

A SEMIHYPERGROUP ASSOCIATED WITH A Γ -SEMIGROUP

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Abstract. In this paper we introduce a notion of a semihypergroup associated with a Γ -semigroup and study different properties of semihypergroups based on the associated Γ -semigroup.

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1. Introduction. Hyperstructure theory was born in 1934 [6], when MARTY defined hypergroups and began to analysis their properties. In 1937, WALL [10] introduced the notion of hypergroups and studied different properties. Now hypergroups are widely studied from different point of view and for their applications to many subjects of pure and applied properties. In [9] SEN and SAHA introduced the notion of a Γ -semigroup as follows.

Let S and Γ be two nonempty sets. S is called a Γ -semigroup if there exists a mapping $S \times \Gamma \times S \rightarrow S$, written as $(a, \alpha, b) \rightarrow a\alpha b$, satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Since then many papers have been published in the field of Γ -semigroups. In this paper we study semihypergroups associated with Γ -semigroups and the hyperideal theory of semihypergroups which can be seen as a generalization of the ideal theory in Γ -semigroups.

We recall some definitions and theorems which we need in what follows.

Definition 1.1 ([9]). Let S be a Γ -semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an α -idempotent if $e\alpha e = e$. The set of all α -idempotents is

denoted by E_α . We denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent elements of S .

Definition 1.2 ([9]). Let S be a Γ -semigroup. An element $a \in S$ is said to be regular if $a \in a\Gamma S\Gamma a$, where $a\Gamma S\Gamma a = \{a\alpha b\beta a : b \in S, \alpha, \beta \in \Gamma\}$. S is said to be regular if every element of S is regular.

Example 1.3 ([7]). Let S be the set of all 3×2 matrices and Γ be the set of all 2×3 matrices over a field. Then for $A, B \in S$, the usual matrix multiplication AB can not be defined i.e., S is not a semigroup under the usual matrix multiplication. But for all $A, B, C \in S$ and $P, Q \in \Gamma$ we have $APB \in S$ and since the matrix multiplication is associative, we have $(APB)QC = AP(BQC)$. Hence S is a Γ -semigroup. Moreover it is regular (see [9]).

Definition 1.4 ([9]). A nonempty subset I of a Γ -semigroup S is called a right (resp. left) ideal if $I\Gamma S \subseteq I$ (resp. $S\Gamma I \subseteq I$) where for the subsets U, V of S , $U\Gamma V = \{u\alpha v : u \in U, v \in V, \alpha \in \Gamma\}$. I is said to be an ideal if I is both left and right ideal of S .

Definition 1.5 ([9]). A Γ -semigroup S is called left (right) simple if it has no proper left (right) ideals. S is said to be simple if it has no proper ideals.

Let S be a Γ -semigroup and α be a fixed element of Γ . We define $a.b = a\alpha b$ for all $a, b \in S$. Then $(S, .)$ is semigroup and we denote this semigroup by S_α .

Theorem 1.6 ([9]). Let S be a Γ -semigroup. S_α is a group for some $\alpha \in \Gamma$ if and only if S is both left simple and right simple.

Corollary 1.7 ([9]). If S_α is a group for some $\alpha \in \Gamma$ then S_α is a group for all $\alpha \in \Gamma$.

Definition 1.8 ([9]). A Γ -semigroup S is called a Γ -group if S_α is a group for some $\alpha \in \Gamma$.

2. Some properties of semihypergroups. In this section we recall some definitions of several semihypergroups and mention some basic properties.

Definition 2.1 ([1]). Let H be a set and $P^*(H)$ be the family of all nonempty subsets of H . A mapping ‘ \circ ’ from $H \times H$ to $P^*(H)$ is called a hyperoperation on H . If $(a, b) \in H \times H$, its image under ‘ \circ ’ is denoted by $a \circ b$.

Remark 2.2 ([1]). The hyperoperation is extended to subsets of H in a natural way, so that $A \circ B$ is given by $A \circ B = \bigcup \{a \circ b : a \in A, b \in B\}$. The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally the singleton set $\{a\}$ is identified by its element a .

Definition 2.3 ([1]). Let H be a set and ‘ \circ ’ be a hyperoperation on H . Then the structure (H, \circ) is called a hypergroupoid. A hypergroupoid (H, \circ) which is associative, i.e., $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$ is called a semihypergroup.

Definition 2.4 ([1]). A hypergroup is a semihypergroup in which $x \circ H = H \circ x = H$ for all $x \in H$.

Definition 2.5. Let (S, \circ) be a hypergroupoid. An element $e \in S$ is called an identity if for all $a \in S$, $a \in e \circ a \cap a \circ e$.

Definition 2.6. Let (S, \circ) be a hypergroupoid. An element $b \in S$ is called an inverse of an element $a \in S$ if there exists an identity $e \in S$ such that $e \in a \circ b \cap b \circ a$.

Definition 2.7. A semihypergroup (S, \circ) is called a regular hypergroup or simply r -hypergroup if it has an identity e and for each $a \in S$, there exists an inverse $a' \in S$.

Theorem 2.8. *Every r -hypergroup is a hypergroup.*

Proof. Let $a \in S$. We show that $S \subseteq a \circ S$. Let $b \in S$. Then there exists an identity $e \in S$ such that $b \in e \circ b \cap b \circ e$. Now for $a \in S$, there exists an inverse $a' \in S$ such that $e \in a' \circ a \cap a \circ a'$. Hence $b \in e \circ b \subseteq a \circ a' \circ b \subseteq a \circ S$. This shows that $S \subseteq a \circ S$. Hence $S = a \circ S$. Similarly we can show that $S \circ a = S$. Therefore S is a hypergroup. \square

The following example shows that the converse of the above theorem is not true in general.

Example 2.9. Let $S = \{a, b\}$. We define a hyperoperation \circ on S by $a \circ a = \{a\}$, $a \circ b = \{b\}$, $b \circ a = \{a, b\}$ and $b \circ b = \{a, b\}$. Then it can be

easily verified that $(a \circ a) \circ a = a \circ (a \circ a) = \{a\}$, $a \circ (a \circ b) = (a \circ a) \circ b = \{b\}$, $(a \circ b) \circ a = a \circ (b \circ a) = \{a, b\}$, $(a \circ b) \circ b = a \circ (b \circ b) = \{a, b\}$, $(b \circ a) \circ a = b \circ (a \circ a) = \{a, b\}$, $(b \circ a) \circ b = b \circ (a \circ b) = \{a, b\}$, $(b \circ b) \circ a = b \circ (b \circ a) = \{a, b\}$, $(b \circ b) \circ b = b \circ (b \circ b) = \{a, b\}$. Hence (S, \circ) is a semihypergroup. Moreover $a \circ S = S \circ a = S = b \circ S = S \circ b$. Hence (S, \circ) is a hypergroup. But in this semihypergroup there is no identity element and hence it is not a r-hypergroup.

Definition 2.10. Let (S, \circ) be a r-hypergroup and let I_S denote the set of all identities of S . Clearly $I_S \neq \phi$. If $I_S = S$, then S is called trivial r-hypergroup.

Example 2.11. Let S be a nonempty set such that $|S| \geq 2$. Define a hyperoperation ' \circ ' on S by $a \circ b = \{a, b\}$. Then (S, \circ) is a semihypergroup. Now for all $b \in S$, $b \in b \circ a \cap a \circ b$. Hence a is an identity, which means that every element of S is an identity element. Also $a \in b \circ a \cap a \circ b$ shows that a is an inverse of b . Hence S is a r-hypergroup such that $I_S = S$. Hence S is a trivial r-hypergroup.

Definition 2.12. Let (S, \circ) be a semihypergroup and $\phi \neq A \subseteq S$. A is said to be subsemihypergroup of S if $A \circ A \subseteq A$.

Definition 2.13. An element a of a semihypergroup (S, \circ) is said to be an idempotent if $a \in a \circ a$. (S, \circ) is said to be an idempotent semihypergroup if all its elements are idempotent.

Example 2.14. Let X and Y be two vector spaces over a field F . Then $X \times Y$ and $Y \times X$ are vector spaces over the field F if we define for $(x_1, y_1), (x_2, y_2) \in X \times Y$ and $\alpha \in F$, $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$. Let S be the set of all subspaces of $X \times Y$ and Γ be the set of all subsets of $Y \times X$. For $A, B \in S$ and $P \in \Gamma$, we define $APB = \{(x, y) \in X \times Y : \text{there exists } (y_1, x_1) \in P \text{ for which } (x, y_1) \in A \text{ and } (x_1, y) \in B\}$. Then we can show that S is a Γ -semigroup. Now let $A \in S$. Define $A' = \{(y, x) : (x, y) \in A\}$. We can show that $AA'A = A$. We define a hyperoperation ' \circ ' on S by $A \circ B = A\Gamma B$ for $A, B \in S$. Then (S, \circ) is a semihypergroup. Moreover, for every $A \in S$, there exists $A' \in \Gamma$ such that $A = AA'A \in A \circ A$. Hence (S, \circ) is an idempotent semihypergroup.

Example 2.15. Let S be a nonempty set and c be a fixed element of S . Define a hyperoperation \circ on S by $a \circ b = \{a, b, c\}$. S is a semihypergroup

under this operation and $a \in a \circ a$ for all $a \in S$. Hence S is an idempotent semihypergroup.

Definition 2.16. A nonempty subset A of a semihypergroup (S, \circ) is said to be an idempotent subset if $A \subseteq A \circ A$.

Theorem 2.17. Let S be a semihypergroup and A be an idempotent subset in S . Then $A \circ A$ is a subsemihypergroup of S if and only if $A \circ A \circ A = A \circ A$.

Proof. Let $A \circ A$ be a subsemihypergroup of S . Hence $A \circ b \subseteq A \circ A$ for every $b \in A \circ A$. Thus $A \circ A \circ A \subseteq A \circ A$. Again since $A \subseteq A \circ A$, we have $A \circ A \subseteq A \circ A \circ A$. Thus $A \circ A \circ A = A \circ A$. Conversely let $A \circ A \circ A = A \circ A$. Let $b, c \in A \circ A$. Then $b \circ c \subseteq A \circ A \circ A \circ A = A \circ A \circ A = A \circ A$. Hence the proof is completed. \square

Corollary 2.18. Let S be a semihypergroup and a be an idempotent element in S . Then $a \circ a$ is a subsemihypergroup of S if and only if $a \circ a \circ a = a \circ a$.

In a semigroup S an element a of S is called regular if there exists x in S such that $axa = a$. The semigroup S is called regular if all its elements are regular(see [5]). Here we like to extend this notion to semihypergroups.

Definition 2.19. Let (S, \circ) be a semihypergroup and $a \in S$. a is said to be regular if there exists an element b of S such that $a \in a \circ b \circ a$. The semihypergroup S is called regular if every element of S is regular.

We now establish a relation between regular semihypergroup and regular hypergroup.

Theorem 2.20. Every r -hypergroup is a regular semihypergroup.

Proof. Let (S, \circ) be a r -hypergroup and $a \in S$. By Theorem 2.8, S is a hypergroup. Then $S \circ a = a \circ S = S$. Hence $a \in b \circ a$ for some $b \in S$. Again $b \in a \circ c$ for some $c \in S$. Thus $a \in a \circ c \circ a$. Hence S is a regular semihypergroup. \square

From the proof of the above theorem we see that every hypergroup is a regular semihypergroup.

Definition 2.21. Let (S, \circ) be a semihypergroup and $a \in S$. A subset B of S is said to be an inverse subset of a in S if $a \in a \circ B \circ a$ and $B \subseteq B \circ a \circ B$.

Theorem 2.22. *Let (S, \circ) be a semihypergroup. If a is regular then it has an inverse subset.*

Proof. Let $a \in S$ be regular. Then there exists an element b of S such that $a \in a \circ b \circ a$ for some $b \in A$. Let us now consider the set $B = b \circ a \circ b$. Now $a \in a \circ b \circ a \subseteq a \circ b \circ a \circ b \circ a = a \circ B \circ a$ and $B = b \circ a \circ b \subseteq b \circ a \circ b \circ a \circ b \subseteq (b \circ a \circ b) \circ a \circ b \circ a \circ b = (b \circ a \circ b) \circ a \circ (b \circ a \circ b) = B \circ a \circ B$. Hence B is an inverse subset of a . \square

Theorem 2.23. *For a semihypergroup (S, \circ) , the following conditions are equivalent:*

- (i) $a \in S$ is regular.
- (ii) There exists an idempotent set $A \subseteq S$ such that $a \in a \circ A$ and $S \circ a = S \circ A$.

Proof. (i) \Rightarrow (ii): Let $a \in S$ be regular. Then there exists an element $b \in S$ such that $a \in a \circ b \circ a$. Let us now consider the set $A = b \circ a$. Clearly $A \subseteq A = b \circ a \subseteq b \circ a \circ b \circ a = A \circ A$ and $a \in a \circ A$. Again $S \circ a \subseteq S \circ a \circ b \circ a = S \circ a \circ A \subseteq S \circ A = S \circ b \circ a \subseteq S \circ a$. Hence $S \circ a = S \circ A$.

(ii) \Rightarrow (i): Let us suppose that the given conditions hold. Then $a \in a \circ A \subseteq a \circ A \circ A \subseteq a \circ S \circ A = a \circ S \circ a$. Hence a is regular. \square

Definition 2.24. A nonempty subset I of a semihypergroup (S, \circ) is said to be a right (resp. left) hyperideal if $I \circ S \subseteq I$ (resp. $S \circ I \subseteq I$). I is said to be a hyperideal if it is left as well as right hyperideal of S .

Theorem 2.25. *Let (S, \circ) be a semihypergroup and I be a hyperideal of S . If (S, \circ) is regular then I is also regular and any hyperideal J of I is a hyperideal of S .*

Proof. Let $a \in I$. Since S is regular, there exists a subset A of S such that $a \in a \circ A \circ a$ and $A \subseteq A \circ a \circ A$. Since I is a hyperideal and $a \in I$ we have $A \circ a \circ A \subseteq I$ and hence $A \subseteq I$. Thus I is regular. Let $a \in J \subseteq I$ where J is an hyperideal of I and $s \in S$. Now $a \circ s \subseteq I$. Let $a \circ s = \{x_i : x_i \in I\}$. Since I is regular, for each $x_i \in a \circ s$, there exists an element y_i of I such that $x_i \in x_i \circ y_i \circ x_i$. Let D be the set consisting of such y_i 's. Then $a \circ s \subseteq (a \circ s) \circ D \circ (a \circ s)$. Since $y_i \in I, D \subseteq I$ and hence $s \circ D \circ a \circ s \subseteq I$. Again since J is a hyperideal of I and $a \in J$, we have $(a \circ s) \circ D \circ (a \circ s) \subseteq J$ i.e., $a \circ s \subseteq J$. Thus J is also a hyperideal of S . \square

Theorem 2.26. *Let (S, \circ) be a semihypergroup. S is regular if and only if for any left hyperideal A and for any right hyperideal B of S , $A \cap B = B \circ A$.*

Proof. Since A is a left hyperideal and B is a right hyperideal of S , we have $B \circ A \subseteq S \circ A \subseteq A$ and $B \circ A \subseteq B \circ S \subseteq B$. Hence $B \circ A \subseteq A \cap B$. Again since S is regular, for $a \in A \cap B$, there exists $a' \in S$ such that $a \in a \circ a' \circ a \subseteq a \circ A \subseteq B \circ A$. i.e., $A \cap B \subseteq B \circ A$. Hence we have $A \cap B = B \circ A$.

Conversely let us suppose that the given condition holds in S . Let $a \in S$. Now $A = \{a\} \cup S \circ a$ is a left hyperideal and $B = a \circ S \cup \{a\}$ is a right hyperideal of S . Hence $A = A \cap B = S \circ (S \circ a \cup \{a\}) = S \circ S \circ a \cup S \circ a \subseteq S \circ a \cup S \circ a = S \circ a$. Similarly we can show that $B \subseteq a \circ S$. Thus we have $a \in A \cap B \subseteq (S \circ a) \cap (a \circ S) = (a \circ S) \circ (S \circ a) = a \circ S \circ a$. Hence a is a regular element in (S, \circ) . \square

Theorem 2.27. *If I is a one sided hyperideal in a regular semihypergroup then $I \circ I = I$.*

Proof. Let I be a right hyperideal of S . Let $a \in I$. Since S is a regular semihypergroup, we have $a \in a \circ x \circ a$ for some $x \in S$. We have $a \circ x \subseteq I$. Hence $a \in a \circ x \circ a \subseteq I$. This shows that $I \subseteq I \circ I$. Again since I is a right hyperideal we have $I \circ I \subseteq I$. Hence $I \circ I = I$. \square

Corollary 2.28. *All one sided hyperideals in a regular semihypergroup are idempotent.*

Corollary 2.29. *All one sided hyperideals in a regular semihypergroup form an idempotent semigroup with respect to the binary operation defined by $A.B = A \circ B$.*

Definition 2.30. Let (S, \circ) be a semihypergroup. For $a \in S$, the smallest left hyperideal of S containing a is called the principal left hyperideal of S generated by a and it is denoted by $L(a)$. Similarly we can define $R(a)$ the principal right hyperideal of S generated by a . Finally, we denote the principal hyperideal generated a by $\langle a \rangle$ which is the smallest hyperideal containing a .

By the above definition we can show that $L(a) = S \circ a \cup \{a\}$, $R(a) = a \circ S \cup \{a\}$ and $\langle a \rangle = \{a\} \cup a \circ S \cup S \circ a \cup S \circ a \circ S$. Note that in a regular semihypergroup $a \in a \circ b \circ a \subseteq a \circ b \circ a \circ b \circ a$ for some $b \in S$. i.e.,

$a \in a \circ S \cap S \circ a \cap S \circ a \circ a$. Thus in a regular semihypergroup $L(a) = S \circ a$, $R(a) = a \circ S$ and $\langle a \rangle = a \circ S \cup S \circ a \cup S \circ a \circ S$.

Theorem 2.31. *Let (S, \circ) be a semihypergroup. Then the following conditions are equivalent:*

(i) *The principal hyperideals in S form a chain.*

(ii) *The hyperideals in S form a chain.*

Proof. (i) \Rightarrow (ii): Let A and B be two hyperideals of S such that $A \not\subseteq B$. Then there exists an element $a \in A$ such that $a \notin B$. Then $\langle a \rangle \not\subseteq \langle b \rangle$ for any $b \in B$. Now from (i) it follows that $\langle b \rangle \subseteq \langle a \rangle$ for any $b \in B$. Thus $B \subseteq A$. Hence the hyperideals in S form a chain. (ii) \Rightarrow (i) is clear. \square

Definition 2.32. Let S be a semihypergroup. S is said to be left (right) regular if for any element $a \in S$, there exists $x \in S$ such that $a \in x \circ a \circ a$ ($x \in a \circ a \circ x$).

Definition 2.33. Let S be a semihypergroup. S is said to be intra regular if for any element $a \in S$, there exist $x, y \in S$ such that $a \in x \circ a \circ a \circ y$.

Theorem 2.34. *Let (S, \circ) be a semihypergroup. S is left (right, intra) regular if and only if for every left (right, both sided) hyperideal I of S , $a \circ a \subseteq I$ implies $a \in I$ for all $a \in S$.*

Proof. Let S be left regular and L be a left hyperideal of S . Let us suppose that $a \circ a \subseteq L$. Now since S is left regular, there exists an element $x \in S$ such that $a \in x \circ a \circ a$. Since L is a left hyperideal of S , $x \circ a \circ a \subseteq L$ i.e., $a \in L$.

Conversely, let the condition hold for every left hyperideal of S . Let $a \in S$. Then $S \circ a \circ a$ is a left hyperideal of S and $a \circ a \subseteq S \circ a \circ a$. By the given condition we have $a \in S \circ a \circ a$. Hence S is left regular. \square

3. The semihypergroup associated with a Γ -semigroup. Let S be a Γ -semigroup. We define a mapping $\circ_\Gamma: S \times S \rightarrow P(S)$, the power set of S by $a \circ_\Gamma b = a\Gamma b$, where $P(S)$ is the power set of S . We have $(a \circ_\Gamma b) \circ_\Gamma c = (a\Gamma b) \circ_\Gamma c = a\Gamma b\Gamma c = a\Gamma(b\Gamma c) = a \circ_\Gamma (b \circ_\Gamma c)$. Thus (S, \circ_Γ) is a semihypergroup. This semihypergroup is called the semihypergroup associated with the Γ -semigroup S . In this section we study different properties of this type of semihypergroups.

Theorem 3.1. *Let (S, \circ_Γ) be a semihypergroup associated with a Γ -semigroup S . Then:*

- (i) *an element a is an idempotent in (S, \circ_Γ) if and only if a is an idempotent in the Γ - semigroup S .*
- (ii) *an element a is regular in (S, \circ_Γ) if and only if it is regular in the Γ - semigroup S .*
- (iii) *(S, \circ_Γ) is a hypergroup if and only if S is a Γ - group.*
- (iv) *A nonempty subset I of S is an hyperideal of the semihypergroup (S, \circ_Γ) if and only if I is an ideal of the Γ - semigroup S .*

Proof. (i) Let a be an idempotent in (S, \circ_Γ) . Then $a \in a \circ_\Gamma a$ i.e., $a \in a\Gamma a$. Thus $a = a\alpha a$ for some $\alpha \in \Gamma$. Hence a is an α - idempotent. The converse follows similarly.

(ii) Let a be regular in (S, \circ_Γ) . Then there exists $b \in S$ such that $a \in a \circ_\Gamma b \circ_\Gamma a$. Thus $a \in a\Gamma b\Gamma a$ i.e., $a = a\alpha b\beta a$ for some $\alpha, \beta \in \Gamma$. Hence a is a regular element in the Γ - semigroup S .

Conversely let a be a regular element of the Γ - semigroup S . Then there exist $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha b\beta a$ i.e., $a \in a\Gamma b\Gamma a$ i.e., $a \in a \circ_\Gamma b \circ_\Gamma a$. Hence a is regular in (S, \circ_Γ) .

(iii) Let us suppose that (S, \circ_Γ) be a hypergroup. Then for every element $a \in S$ we have $a \circ_\Gamma S = S \circ_\Gamma a = S$. Thus $a\Gamma S = S\Gamma a = S$ for every $a \in S$. Hence the Γ - semigroup is left simple as well as right simple. Thus S is a Γ - group.

Conversely suppose that S is a Γ - group. Then S is both left simple and right simple. Thus $a\Gamma S = S\Gamma a = S$ for all $a \in S$ i.e., $a \circ_\Gamma S = S \circ_\Gamma a = S$ for every $a \in S$. Hence (S, \circ_Γ) is a hypergroup.

(iv) Let I be a hyperideal of (S, \circ_Γ) . Then $a \circ_\Gamma I \subseteq I$ and $I \circ_\Gamma a \subseteq I$ for all $a \in S$. i.e., $a\Gamma I \subseteq I$ and $I\Gamma a \subseteq I$ for all $a \in S$. Hence I is an ideal of the Γ - semigroup S . Conversely let I be an ideal of the Γ - semigroup S . Then for every element $a \in S$ we have $a\Gamma I \subseteq I$ and $I\Gamma a \subseteq I$. Thus $I \circ_\Gamma a \subseteq I$ and $a \circ_\Gamma I \subseteq I$ for all $a \in S$. Hence I is a hyperideal of (S, \circ_Γ) . \square

Theorem 3.2. *A Γ - semigroup S is a Γ - group if and only if its associated semihypergroup (S, \circ_Γ) is a r -hypergroup.*

Proof. Let S be a Γ -group. Suppose $\alpha \in \Gamma$. Then S_α is a group. Hence there exists an identity element $e \in S_\alpha$ and for each $a \in S_\alpha$ there exists an inverse $a' \in S_\alpha$ such that $e \cdot a = a \cdot e = a$ and $a \cdot a' = a' \cdot a = e$. Then $a = e\alpha a = a\alpha e$ and $aaa' = a'\alpha a = e$. So we find that $a = e\alpha a \in e\Gamma a = e \circ a$, $a\alpha e \in a\Gamma e = a \circ e$ and $e = aaa' \in a\Gamma a' = a \circ a'$ and similarly $e \in a' \circ a$. Thus we find that (S, \circ_Γ) is a r-hypergroup.

Conversely suppose that (S, \circ_Γ) is a W -hypergroup. Then for $s \in S$, there exists an identity $e \in S$ and an element $s' \in S$ such that $s \in s \circ_\Gamma e \cap e \circ_\Gamma s$ and $e \in s \circ_\Gamma s' \cap s' \circ_\Gamma s$. Let $a, b \in S$ and a' be an inverse of a . Now $b \in b \circ_\Gamma e \cap e \circ_\Gamma b = b\Gamma e \cap e\Gamma b$. Hence $b = eab$ for some $\alpha \in \Gamma$. Thus $b = eab \subseteq a\Gamma a' \alpha b \subseteq a\Gamma S$. Hence $S \subseteq a\Gamma S$ which implies that $S = a\Gamma S$. Similarly we can prove that $S = S\Gamma a$. Hence S is a Γ -group. \square

From Theorem 3.1 and Theorem 3.2 we can conclude the following.

Corollary 3.3. *Let S be a Γ -semigroup. Then (S, \circ_Γ) is a r-hypergroup if and only if it is a hypergroup.*

Example 3.4. Let $S = \{4n + 1 : n \in \mathbb{Z}, \text{ the set of all integers } \}$ and $\Gamma = \{3, 7\}$. Then S is a Γ -semigroup. Again in S_3 , -3 plays the role of identity and $4(-n - 2)$ is an inverse of $4n + 1$. So S is a Γ -group and by Theorem 3.2, (S, \circ_Γ) is a r-hypergroup. Let e be an identity element of (S, \circ_Γ) . Then for every $a \in S$, $a + \alpha + e = e + \alpha + a = a$ for some $\alpha \in \Gamma$. This implies $e = -3$ or $e = -7$. Hence S is a nontrivial r-hypergroup.

Here we see that for a Γ -semigroup S we can construct a semihypergroup (S, \circ_Γ) . Applying Theorem 2.26 and Theorem 3.1 we can prove the Theorem 3.1 of [9].

Corollary 3.5. *Let S be a Γ -semigroup. S is regular if and only if for any left ideal A and for any right ideal B of S , $A \cap B = B\Gamma A$.*

4. Prime hyperideals and related concepts in semihypergroups.

In the last section we see that we can construct a semihypergroup, using a Γ -semigroup which is called the semihypergroup associated with Γ -semigroup. We see also that some basic structures of a Γ -semigroup can be studied with the help of the semihypergroup associated with this Γ -semigroup and vice versa. Ideal theory is one of such structure. In this section our aim is to generalize ideal theory in semigroups to semihypergroups.

We give now the definition of a prime hyperideal in a semihypergroup and study such hyperideals. First we recall the definition of one-sided prime ideals in Γ - semigroups.

Definition 4.1. A one-sided ideal P of a Γ - semigroup S is called prime if $a\Gamma Sb \subseteq P$ implies $a \in P$ or $b \in P$ for any two elements a, b of S .

We define the above notion in a semihypergroup.

Definition 4.2. A one-sided hyperideal P of a semihypergroup (S, \circ) is called prime if $a \circ S \circ b \subseteq P$ implies $a \in P$ or $b \in P$ for any two elements a, b of S .

The Theorem 4.3 follows immediately.

Theorem 4.3. Let S be a Γ - semigroup. Then $P \subseteq S$ is a left (resp. right) prime ideal of the Γ - semigroup S if and only if P is a left (resp. right) prime ideal of (S, \circ_Γ) .

Theorem 4.4. Let (S, \circ) be a semihypergroup. For a left hyperideal P of S , the following statements are equivalent:

- (i) P is prime.
- (ii) $I \circ J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$ where I and J are two left ideals of S .

Proof. (i) \Rightarrow (ii): Let $I, J \not\subseteq P$ for two left ideals I and J of S , where $I \circ J \subseteq P$. So there exist $x \in I$ and $y \in J$ such that $x, y \notin P$. Since P is prime there exists $s \in S$ such that $x \circ s \circ y \notin P$. But $x \circ s \circ y \in I \circ J$. Hence $I \circ J \not\subseteq P$, which is a contradiction. Hence (ii) holds.

(ii) \Rightarrow (i): Let $x, y \in S$ be such that $x \circ S \circ y \subseteq P$. Then $S \circ x \circ S \circ y \subseteq S \circ P \subseteq P$. Again by (ii) we have $S \circ x \subseteq P$ or $S \circ y \subseteq P$. Let $S \circ x \subseteq P$ and $I = S \circ x \cup \{x\}$ be the left ideal of S generated by x . Then $I \circ I \subseteq S \circ x \subseteq P$ which implies that $I \subseteq P$. Hence $x \in P$. Similarly we can show that if $S \circ y \subseteq P$ then $y \in P$. \square

Note that we obtain Theorem 3.2 of [3] as a corollary of the above theorem.

Corollary 4.5. Let S be a Γ - semigroup. For a left ideal P of S , the following statements are equivalent:

- (i) P is prime.
- (ii) $I\Gamma P$ implies that $I \subseteq P$ or $J \subseteq P$ where I and J are two left ideals of S .

Definition 4.6. A proper hyperideal P of a semihypergroup (S, \circ) is called a prime hyperideal if $A \circ B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any two hyperideals A, B of S .

We recall the definition of prime ideals in Γ - semigroups.

Definition 4.7 ([4]). An ideal P of a Γ - semigroup S is called a prime ideal if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any two ideals A, B of S .

From Theorem 3.1 we can conclude the following result.

Corollary 4.8. Let S be a Γ - semigroup. Then $P \subseteq S$ is a prime ideal of the Γ - semigroup S if and only if P is a prime ideal of (S, \circ_Γ) .

In a semihypergroup we can prove the following characterization theorem of prime ideals.

Theorem 4.9. Let (S, \circ) be a semihypergroup. For a hyperideal P of S , the following statements are equivalent:

- (i) If A and B are hyperideals of S such that $A \circ B \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.
- (ii) If $\langle a \rangle, \langle b \rangle$ are principal hyperideals of S such that $\langle a \rangle \circ \langle b \rangle \subseteq P$ then either $a \in P$ or $b \in P$.
- (iii) If $a \circ S \circ b \subseteq P$ then either $a \in P$ or $b \in P$ ($a, b \in S$).
- (iv) If I_1 and I_2 are two right hyperideals of S such that $I_1 \circ I_2 \subseteq P$ then either $I_1 \subseteq P$ or $I_2 \subseteq P$.
- (v) If J_1 and J_2 are two left hyperideals of S such that $J_1 \circ J_2 \subseteq P$ then either $J_1 \subseteq P$ or $J_2 \subseteq P$.

Definition 4.10. An ideal M of a semihypergroup S is called maximal if $M \neq S$ and there does not exist any ideal M_1 of S such that $M \subset M_1 \subset S$.

Theorem 4.11. *Let (S, \circ) be a semihypergroup. If I is a hyperideal of S and P is a prime hyperideal of S then $I \cap P$ is a prime hyperideal of I considering I as a semihypergroup.*

Proof. Proof is similar to the proof in semigroup theory. \square

We observe now that Proposition 3.5 of [2] is obtained as a corollary of the above theorem.

Corollary 4.12. *Let S be a Γ - semigroup. If I is an ideal of S and P is a prime ideal of S then $I \cap P$ is a prime ideal of I considering I as a Γ - semigroup.*

We recall now the definition of an m -system in a Γ - semigroup.

Definition 4.13 ([2]). Let S be a Γ - semigroup. A subset H of S is said to be an m -system of S if and only if $c \in H, d \in H$ imply that there exist elements $p \in S$ and $\alpha_1, \alpha_2 \in \Gamma$ such that $c\alpha_1 p \alpha_2 d \in H$. The empty set is considered to be an m -system.

Definition 4.14. A subset M of a semihypergroup (S, \circ) is said to be an m -system of S if for $a, b \in M$ there exists an element $s \in S$ such that $a \circ s \circ b \cap M \neq \phi$. The empty set is considered to be an m -system.

Corollary 4.15. *Let S be a Γ - semigroup. Then $M \subseteq S$ is an m -system of the Γ - semigroup S if and only if M is an m -system of (S, \circ_Γ) .*

Theorem 4.16. *A hyperideal I of a semihypergroup (S, \circ) is prime if and only if its complement I^c is an m -system.*

Proof. Let I be a prime hyperideal of S . Then for $a, b \in I^c, a \circ S \circ b \not\subseteq I$ i.e., there exists $p \in S$ such that $a \circ p \circ b \cap I^c \neq \phi$. Hence I^c is an m -system. Now let I^c be an m -system. \square

The following result of [2] is obtained as corollary of the above theorem.

Corollary 4.17. *An ideal I of a Γ - semigroup S is prime if and only if its complement I^c is an m -system.*

Applying Zorn's lemma we can prove the following theorem

Theorem 4.18. *Let M be an m -system disjoint from a hyperideal I of a semihypergroup (S, \circ) . Then there exists an m -system $N \supseteq M$ of S which is maximal in the class of m -systems of S disjoint from I .*

Theorem 4.19. *Let (S, \circ) be a semihypergroup. Let A be a hyperideal of S disjoint from an m -system M of S . Then there exists a hyperideal $P \supseteq A$ which is maximal in the class of hyperideals containing A and disjoint from M . Moreover P is prime.*

Proof. Applying Zorn's lemma we have a hyperideal P which is maximal in the class of hyperideals containing A and disjoint from M . We prove now that P is prime. Let $a \notin P$ and $b \notin P$. Since P is maximal, $P \cup \{a\}$ contains an element m_1 of M and $P \cup \{b\}$ contains an element m_2 of M . Since M is an m -system there exists $c \in S$ such that $m_1 \circ c \circ m_2 \cap M \neq \phi$. We have $m_1 \circ c \circ m_2 \subseteq (p \cup \{a\}) \circ (p \cup \{b\}) \subseteq P$ and hence $m_1 \circ c \circ m_2 \subseteq P$ i.e., $M \cap P \neq \phi$ which is a contradiction. Hence $\langle a \rangle \circ \langle b \rangle \not\subseteq P$. Thus P is prime. \square

In [8] Stefan Schwarz studied prime ideals and maximal ideals in semigroups. Here we generalize the results in semihypergroup theory.

Theorem 4.20. *Let (S, \circ) be a semihypergroup with $S = S^2$. Then every maximal ideal of S is a prime ideal of S .*

Proof. Let M be a maximal ideal of S . Let $S - M = P$. Now $S = (M \cup P) \circ (M \cup P) = M \circ M \cup MP \cup PM \cup P \circ P \subseteq M \cup P \circ P$. Since $M \cap P = \phi$, we have $P \subseteq P \circ P$. Hence we get $P = P \circ P$. Let us suppose now that $A, B \not\subseteq M$ are two ideals of S such that $A \circ B \subseteq M$. Since $A \not\subseteq M$ and M is maximal, we have $A \cup M = S$. Hence $P \subseteq A$. Similarly we can say that $P \subseteq B$. Hence $P \circ P \subseteq A \circ B$, i.e., $P \subseteq A \circ B$ which contradicts the fact that $A \circ B \subseteq M$. Hence M is a prime ideal of S . \square

Theorem 4.21. *If M is a maximal ideal of a semihypergroup S such that $S - M$ contains an idempotent, then M is a prime ideal of S .*

Proof. Let M be a maximal ideal of S such that $S - M$ contains an idempotent element e . Let $A \not\subseteq M$ and $B \not\subseteq M$ be two hyperideals such that $A \circ B \subseteq M$. Since M is maximal, we have $M \cup A = S = M \cup B$. Now $S \circ S = (M \cup A) \circ (M \cup B) = M \circ M \cup M \circ B \cup A \circ M \cup A \circ B \subseteq M$. Thus we have $e \in e \circ e \subseteq S \circ S \subseteq M$ which is a contradiction. Hence M is prime. \square

Theorem 4.22. *Let $\{M_\alpha : \alpha \in \Lambda\}$ be the set of all different maximal ideals of a semihypergroup S . Let $\text{card } \Lambda \geq 2$ and let $P_\alpha = S - M_\alpha$ and $M^* = \bigcap_\alpha M_\alpha$. Then we have:*

- (i) $P_\alpha \cap P_\beta = \phi$ for $\alpha \neq \beta$.

- (ii) $S = [\bigcup_{\alpha \in \Lambda} P_\alpha] \cup M^*$.
- (iii) For every $\nu \neq \alpha$ we have $P_\alpha \subseteq M_\nu$.
- (iv) If I is an ideal of S and $I \cap P_\alpha \neq \phi$, then $P_\alpha \subseteq I$.
- (v) For $\alpha \neq \beta$ we have $P_\alpha \circ P_\beta \subseteq M^*$, so that M^* is nonempty.

Proof. The proof is almost similar to the proof of the Theorem 2 of [8]. \square

Theorem 4.23. *Let S be a semihypergroup containing maximal ideals and let M^* be the intersection of all maximal ideals of S . Then every prime ideal of S containing M^* and different from S is a maximal ideal of S .*

Proof. The proof is almost similar to the proof of the Theorem 3 of [8]. \square

5. Semiprime hyperideals. In this section we introduce the concept of semiprime hyperideals and study some of its properties.

Definition 5.1. Let (S, \circ) be a semihypergroup. A hyperideal P of S is said to be semiprime if for any hyperideal A of S , $A \circ A \subseteq P$ implies $A \subseteq P$.

We note that every prime hyperideal is semiprime.

Theorem 5.2. *All hyperideals are semiprime in a regular semihypergroup.*

Proof. Let I be a hyperideal of S and $A \circ A \subseteq I$ for some hyperideal A of S . Now by Theorem 2.27, $A \circ A = A$. So we have $A \subseteq I$ and hence I is semiprime. \square

We can prove the following characterization theorem of semiprime ideals.

Theorem 5.3. *If P is a hyperideal of a semihypergroup (S, \circ) , then the following statements are equivalent:*

- (i) P is semiprime.
- (ii) If $a \in S$ such that $a \circ S \circ a \subseteq P$ then $a \in P$.
- (iii) If $\langle a \rangle$ is a principal hyperideal of S such that $\langle a \rangle \circ \langle a \rangle \subseteq P$ then $a \in P$.

(iv) If U is a right hyperideal in S such that $U \circ U \subseteq P$ then $U \subseteq P$.

(v) If U is a left hyperideal in S such that $U \circ U \subseteq P$ then $U \subseteq P$.

Definition 5.4. Let (S, \circ) be a semihypergroup. A set N of S is said to be an n -system of S if for every $a \in N$, there exists $x \in S$ such that $a \circ x \circ a \cap N \neq \phi$. The empty set is considered as an n -system.

From the above definition it is clear that every m -system is an n -system.

Theorem 5.5. Let Q be a hyperideal of a semihypergroup (S, \circ) . Then Q is semiprime if and only if Q^c is an n -system of S .

Proof. Let Q be a semiprime hyperideal of S . Let $a \notin Q$. Then $a \circ S \circ a \not\subseteq Q$. i.e., there exists an element $x \in S$ such that $a \circ x \circ a \cap Q^c \neq \phi$. Hence Q^c is an n -system. Next let Q^c be an n -system and $a \in Q^c$. Thus there exists $x \in S$ such that $a \circ x \circ a \cap Q^c \neq \phi$ i.e., for $a \notin Q$, $a \circ S \circ a \not\subseteq Q$. Hence Q is semiprime. \square

Theorem 5.6. Let (S, \circ) be a semihypergroup. If N is an n -system in S and $a \in N$ there exists an m -system M in S such that $a \in M$ and $M \subseteq N$.

Proof. Let $a_1 = a$. Since $a_1 \in N$, there exists $b_1 \in S$ such that $a_1 \circ b_1 \circ a_1 \cap N \neq \phi$. We now choose a_2 as some element of $a_1 \circ b_1 \circ a_1 \cap N$. By similar argument we can choose a_i from the set $a_{i-1} \circ b_{i-1} \circ a_{i-1} \cap N$. We now take $M = \{a_1, a_2, a_3, \dots\}$. Then $a \in M$ and $M \subseteq N$. We now show that M is an m -system. Let $a_i, a_j \in M$ with $i \leq j$. Then $a_{j+1} \in a_j \circ b_j \circ a_j \subseteq a_i \circ a_j$. Thus we have an element $s \in S$ such that $a_{j+1} \in a_i \circ s \circ a_j \cap M$ i.e., $a_i \circ s \circ a_j \cap M \neq \phi$. Hence M is an m -system. \square

Now as in semigroup theory we can prove the following theorem.

Theorem 5.7. A hyperideal Q of a semihypergroup (S, \circ) is semiprime if and only if $r(Q) = Q$ where $r(Q)$ is the intersection of prime ideals of S containing Q .

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