ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVI, 2010, f.1

A COMMUTATIVITY STUDY FOR CERTAIN RINGS

BY

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Abstract. In this paper, we discuss with the polynomial identities of the form $x^s[x,y]x^t - y^p[x^n,y^m]^r y^q = 0$ and $x^s[x,y]x^t + y^p[x^n,y^m]^r y^q = 0$, where $s \geq 0$, $t \geq 0$, $n \geq 0$, $p \geq 0$, $q \geq 0$, $r > 0$, $m > 1$ are fixed integers, and also they are different in the noncommutative situation. Firstly, it is shown that a semiprime ring is commutative if and only if it satisfies the above conditions. Secondly, commutativity of associative rings with unity 1 and without unit 1 have also been obtained if they satisfy above and related polynomial identities. Thirdly, the result for rings with unity 1 is extended to one-sided s-unital rings. Also, we give some examples that appreciate our results. Finally, we propose a problem for future endeavor.

Mathematics Subject Classification 2000: 16U80, 16U99.

Key words: commutativity, commutator, polynomial identity, semiprime ring, s-unital ring.

1. Introduction.

Throughout, R will represent an associative ring. The center of R, the commutator ideal of R, and the set of all nilpotent elements of R will be denoted $Z(R), C(R)$ and $N(R)$, respectively. For any $x, y \in R$, the symbol $[x,y]$ will stand for the commutator $xy-yx$. By $GF(q)$, we mean the Galois field (finite field) with q elements and by $(GF(q))_2$, the ring of all 2×2 matrices over $GF(q)$. As usual, $\mathbf{Z[X]}$ is the totality of polynomials in X with coefficients in Z, the ring of integers.

In 1945, Jacobson [6] proved that a classical theorem which asserts that if every element $x \in R$ satisfies the relation $x^{n(x)} = x$, where $n(x) > 1$ is a positive integer, then R is commutative. This result at the same time extends the theorem of Wedderburn that every finite division ring is commutative. A celebrated theorem of Herstein [5, Theorem 2] generalized Jacobson's result as follows: let R be a ring in which for every $x, y \in R$, there exists a positive integer $m = m(x, y) > 1$ such that $[x, y] = [x^m, y]$, then R is commutative. Inspired by these works we define the ring property as given below.

(P) "For all $x, y \in R$", there exist $n \geq 0,\ p \geq 0,\ s \geq 0,\ t \geq 0\ q \geq 0, r > 0$, and $m > 1$ such that $x^s[x, y]x^t = \pm y^p[x^n, y^m]^r y^q$ holds.

There are several results in the existing literature [1, 2, 3, 4, 9, 10, 11, 12, 13, 14, 15, 18] concerning the commutativity of rings satisfying special cases of the above ring property. In Sections 2 and 3, we shall prove the commutativity of semi prime rings and rings with unity 1 satisfying the property (P). Section 4 devotes these results to the wider class of rings that are one-sided s-unital. In these sections, we give some examples that justify our results.

We begin with

2. Commutativity of semi prime rings. In what follows, a ring R is called semi prime if for $x \in R, xyx = 0$ for all $y \in R$ implies $x = 0$; equivalently, R is semi prime if for any $x \neq 0$, there is $y_0 \in R$ such that $xy_0x \neq 0$. The set of all real numbers is a semi prime ring.The class of semi prime rings is fairly large and we refer the reader to [16, page 591] for some details about semi prime rings.Further, we provide some examples of semi prime rings and discuss commutativity of rings. These examples also motivate and illustrate our main result of this section (see Theorem 2.1).

Example 2.1. Take the ring $R = \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \backslash a, b \in Z_2\}$ where $Z_2 = \{0, 1\}$ is the set of integer modulo 2, that is

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \backslash\, 0, 1 \in Z_2 \right\}.$$

Then R is a commutative semiprime ring.

Example 2.2. Let R is the set of all real numbers. Construct the ring $S = \{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \backslash\, a \in R\}$. Then S is a commutative ring but not semi prime ring.

Motivated from the above examples, one can establish the following.

Theorem 2.1. *Let R be a semi prime. Then R is commutative if R satisfy (P) together with at least one of s or t is zero.*

For developing the proof of the above theorem we state the following known result.

Lemma 2.1 [8, Theorem]. *Let f be a polynomial in n non-commuting indeterminate $x_1, x_2, \ldots, x_n$ with relatively prime integral coefficients. Then the following are equivalent*:

(i) For any ring satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.

(ii) For every prime $p, (GF(p))_2$ the ring of all 2×2 matrices over $GF(p)$, fails to satisfy $f = 0$.

(iii) Every semi prime ring satisfying $f = 0$ is commutative.

We prove the following.

Claim 2.1. Let R be a ring with 1 satisfying (P). Then $C(R) \subseteq N(R)$.

Proof. Let R satisfy (P). If $t = 0$, then we observe that $x = e_{11}$, $y = e_{12}$ fail to satisfy (P) in $(GF(p))_2$, p a prime; and again if s = 0, then $x = e_{11} + e_{12}$ and $y = e_{12}$ fail to satisfy (P) in $(GF(p))_2$. In both cases using Lemma 2.1, we obtain the required result. □

Proof of Theorem 2.1. Clearly, every commutative ring R satisfies the condition (P). Conversely, if a semi prime ring R satisfies the property (P) then the above Claim 2.1 together with Lemma 2.1 yield that R is commutative. □

Remark 2.1. In general, Example 2.1 shows that the ring of all real numbers R is a commutative semi prime ring. The ring considered in Example 2.2 is commutative but it is not semi prime. So, the assumption that R is semi prime in the Theorem 2.1 is sufficient but not necessary.

Remark 2.2. The ring of $n \times n$ strictly upper (or lower) triangular matrices over an associative ring satisfies the hypotheses of Theorem 2.1 but need not be commutative for $n > 2$. This indicates that the above theorem is not valid for arbitrary rings. If $n = 2$, then it contradicts claim of the above Example 2.2.

In the sequel, we have

3. Commutativity of rings with unity 1

Theorem 3.1. *Let R be a ring with unity 1. Then R is commutative if R satisfy (P).*

To the proof of the above theorem we begin with the known results as given below.

Lemma 3.1 [7, p. 221]. *If $[x, y]$ commutes with x, then for any positive integer k, $[x^k, y] = kx^{k-1}[x, y]$.*

Lemma 3.2 [15, Lemma 4]. *Let R be a ring with unity 1 and let $f : R \to R$ be any polynomial function of two variables with the property $f(x+1, y) = f(x, y)$, for all x, y in R. If for all x, y in R there exists a positive integer $n = n(x, y)$ such that $x^n f(x, y) = 0$ (or $f(x, y)x^n = 0$), then necessarily $f(x, y) = 0$.*

Next, we prove.

Claim 3.1. Let R be a ring with 1 satisfying (P). Then $N(R) \subseteq Z(R)$.

Proof. Let R satisfy the property (P) and let a be an arbitrary element in $N(R)$. Then there exists an integer $l \geq 1$ such that

$$a^k \in Z(R), \text{ for all integers } k > l, \ l \text{ minimal.} \tag{3.1}$$

If $l = 1$, then $a \in Z(R)$, that is $N(R) \subseteq Z(R)$. Suppose that $l > 1$. Re placing x by a^{l-1} in (P), we get

$$(a^{(l-1)s}[a^{(l-1)}, y]a^{(l-1)t} = \pm y^p[a^{n(l-1)}, y^m]^r y^q.$$

In view of (3.1), and the fact that $t(l-1) \geq l$, for integer $l > 1$, we get

$$y^p[a^{n(l-1)}, y^m]^r y^q = 0, \text{ for all } x \text{ in } R. \tag{3.2}$$

Replacing x by $1 + a^{l-1}$ in (P), we get

$$(1 + a^{(l-1)})^s[a^{(l-1)}, y](1 + a^{(l-1)})^t = \pm y^p[(1 + a^{(l-1)})^n, y^m]^r y^q.$$

In view of (3.2), we obtain

$$(1 + a^{(l-1)})^s[a^{(l-1)}, y](1 + a^{(l-1)})^t = 0.$$

But since $(1+a^{(l-1)})$ is invertible, the last equation implies that

$$[a^{l-1}, y] = 0.$$

that is, $a^{l-1} \in Z(R)$, which contradicts the minimality of l in (3.1). Hence $l = 1$ and $a \in Z(R)$ yield the required result. □

Proof of Theorem 3.1. In view of Claims 2.1 and 3.1, we have

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{3.3}$$

Further, we break the proof in two steps.

Step 1. Let $r = 1$ in (P). Then R is commutative.

Proof. By our assumptions, (P) can be written as

$$x^{(s+t)}[x, y] = \pm[x^n, y^m]y^{p+q}. \tag{3.4}$$

Replacing y by $2y$ in (3.4) and using (3.4), we get

$$2x^{(s+t)}[x, y] = (\pm[x^n, y^m]y^{p+q})2^{p+q+m}.$$

This implies that

$$2x^{(s+t)}[x, y] = x^{(s+t)}[x, y]2^{p+q+m}.$$

And

$$x^{(s+t)}[x, y](2^{p+q+m} - 2) = 0.$$

Let $k = 2^{p+q+m} - 2 > 0$. Then $kx^{(s+t)}[x, y] = 0$ and using Lemma 3.2, we obtain $k[x, y] = 0$ By Lemma 3.1, this yields

$$x^k \in Z(R), \text{for all} x \in R,\ k = 2^{p+q+m} - 2. \tag{3.5}$$

Next using Lemma 3.1 and equations (3.3), (3.4) repeatedly, we get

$$\begin{aligned}
&(1 - y^{(m-1)(p+q+m-1)})x^{(s+t)})[x, y^m] \\
&= x^{(s+t)}[x, y^m] - y^{(m-1)(p+q+m-1)}my^{(m-1)}[x, y]x^{(s+t)} \\
&= x^{(s+t)}[x, y^m] - my^{(m-1)^2}y^{(m-1)(p+q)}(\pm y^p[x^n, y^m]y^q) \\
&= x^{(s+t)}[x, y^m] \mp y^{m(m-1)}y^{mp}[x^n, y^m]x^{mq} \\
&= x^{(s+t)}[x, y^m] \mp y^{mp}[x^n, y^{m^2}]y^{mq} = 0.
\end{aligned}$$

In view of Lemma 3.1, this gives that $(1-y^{(m-1)(p+q+m-1)})[x,y^m]=0$. Replacing x by x^n in the last equation, we get

$$(1-y^{(m-1)(p+q+m-1)})[x^n,y^m]=0.$$

This implies that

$$(1-y^{(m-1)(p+q+m-1)})(\pm[x^n,y^m]y^{p+q})=0$$

that is

$$(1-y^{(m-1)(p+q+m-1)})[x,y]x^{(s+t)}=0.$$

By an application of Lemma 3.2 yields that $(1-y^{(m-1)(p+q+m-1)})[x,y]=0$, and hence $(1-y^{k(m-1)(p+q+m-1)})[x,y]=0$, where k considered as in (3.5). In view of (3.5) , the last identity can be rewritten as

$$[x,y-y^{k(m-1)(p+q+m-1)+1}]=0, \text{ for all } x,y\in R.$$

By an application of Herstein's result [5, Theorem 2], R is commutative. □

Step 2. Let $r>1in(P)$. Then R is commutative.

Proof. Continuing to use our polynomial identity and (3.5) repeatedly, one observes that for any positive integer γ,

$$\begin{aligned}
&x^{(s+t)+rn(s+t)+r^2n^2(s+t)+\ldots+r^{\gamma-1}n^{\gamma-1}(s+t)}[x,y]\\
&=\pm x^{rn(s+t)+r^2n^2(s+t)+\ldots+r^{\gamma-1}n^{\gamma-1}(s+t)}[x^n,y^m]^r y^{(p+q)}\\
&=(\pm)^{1+r}x^{r^2n^2(s+t)+\ldots+r^{\gamma-1}n^{\gamma-1}(s+t)}(x^{n(s+t)}[x^n,y^m])^r y^{(p+q)}\\
&=(\pm)^{1+2r}x^{r^2n^2(s+t)+\ldots+r^{\gamma-1}n^{\gamma-1}(s+t)}([x^{n^2},y^{m^2}]^r)y^{m(p+q)})^r y^{(p+q)}\\
&=(\pm)^{1+2r}x^{r^2n^2(s+t)+\ldots+r^{\gamma-1}n^{\gamma-1}(s+t)}([x^{n^2},y^{m^2}]^{r^2})y^{(p+q)+mr(p+q)}\\
&=(\pm)^{1+3r}x^{r^3n^3(s+t)+\ldots+r^{\gamma-1}n^{\gamma-1}(s+t)}([x^{n^3},y^{m^3}]^{r^3})\\
&\qquad\cdot y^{(p+q)+mr(p+q)+r^2m^2(p+q)}\\
&\ldots\\
&=(\pm)^{(1+\gamma r)}[x^{n^\gamma},y^{m^\gamma}]^{t^\gamma}y^{(p+q)+mr(p+q)+r^2m^2(p+q)+\ldots+r^{\gamma-1}m^{\gamma-1}(p+q)}.
\end{aligned}$$

In view (3.5) and Lemma 3.1 this gives

$$x^{(s+t)+rn(s+t)+r^2n^2(s+t)+\ldots+r^{\gamma-1}n^{\gamma-1}(s+t)}[x,y]$$

$= (\pm)^{(1+\gamma r)}(n^{\gamma}m^{\gamma}x^{n^{\gamma}-1}y^{m^{\gamma}-1})^{r^{\gamma}}y^{(p+q)+mr(p+q)+r^2m^2(p+q)+\ldots+r^{\gamma-1}m^{\gamma-1}(p+q)}$.

Since commutator are nilpotent, so we obtain

$$x^{(s+t)+rn(s+t)+r^2n^2(s+t)+\ldots+r^{\gamma-1}n^{\gamma-1}(s+t)}[x, y] = 0$$

and using Lemma 3.2 we get the required result. □

Remark 3.1. Theorem 3.1 is false for rings without identity 1, because any nilpotent ring of index ≤ 4 and nil ring of index 2 will easily satisfies (P) but such rings need not be commutative (see [3] for details).

Particularly, if we take $s = 0$, $t = 0$ in the property (P), then we obtain a property.

(P_1) "For all $x, y \in R$", there exist $n \geq 0$, $p \geq 0$, $q \geq 0, r > 0$, and $m > 1$ such that $[x, y] = \pm y^p[x^n, y^m]^r y^q$ holds.

Remark 3.2. In the proof of Theorem 3.1, (P) implies (P_1) could have been obtained by [12, Theorem]. Further, we would like to give a direct proof with a view of preparing some ground work for the following result, which proves commutativity of arbitrary rings.

Theorem 3.2. *An associative ring R (may be without unity 1) is commutative if and only if R satisfies the property (P_1).*

Proof. It is easy to check that every commutative ring R satisfies (P_1). Let us prove the converse. Let R satisfies (P_1). A careful scrutiny of the proof of Claims 2.1, and 3.1 demonstrates that Equation 3.3 are still valid in the present situation, and hence we have

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{3.6}$$

Taking $r = 1$ in (P_1), then we have $[x, y] = \pm y^p[x^n, y^m]y^q$. Applying similar techniques as used to obtain (3.5), we get

$$x^w \in Z(R), \text{ for all } x \in R, \ w = 2^{p+q+m} - 2. \tag{3.7}$$

Next using Lemma 3.1, and equations (3.6), (3.7) repeatedly, we get

$$\begin{aligned}
&[x, y^m] - y^{(m-1)(p+q+m-1)}[x, y^m] \\
&= [x, y^m] - y^{(m-1)(p+q+m-1)}my^{(m-1)}[x, y] \\
&= [x, y^m] \mp y^{m(m-1)}y^m p[x^n, y^{m^2}]y^m q = 0.
\end{aligned}$$

This gives that $y^p\{[x, y^m]-y^{(m-1)(p+q+m-1)})[x, y^m]\}y^q = 0$. In view of (3.7), we obtain that $[x, y] = y^{(m-1)(p+q+m-1)}[x, y]$. This implies that $[x, y] = y^{w(m-1)(p+q+m-1)}[x, y]$. where w is considered as in (3.7). In view of (3.7), so the obtained result can be rewritten as

$$[x, y - y^{w(m-1)(p+q+m-1)+1}] = 0, \text{for all } \ x, y \in R,$$

and hence R is commutative by Herstein [5, Theorem 2].

Letting $r > 1$ in (P_1). Then for an arbitrary integer γ, Using similar techniques as used in the proof of Theorem 3.1, we have

$$[x, y] = (\pm)^{(\gamma)}[x^{n^\gamma}, y^{m^\gamma}]^{t^\gamma} y^{(p+q)+mr(p+q)+r^2m^2(p+q)+\ldots+r^{\gamma-1}m^{\gamma-1}(p+q)}.$$

An application of Lemma 3.1, gives that

$$\begin{aligned}[x, y] &= (\pm)^{(\gamma)}(n^\gamma m^\gamma x^{n^\gamma-1} y^{m^\gamma-1})^{r^\gamma}[x, y]^{r\gamma} \\ &\quad \cdot y^{(p+q)+mr(p+q)+r^2m^2(p+q)+\ldots+r^{\gamma-1}m^{\gamma-1}(p+q)}.\end{aligned}$$

But since commutators are nilpotent, we obtain $[x, y] = 0$. This complete the proof. □

Remark 3.3. As an application of Theorem 3.2; if $s = 0,\ t = 0,\ p = 0,\ q = 0$ in P_1 then the above theorem reduces to Theorem 2 of [2]. Next, in view of Remark 3.2, Theorem 3.1 cannot be generalized to arbitrary rings. So, we generalize Theorem 3.1 to a wider class of rings called s-unital rings.

4. Extension to one-sided s-unital rings. The above result requires that the ring contain a unity, but ring need not have a unit and researchers still want to be able to work with them. In an efforts to have a class of rings that are close to having unit, the concept of s-unital was developed. A ring R is said to be left (resp. right) s-unital, if $x \in Rx$ (resp., $x \in xR$) for each element $x \in R$, R is called s-unital if it is both left as well as right s-unital,that is, $x \in xR \cap\ Rx$ for each $x \in R$. If R is s-unital (resp. left or right s-unital), then for any finite subset F of R there exists an element $e \in R$ such that $ex = x$ or $xe = x$ for all $x \in F$. such an element e is called a pseudo-identity (resp. pseudo-left identity or pseudo-right identity) of F in R (for references see [4]).

The existence of non-commutative ring R with R^2 being central rules out the possible generalization of the theorem 3.1 for arbitrary rings. In this context, Example 2.1 strengthens the existence of unity 1 in the hypothesis of the theorems obtained in Section 3, nevertheless the same theorems may be extended to one sided s-unital rings. There are several examples in the existing literature (see [17, Remark 2]), which show that these classes of rings are the generalizations of the class of rings with unity 1, particularly a number of commutativity theorems have been extended to one sided s-unital. Before we go ahead with our task, we pause to recall a few results in order to make our paper self contained as possible. Clearly, in [4, Proposition 1], if P is a ring property (i.e., P is inherited by every subring and every homomorphic image), then P is called an h-property. More weakly, if P is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then P is called an H-property. A ring property P such that a ring R has the property P if and only if all its finitely generated subrings have P is called an F-property.

Lemma 4.1 [4, Proposition 1]. *Suppose that P is an H-property, and P' is an F-property. If every ring R with unity 1 having the property P has the property P', then every s-unital ring having P has P'.*

Indeed, we have

Theorem 4.1. *Let R be a left s-unital ring satisfying (P) with $t = 0$. Then R is commutative.*

Proof. Let R be a left s-unital ring satisfying the above hypothesis. If $x, y \in R$ then there exists an element $e \in R$ such that $ex = x,\ \ ey = y$. Put $x = e$ in (P) with $t = 0$ to get $e^s[e, y] = \pm y^p[e^n, y^m]^r y^q$. Taking $r = 1$, this gives $y = y(e + y^{p+q+m-1} - y^{p+q-1}e^n y^p) \in yR$, when $m > 1$. If $r > 1$, we find that $e^s[e, y] = 0$. that is, $y = ye \in yR$ for all $y \in R$. Thus R is right s-unital. Therefore, R is s-unital. In view of Lemma 4.1, R is commutative, if R with unity is commutative and this is guaranteed by Theorem 3.1.

Using the similar techniques to establish the result as given below.

Theorem 4.2. *Let R be a right s-unital ring satisfying the (P) with $s = 0$. Then R is commutative.*

Remark 4.1. As a consequence of Theorems 4.1 and 4.2, we get the following corollary which includes [3, Theorem], [10, Theorems 1-4], [11, Theorems 2 and 3] [13, Theorem], [14, Theorem], [15, Theorem] and [17, Theorem].

Corollary 4.1. *Let $m > 1, p, q, n, s$ and t be fixed non-negative integers and R a left (resp. right) s-unital ring in which for every $y \in R$ there exist integers $p = p(y) \geq 0$, $q = q(y) \geq 0$, $m = m(y) \geq 0$ such that $x^s[x, y] = y^p[x^n, y^m]y^q$ (resp. $[x, y]x^s = y^p[x^n, y^m]y^q$) for all $x \in R$. Then R is commutative.*

Example 4.1. If we drop the restriction that R is left (or right) s-unital, then R may be badly non commutative. Indeed the following example demonstrate this constraints: Let D_k be the ring of all $k \times k$ matrices over a division ring D, and let $A_k = \{[a_{ij}]|a_{ij} = 0, j \geq i\}$. Then A_k is a noncommutative ring for any positive integer $k > 2$. It is obvious to see that a noncommutative ring A_3 satisfies (P).

Remark 4.2. The following example shows that a left (resp. right) s-unital ring with the property $[x, y]x^s = y^p[x^n, y^m]y^q$ (resp. $x^t[x, y] = y^p[x^n, y^m]y^q$) need not be commutative.

Example 4.2. Let

$$R_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

(resp.

$$R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

be subring of 2×2 matrices over $GF(2)$. Then R_1 (resp. R_2 is a non commutative left (resp. right) s-unital ring satisfying the property $[x, y]x^s = y^p[x^n, y^m]y^q$ (resp. $x^s[x, y] = y^p[x^n, y^m]y^q$) for any fixed integers s, t, p, q, n and $m > 1$.

Remark 4.3. One might be remarked in the above theorems the exponents in the underlying conditions are assumed to be global. It would be interested to further generalize these results for the case when they are assumed to be local one.

In retrospect, it is tempting to conjecture as given below.

Conjecture. Let $k > 0, p \geq 0$ and $q \geq 0$ be fixed non-negative integers and let R be an arbitrary ring in which for every $x, y \in R$ there exist polynomials $f(X), g(X), h(X) \in Z[X]$ such that

$$x^p[x, y]y^q = g(y)\,[x, f(y)]^k\,h(y).$$

Then R is commutative (and conversely).

Acknowledgement. The author would like to thank the referee for his valuable suggestions. The author also would like to thank Dr. Rafikul Alam, Head Department of mathematics, Indian Institute of Technology (IIT) Guwhati, for inviting him to visit and deliver a talk on this paper.

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Received: 28.VIII.2008
Revised: 25.IX.2009

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