider the 1-forms A, B, D as follows:

$$\begin{aligned} A(\partial_i) = A_i &= \frac{1}{6} \text{ for } \quad i = 1, \\ &= -\frac{5}{1+x^3} \text{ for } \quad i = 3, \\ &= 0 \text{ otherwise }, \\ B(\partial_i) = B_i &= \frac{1}{3} \text{ for } \quad i = 1, \\ &= \frac{3}{1+x^3} \text{ for } \quad i = 3, \\ &= 0 \text{ otherwise }, \\ D(\partial_i) = D_i &= \frac{1}{2} \text{ for } \quad i = 1, \end{aligned} \tag{5.4}$$

$$= \frac{1}{1+x^3} \text{ for } \quad i=3,$$
$$= 0 \text{ otherwise },$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

In terms of local coordinate system, the defining condition (1.4) of a $(W\tilde{C}S)_n$ can be written as

$$\tilde{C}_{ijkl,p} = A_p\tilde{C}_{ijkl} + B_i\tilde{C}_{pjkl} + B_j\tilde{C}_{ipkl} + D_k\tilde{C}_{ijpl} + D_l\tilde{C}_{ijkp}, \tag{5.5}$$

which reduces with these 1-forms to the following equations:

$$\tilde{C}_{1331,i} = A_i\tilde{C}_{1331} + B_1\tilde{C}_{i331} + B_3\tilde{C}_{1i31} + D_3\tilde{C}_{13i1} + D_1\tilde{C}_{133i}, \tag{5.6}$$
$$\tilde{C}_{1332,i} = A_i\tilde{C}_{1332} + B_1\tilde{C}_{i332} + B_3\tilde{C}_{1i32} + D_3\tilde{C}_{13i2} + D_2\tilde{C}_{133i}, \tag{5.7}$$
$$\tilde{C}_{1334,i} = A_i\tilde{C}_{1334} + B_1\tilde{C}_{i334} + B_3\tilde{C}_{1i34} + D_3\tilde{C}_{13i4} + D_4\tilde{C}_{133i}, \tag{5.8}$$
$$\tilde{C}_{3112,i} = A_i\tilde{C}_{3112} + B_3\tilde{C}_{i112} + B_1\tilde{C}_{3i12} + D_1\tilde{C}_{31i2} + D_2\tilde{C}_{311i}, \tag{5.9}$$
$$\tilde{C}_{3114,i} = A_i\tilde{C}_{3114} + B_3\tilde{C}_{i114} + B_1\tilde{C}_{3i14} + D_1\tilde{C}_{31i4} + D_4\tilde{C}_{311i}, \tag{5.10}$$

where $i = 1, 2, 3, 4$, since for the cases other than (5.5) and the components of each term of (5.5) either vanishes identically or the relation (5.5) holds trivially using the skew-symmetry property of $\tilde{C}$.

Now, from (5.4) and (5.2), it follows that, for $i = 1$, right hand side of (5.6) $= (A_1 + B_1 + D_1)\tilde{C}_{1331} = \frac{e^{x^1}}{2(1+x^3)} = \tilde{C}_{1331,1} =$ left hand side of (5.6). Proceeding similarly it can be easily shown that the relation (5.6) holds for $i = 2, 3, 4$. By the similar argument it can be easily shown that the relation (5.7) – (5.10) hold. Thus, the manifold under consideration is weakly concircular symmetric manifold. Hence we can state the following:

Theorem 5.1. *Let* (M^4, g) *be a Riemannian manifold endowed with the metric given in* (5.1). *Then* (M^4, g) *is a weakly concircular symmetric manifold with vanishing scalar curvature which is neither concircularly symmetric nor concircularly recurrent.*

Note. It can be easily shown that the above Riemannian manifold (M^4, g) endowed with the metric given in (5.1) is a weakly symmetric manifold with vanishing scalar curvature which is neither symmetric nor recurrent. This result verifies the Theorem 3.1.

Let (M_1^4, g_1) be a Riemannian manifold in Example 5.1, where g_1 is the metric given in (5.1). Let (R^{n-4}, g_0) be an $(n-4)$ dimensional Euclidean manifold with the flat metric g_0. Then (M^n, g) is a product manifold of (M_1^4, g_1) and (R^{n-4}, g_0). Thus we can state the following:

Theorem 5.2. *Let $(M^n, g)(n \geq 4)$ be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = f(dx^1)^2 + 2dx^1dx^2 + \sum_{k=3}^{n}(dx^k)^2, (i, j = 1, 2, \cdots, n),$$

where $f = a_0 + a_1x^3 + e^{x^1}\{\frac{(x^3)^2}{2} - \frac{(x^3)^3}{6} + \frac{(x^3)^4}{12} - \cdots + \frac{(-1)^{n-1}(x^3)^{n+1}}{n(n+1)} + \cdots\}$, a_0, a_1 are non-constant functions of x^1 only and $-1 < x^3 \leq 1$. Then $(M^n, g)(n \geq 4)$ is a decomposable weakly concircular symmetric manifold $(M_1^4, g_1) \times (R^{n-4}, g_0)$ with vanishing scalar curvature which is neither concircularly symmetric nor concircularly recurrent.

Example 5.2. Let M be an open subset of R^n endowed with the metric

$$ds^2 = x^3e^{x^1}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + x^4(dx^4)^2 + \sum_{k=5}^{n}(dx^k)^2, \tag{5.11}$$

where x^3 is non-zero finite and $x^4 \neq 0$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, scalar curvature, concircular curvature tensor and its covariant derivatives are given by

$$\begin{aligned}
&\Gamma^1_{11} = \frac{1}{2}, \Gamma^3_{11} = -\frac{1}{2}e^{x^1}, \Gamma^1_{13} = \frac{1}{2x^3}, \Gamma^4_{44} = \frac{1}{2x^4},\\
&R_{1331} = -\frac{1}{4x^3}e^{x^1} \neq 0, r = -\frac{1}{2(x^3)^2} \neq 0,\\
&\tilde{C}_{1221} = \frac{e^{x^1}}{2n(n-1)x^3} = \tilde{C}_{1pp1}, \tilde{C}_{1331} = -\frac{(n-2)(n+1)e^{x^1}}{4n(n-1)x^3},\\
&\tilde{C}_{1441} = \frac{x^4e^{x^1}}{2n(n-1)x^3}, \tilde{C}_{2442} = \frac{x^4}{2n(n-1)(x^3)^2} = \tilde{C}_{3443} = \tilde{C}_{4pp4},\\
&\tilde{C}_{2332} = \frac{1}{2n(n-1)(x^3)^2} = \tilde{C}_{2pp2} = \tilde{C}_{3pp3} = \tilde{C}_{pqqp},
\end{aligned}$$

$$(5.12)\qquad \tilde{C}_{1221,3} = -\frac{e^{x^1}}{n(n-1)(x^3)^2} = \tilde{C}_{1pp1,3},$$

$$(5.13)\qquad \tilde{C}_{1331,3} = \frac{(n-2)(n+1)e^{x^1}}{2n(n-1)(x^3)^2},$$

$$(5.14)\qquad \tilde{C}_{1441,3} = -\frac{x^4 e^{x^1}}{n(n-1)(x^3)^2},$$

$$(5.15)\qquad \tilde{C}_{2332,3} = -\frac{1}{n(n-1)(x^3)^3} = \tilde{C}_{2pp2,3} = \tilde{C}_{3pp3,3} = \tilde{C}_{pqqp,3},$$

$$(5.16)\qquad \tilde{C}_{2442,3} = -\frac{x^4}{n(n-1)(x^3)^3} = \tilde{C}_{3443,3} = \tilde{C}_{4pp4,3},$$

$$\text{for } 5 \le p \le n, 5 \le q \le n, p \ne q,$$

and the components that can be obtained from these by the symmetry properties, where ',' denotes the covariant differentiation with respect to the metric tensor g. From the above relations, it follows that the manifold (M^n, g) under consideration is a Riemannian manifold of non-vanishing scalar curvature which is not concircularly flat.

We shall now show that this (M^n, g) is a $(W\tilde{C}S)_n$, that is, it satisfies (1.4).

In terms of local coordinate system, we consider the components of the 1-forms A, B, D as follows:

$$(5.17)\qquad \begin{aligned} A(\partial_i) = A_i &= -\frac{2}{x^3} \text{ for } \quad i = 3, \\ &= 0 \text{ otherwise }, \\ B(\partial_i) = B_i &= 0 \text{ for } \quad i = 1, 2, \cdots, n, \\ D(\partial_i) = D_i &= 0 \text{ for } \quad i = 1, 2, \cdots, n, \end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x_i}$.
In terms of local coordinate system, the defining condition (1.4) of a $(W\tilde{C}S)_n$ can be written as (5.5), which reduces with these 1-forms to the following equations:

$$(5.18)\qquad \tilde{C}_{1221,i} = A_i\tilde{C}_{1221} + B_1\tilde{C}_{i221} + B_2\tilde{C}_{1i21} + D_2\tilde{C}_{12i1} + D_1\tilde{C}_{122i} \quad ,$$

$$(5.19)\qquad \tilde{C}_{1331,i} = A_i\tilde{C}_{1331} + B_1\tilde{C}_{i331} + B_3\tilde{C}_{1i31} + D_3\tilde{C}_{13i1} + D_1\tilde{C}_{133i} \quad ,$$

$$\tilde{C}_{1441,i} = A_i\tilde{C}_{1441} + B_1\tilde{C}_{i441} + B_4\tilde{C}_{1i41} + D_4\tilde{C}_{14i1} + D_1\tilde{C}_{144i} \quad , \tag{5.20}$$

$$\tilde{C}_{2332,i} = A_i\tilde{C}_{2332} + B_2\tilde{C}_{i332} + B_3\tilde{C}_{2i32} + D_3\tilde{C}_{23i2} + D_2\tilde{C}_{233i} \quad , \tag{5.21}$$

$$\tilde{C}_{2442,i} = A_i\tilde{C}_{2442} + B_2\tilde{C}_{i442} + B_4\tilde{C}_{2i42} + D_4\tilde{C}_{24i2} + D_2\tilde{C}_{244i} \quad , \tag{5.22}$$

$$\tilde{C}_{3443,i} = A_i\tilde{C}_{3443} + B_3\tilde{C}_{i443} + B_4\tilde{C}_{3i43} + D_4\tilde{C}_{34i3} + D_3\tilde{C}_{344i}, \tag{5.23}$$

$$\tilde{C}_{1pp1,i} = A_i\tilde{C}_{1pp1} + B_1\tilde{C}_{ipp1} + B_p\tilde{C}_{1ip1} + D_p\tilde{C}_{1pi1} + D_1\tilde{C}_{1ppi} \quad , \tag{5.24}$$

$$\tilde{C}_{2pp2,i} = A_i\tilde{C}_{2pp2} + B_2\tilde{C}_{ipp2} + B_p\tilde{C}_{2ip2} + D_p\tilde{C}_{2pi2} + D_2\tilde{C}_{2ppi} \quad , \tag{5.25}$$

$$\tilde{C}_{3pp3,i} = A_i\tilde{C}_{3pp3} + B_3\tilde{C}_{ipp3} + B_p\tilde{C}_{3ip3} + D_p\tilde{C}_{3pi3} + D_3\tilde{C}_{3ppi}, \tag{5.26}$$

$$\tilde{C}_{4pp4,i} = A_i\tilde{C}_{4pp4} + B_4\tilde{C}_{ipp4} + B_p\tilde{C}_{4ip4} + D_p\tilde{C}_{4pi4} + D_4\tilde{C}_{4ppi}, \tag{5.27}$$

$$\tilde{C}_{pqqp,i} = A_i\tilde{C}_{pqqp} + B_p\tilde{C}_{iqqp} + B_q\tilde{C}_{piqp} + D_q\tilde{C}_{pqip} + D_p\tilde{C}_{pqqi}, \tag{5.28}$$

$$\tilde{C}_{1223,i} = A_i\tilde{C}_{1223} + B_1\tilde{C}_{i223} + B_2\tilde{C}_{1i23} + D_2\tilde{C}_{12i3} + D_3\tilde{C}_{122i}, \tag{5.29}$$

$$\tilde{C}_{1224,i} = A_i\tilde{C}_{1224} + B_1\tilde{C}_{i224} + B_2\tilde{C}_{1i24} + D_2\tilde{C}_{12i4} + D_4\tilde{C}_{122i}, \tag{5.30}$$

$$\tilde{C}_{122p,i} = A_i\tilde{C}_{122p} + B_1\tilde{C}_{i22p} + B_2\tilde{C}_{1i2p} + D_2\tilde{C}_{12ip} + D_p\tilde{C}_{122i}, \tag{5.31}$$

$$\tilde{C}_{2113,i} = A_i\tilde{C}_{2113} + B_2\tilde{C}_{i113} + B_1\tilde{C}_{2i13} + D_1\tilde{C}_{21i3} + D_3\tilde{C}_{211i}, \tag{5.32}$$

$$\tilde{C}_{2114,i} = A_i\tilde{C}_{2114} + B_2\tilde{C}_{i114} + B_1\tilde{C}_{2i14} + D_1\tilde{C}_{21i4} + D_4\tilde{C}_{211i}, \tag{5.33}$$

$$\tilde{C}_{211p,i} = A_i\tilde{C}_{211p} + B_2\tilde{C}_{i11p} + B_1\tilde{C}_{2i1p} + D_1\tilde{C}_{21ip} + D_p\tilde{C}_{211i}, \tag{5.34}$$

$$\tilde{C}_{1332,i} = A_i\tilde{C}_{1332} + B_1\tilde{C}_{i332} + B_3\tilde{C}_{1i32} + D_3\tilde{C}_{13i2} + D_2\tilde{C}_{133i}, \tag{5.35}$$

$$\tilde{C}_{1334,i} = A_i\tilde{C}_{1334} + B_1\tilde{C}_{i334} + B_3\tilde{C}_{1i34} + D_3\tilde{C}_{13i4} + D_4\tilde{C}_{133i}, \tag{5.36}$$

$$\tilde{C}_{133p,i} = A_i\tilde{C}_{133p} + B_1\tilde{C}_{i33p} + B_3\tilde{C}_{1i3p} + D_3\tilde{C}_{13ip} + D_p\tilde{C}_{133i}, \tag{5.37}$$

$$\tilde{C}_{3114,i} = A_i\tilde{C}_{3114} + B_3\tilde{C}_{i114} + B_1\tilde{C}_{3i14} + D_1\tilde{C}_{31i4} + D_4\tilde{C}_{311i}, \tag{5.38}$$

$$\tilde{C}_{311p,i} = A_i\tilde{C}_{311p} + B_3\tilde{C}_{i11p} + B_1\tilde{C}_{3i1p} + D_1\tilde{C}_{31ip} + D_p\tilde{C}_{311i}, \tag{5.39}$$

$$\tilde{C}_{1442,i} = A_i\tilde{C}_{1442} + B_1\tilde{C}_{i442} + B_4\tilde{C}_{1i42} + D_4\tilde{C}_{14i2} + D_2\tilde{C}_{144i}, \tag{5.40}$$

$$\tilde{C}_{1443,i} = A_i\tilde{C}_{1443} + B_1\tilde{C}_{i443} + B_4\tilde{C}_{1i43} + D_4\tilde{C}_{14i3} + D_3\tilde{C}_{144i}, \tag{5.41}$$

$$\tilde{C}_{144p,i} = A_i\tilde{C}_{144p} + B_1\tilde{C}_{i44p} + B_4\tilde{C}_{1i4p} + D_4\tilde{C}_{14ip} + D_p\tilde{C}_{144i}, \tag{5.42}$$

$$\tilde{C}_{411p,i} = A_i\tilde{C}_{411p} + B_4\tilde{C}_{i11p} + B_1\tilde{C}_{4i1p} + D_1\tilde{C}_{41ip} + D_p\tilde{C}_{411i}, \tag{5.43}$$

$$\tilde{C}_{2334,i} = A_i\tilde{C}_{2334} + B_2\tilde{C}_{i334} + B_3\tilde{C}_{2i34} + D_3\tilde{C}_{23i4} + D_4\tilde{C}_{233i}, \tag{5.44}$$

$$\tilde{C}_{233p,i} = A_i\tilde{C}_{233p} + B_2\tilde{C}_{i33p} + B_3\tilde{C}_{2i3p} + D_3\tilde{C}_{23ip} + D_p\tilde{C}_{233i}, \tag{5.45}$$

$$\tilde{C}_{3224,i} = A_i\tilde{C}_{3224} + B_3\tilde{C}_{i224} + B_2\tilde{C}_{3i24} + D_2\tilde{C}_{32i4} + D_4\tilde{C}_{322i}, \tag{5.46}$$

$$\tilde{C}_{322p,i} = A_i\tilde{C}_{322p} + B_3\tilde{C}_{i22p} + B_2\tilde{C}_{3i2p} + D_2\tilde{C}_{32ip} + D_p\tilde{C}_{322i}, \tag{5.47}$$

$$\tilde{C}_{2443,i} = A_i\tilde{C}_{2443} + B_2\tilde{C}_{i443} + B_4\tilde{C}_{2i43} + D_4\tilde{C}_{24i3} + D_3\tilde{C}_{244i}, \tag{5.48}$$

$$\tilde{C}_{244p,i} = A_i\tilde{C}_{244p} + B_2\tilde{C}_{i44p} + B_4\tilde{C}_{2i4p} + D_4\tilde{C}_{24ip} + D_p\tilde{C}_{244i}, \tag{5.49}$$

$$\tilde{C}_{422p,i} = A_i\tilde{C}_{422p} + B_4\tilde{C}_{i22p} + B_2\tilde{C}_{4i2p} + D_2\tilde{C}_{42ip} + D_p\tilde{C}_{422i}, \tag{5.50}$$

$$\tilde{C}_{344p,i} = A_i\tilde{C}_{344p} + B_3\tilde{C}_{i44p} + B_4\tilde{C}_{3i4p} + D_4\tilde{C}_{34ip} + D_p\tilde{C}_{344i}, \tag{5.51}$$

$$\tilde{C}_{433p,i} = A_i\tilde{C}_{433p} + B_4\tilde{C}_{i33p} + B_3\tilde{C}_{4i3p} + D_3\tilde{C}_{43ip} + D_p\tilde{C}_{433i}, \tag{5.52}$$

$$\tilde{C}_{1pp2,i} = A_i\tilde{C}_{1pp2} + B_1\tilde{C}_{ipp2} + B_p\tilde{C}_{1ip2} + D_p\tilde{C}_{1pi2} + D_2\tilde{C}_{1ppi}, \tag{5.53}$$

$$\tilde{C}_{1pp3,i} = A_i\tilde{C}_{1pp3} + B_1\tilde{C}_{ipp3} + B_p\tilde{C}_{1ip3} + D_p\tilde{C}_{1pi3} + D_3\tilde{C}_{1ppi}, \tag{5.54}$$

$$\tilde{C}_{1pp4,i} = A_i\tilde{C}_{1pp4} + B_1\tilde{C}_{ipp4} + B_p\tilde{C}_{1ip4} + D_p\tilde{C}_{1pi4} + D_4\tilde{C}_{1ppi}, \tag{5.55}$$

$$\tilde{C}_{1ppq,i} = A_i\tilde{C}_{1ppq} + B_1\tilde{C}_{ippq} + B_p\tilde{C}_{1ipq} + D_p\tilde{C}_{1piq} + D_q\tilde{C}_{1ppi}, \tag{5.56}$$

$$\tilde{C}_{p11q,i} = A_i\tilde{C}_{p11q} + B_p\tilde{C}_{i11q} + B_1\tilde{C}_{pi1q} + D_1\tilde{C}_{p1iq} + D_q\tilde{C}_{p11i}, \tag{5.57}$$

$$\tilde{C}_{2pp3,i} = A_i\tilde{C}_{2pp3} + B_2\tilde{C}_{ipp3} + B_p\tilde{C}_{2ik3} + D_p\tilde{C}_{2pi3} + D_3\tilde{C}_{2ppi}, \tag{5.58}$$

$$\tilde{C}_{2pp4,i} = A_i\tilde{C}_{2pp4} + B_2\tilde{C}_{ipp4} + B_p\tilde{C}_{2ip4} + D_p\tilde{C}_{2pi4} + D_4\tilde{C}_{2ppi}, \tag{5.59}$$

$$\tilde{C}_{2ppq,i} = A_i\tilde{C}_{2ppq} + B_2\tilde{C}_{ippq} + B_p\tilde{C}_{2ipq} + D_p\tilde{C}_{2piq} + D_q\tilde{C}_{2ppi}, \tag{5.60}$$

$$\tilde{C}_{p22q,i} = A_i\tilde{C}_{p22q} + B_p\tilde{C}_{i22q} + B_2\tilde{C}_{pi2q} + D_2\tilde{C}_{p2iq} + D_q\tilde{C}_{p22i}, \tag{5.61}$$

$$\tilde{C}_{3pp4,i} = A_i\tilde{C}_{3pp4} + B_3\tilde{C}_{ipp4} + B_p\tilde{C}_{3ip4} + D_p\tilde{C}_{3pi4} + D_4\tilde{C}_{3ppi}, \tag{5.62}$$

$$\tilde{C}_{3ppq,i} = A_i\tilde{C}_{3ppq} + B_3\tilde{C}_{ippq} + B_p\tilde{C}_{3ipq} + D_p\tilde{C}_{3piq} + D_q\tilde{C}_{3ppi}, \tag{5.63}$$

$$\tilde{C}_{p33q,i} = A_i\tilde{C}_{p33q} + B_p\tilde{C}_{i33q} + B_3\tilde{C}_{pi3q} + D_3\tilde{C}_{p3iq} + D_q\tilde{C}_{p33i}, \tag{5.64}$$

$$\tilde{C}_{4ppq,i} = A_i\tilde{C}_{4ppq} + B_4\tilde{C}_{ippq} + B_p\tilde{C}_{4ipq} + D_p\tilde{C}_{4piq} + D_q\tilde{C}_{4ppi}, \tag{5.65}$$

$$\tilde{C}_{p44q,i} = A_i\tilde{C}_{p44q} + B_p\tilde{C}_{i44q} + B_4\tilde{C}_{pi4q} + D_4\tilde{C}_{p4iq} + D_q\tilde{C}_{p44i}, \tag{5.66}$$

$$\tilde{C}_{qpp1,i} = A_i\tilde{C}_{qpp1} + B_q\tilde{C}_{ipp1} + B_p\tilde{C}_{qip1} + D_p\tilde{C}_{qpi1} + D_1\tilde{C}_{qppi}, \tag{5.67}$$

$$\tilde{C}_{qpp2,i} = A_i\tilde{C}_{qpp2} + B_q\tilde{C}_{ipp2} + B_p\tilde{C}_{qip2} + D_p\tilde{C}_{qpi2} + D_2\tilde{C}_{qppi}, \tag{5.68}$$

$$\tilde{C}_{qpp3,i} = A_i\tilde{C}_{qpp3} + B_q\tilde{C}_{ipp3} + B_p\tilde{C}_{qip3} + D_p\tilde{C}_{qpi3} + D_3\tilde{C}_{qppi}, \tag{5.69}$$

$$\tilde{C}_{qpp4,i} = A_i\tilde{C}_{qpp4} + B_q\tilde{C}_{ipp4} + B_p\tilde{C}_{qip4} + D_p\tilde{C}_{qpi4} + D_4\tilde{C}_{qppi}, \tag{5.70}$$

$$\tilde{C}_{qppm,i} = A_i\tilde{C}_{qppm} + B_q\tilde{C}_{ippm} + B_p\tilde{C}_{qipm} + D_p\tilde{C}_{qpim} + D_m\tilde{C}_{qppi}, \tag{5.71}$$

for $5 \leq p \leq n, 5 \leq q \leq n, 5 \leq m \leq n, p \neq q \neq m$, where $i = 1, 2, \cdots, n$, since for the cases other than (5.18)–(5.71), the components of each term of (5.5) either vanishes identically or the relation (5.5) holds trivially using the skew-symmetry property of $\tilde{C}$. Now, from (5.12) and (5.17), it follows, for $i = 3$, that right hand side of (5.18) $= A_3\tilde{C}_{1221} = -\frac{e^{x^1}}{n(n-1)(x^3)^2} = \tilde{C}_{1221,3}$ = left hand side of (5.18).

For $i = 1, 2, 4, \cdots, n$, the relation (5.17) implies that both sides of equation (5.18) are equal. By the similar argument, it can be easily shown that the equation (5.19) – (5.71) hold. Hence the manifold under consideration is a $(W\tilde{C}S)_n$. Thus we can state the following:

Theorem 5.3. *Let (M^n, g) be a Riemannian manifold equipped with the metric given in* (5.11)*. Then (M^n, g) is a weakly concircular symmetric manifold with non-vanishing scalar curvature which is neither concircularly flat nor concircularly symmetric.*

Let (M_3^4, g_3) be a Riemannian manifold, where g_3 is the metric given by

$$ds^2 = x^3e^{x^1}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + x^4(dx^4)^2,$$

where x^3 is non-zero finite and $x^4 \neq 0$. Let (R^{n-4}, g_2) be an $(n-4)$ dimensional Euclidean manifold with the flat metric g_2. Then (M^n, g) in Example 5.2. is a product manifold of (M_3^4, g_3) and (R^{n-4}, g_2). Thus we can state the following:

Theorem 5.4. *Let $(M^n, g)(n \geq 4)$ be a Riemannian manifold endowed with the metric given in* (5.11)*. Then $(M^n, g)(n \geq 4)$ is a decomposable weakly concircular symmetric manifold $(M_3^4, g_3) \times (R^{n-4}, g_2)$ with non-vanishing scalar curvature which is neither concircularly flat nor concircularly symmetric.*

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REFERENCES

1. BINH, T.Q. – *On weakly symmetric Riemannian spaces,* Publ. Math. Debrecen, 42(1993), 103–107.
2. DE, U.C.; BANDYOPADHYAY, S. – *On weakly symmetric Riemannian spaces,* Publ. Math. Debrecen, 54(1999), 377–381.
3. FERUS, D. – *A remark on Codazzi tensors on constant curvature space,* Lecture Notes Math., 838, Global Differential Geometry and Global Analysis, Springer-Verlag, New York, 1981.
4. SCHOUTEN, J.A. – *Ricci Calculus* (2nd Edition), Springer-Verlag, Berlin, 1934.
5. TAMÁSSY, L.; BINH, T.Q. – *On weakly symmetric and weakly projective symmetric Riemannian manifolds,* Coll. Math. Soc., J. Bolyai, 56(1989), 663–670.
6. YANO, K. – *Concirclar geometry I,* Proc. Imp. Acad. Sci. of Japan, 16(1940), 195–200.
7. YANO, K.; KON, M. – *Structure on manifolds,* World Scientific Publ., Singapore, 1986.

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