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A STUDY OF GENERALIZED FRACTIONAL INTEGRATION OPERATORS WITH $\overline{H}$-FUNCTIONS COMPOSITION ON SPACES $F_{p,\mu}$ AND $F'_{p,\mu}$

BY

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Abstract. First we define two fractional operators as

$$\left(R^{(m,M),(n,N)}_{(r,R),(q,Q)}\phi\right)(x)=\frac{1}{x}\int_0^x \overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\,\middle|\,\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,r}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right]$$
$$\overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\,\middle|\,\begin{matrix}\left(a'_j,\alpha'_j;A'_j\right)_{1,N},\left(a'_j,\alpha'_j\right)_{N+1,R}\\\left(b'_j,\beta'_j\right)_{1,M},\left(b'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{matrix}\right]\phi(t)dt(x>0)$$

$$\left(A^{(m,M),(n,N)}_{(r,R),(q,Q)}\phi\right)(x)=\frac{1}{x}\int_x^\infty \overline{H}^{m,n}_{r,q}\left[\frac{x}{t}\,\middle|\,\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,r}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right]$$
$$\overline{H}^{M,N}_{R,Q}\left[\frac{x}{t}\,\middle|\,\begin{matrix}\left(a'_j,\alpha'_j;A'_j\right)_{1,N},\left(a'_j,\alpha'_j\right)_{N+1,R}\\\left(b'_j,\beta'_j\right)_{1,M},\left(b'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{matrix}\right]\phi(t)dt(x>0)$$

and now we will study the properties of above operators in space $F_{p,\mu}$ and $F'_{p,\mu}$.

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1. Introduction. An integral transforms with Fox's H-function as a kernel were considered by many authors, Shalapakov [8]-[10] (see also [13, chapter 5] and [22, §39.2] arise special cases of such transforms which generalize classical fractional integration operators. Transforms having Fox's H-function kernel in $L_p(0,\infty)$ space were considered by Kiryakova [12], [13], Kalla and Kiryakova [5], [6] and in the space $F_{p,\mu}$ and the corresponding space of generalized function $F'_{p,\mu}$ by Raina and Saigo [18], Saigo, Raina and Kilbas [21]. Further, fractional calculus operators with gauss hypergeometric function ${}_2F_1(a, b; c; z)$ in $F_{p,\mu}$ and $F'_{p,\mu}$ were studied by SAIGO and GLAESKE [19], [20].

Here our paper concerns with the fractional integration operators involving $\bar{H}$-function of general kind's composition in space $F_{p,\mu}$ and $F'_{p,\mu}$, where we open $\overline{H}^{M,N}_{R,Q}$ always with its series which helps us in many ways. In equation (1) and (2) we used $\bar{H}$- function, introduced by Inayat Hussain in terms of Mellin-Barnes type contour integral, is defined by

$$(1)\qquad \overline{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\varphi}\,(\xi)z^{\xi}d\xi,$$

where

$$(2)\qquad \overline{\varphi}\,(\xi)=\frac{\prod\limits_{j=1}^{m}\Gamma\,(b_j-\beta_j\xi)\prod\limits_{j=1}^{n}\{\Gamma\,(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod\limits_{j=m+1}^{q}\{\Gamma\,(1-b_j+\beta_j\xi)\}^{B_j}\prod\limits_{j=n+1}^{p}\Gamma\,(a_j-\alpha_j\xi)}$$

which contains the fractional power of some of the gamma functions. Here and throughout the paper $a_j(j = 1, \dots, q)$ and $b_j(j = 1, \dots, q)$ are complex parameters, $\alpha_j \geq 0$, $(j = 1, \dots, p)$, $\beta_j > 0, (j = 1, \dots, p)$ (not all zero simultaneously) and the exponents $A_j(j = 1, \dots, n)$ and $B_j(j = m+1, \dots, q)$ can take non-integer values. The contour in (3) is imaginary axis $Re(\xi) = 0$. It is suitably indented in order to avoid the singularities of the gamma functions and to keep those singularities on appropriate sides. Again, for Aj $(j = 1, , n)$ not an integer, the poles of the gamma functions of the numerator in (2) are converted to the branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j - \beta_j\xi)$, $(j = 1, \dots, m)$ and $\Gamma(1 - a_j + \alpha_j\xi)$, $(j = 1, \dots, n)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

The following sufficient conditions for the absolute convergence of the defining integral for the $\bar{H}$-function given by equation (1) have given by BUSCHMAN and SRIVASTAVA [2].

$$(3)\qquad \Omega=\sum_{j=1}^{m}\beta_j+\sum_{j=1}^{n}|A_j\alpha_j|-\sum_{j=m+1}^{q}|B_j\beta_j|-\sum_{j=n+1}^{p}\alpha_j>0$$

and $|\arg(z)| < \frac{1}{2}\Pi\Omega$. The behaviour of the $\bar{H}$-function for small values of $|z|$ follows easily from a result recently given by Rathie, we have

$$(4)\qquad \overline{H}^{m,n}_{p,q}\,[z]=0\,(|z|^{\rho})\,,\quad \rho=\min_{1\leq j\leq m}[\mathrm{Re}\,(b_j|\,\beta_j)]\,,|z|\to 0.$$

If we take $A_j = 1(j = 1, \dots, n)$, $B_j = 1(j = m+1, \dots, q)$ in (3), the function $\bar{H}$ reduces to the Fox H-function [2]. The following series representation for the $\bar{H}$-function [10, pp. 305-306, Eq. (6.8)] will be required in the sequal:

(5)

$$\overline{H}_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,p} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{matrix}\right]$$

$$= \sum_{h=1}^{m} \sum_{r=0}^{\infty} \frac{\prod_{\substack{j=1 \\ j \neq h}}^{m} \Gamma\left(b_j - \beta_j \xi_{h,r}\right) \prod_{j=1}^{n} \left\{\Gamma\left(1 - a_j + \alpha_j \xi_{h,r}\right)\right\}^{A_j} (-1)^r}{\prod_{j=m+1}^{q} \left\{\Gamma\left(1 - b_j + \beta_j \xi_{h,r}\right)\right\}^{B_j} \prod_{j=n+1}^{q} \Gamma\left(a_j - \alpha_j \xi_{h,r}\right) r!} \cdot \frac{z^{\xi_{h,r}}}{\beta_h}$$

where $\xi_{h,r} = \frac{b_h + r}{\beta_h}$ The paper is organized as follows:

Section 2. Deals with some required results about $\bar{H}$-function, and theory of spaces $F_{p,\mu}$ and $F'_{p,\mu}$.

Section 3. We state and prove conditions for existence of the generalized fractional integration operator (1) and (2) in the space $F_{p,\mu}$ and prove a simple property for them.

Section 4. Contain Mellin transform of such operators and relation of integration by parts.

Section 5. Is devoted to define operators (1) and (2) in the space $F'_{p,\mu}$ and give the results similar to those in section 3.

$\bar{H}$ -Function properties and spaces $F_{p,\mu}$ and $F'_{p,\mu}$:

Firstly, we will state some properties of $\bar{H}$-function

(6)

$$Z^{\lambda} \overline{H}_{r,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{matrix}\right]$$

$$= \overline{H}_{r,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j + \lambda\alpha_j, \alpha_j; A_j)_{1,n}, (a_j + \lambda\alpha_j, \alpha_j)_{n+1,r} \\ (b_j + \lambda\beta_j, \beta_j)_{1,m}, (b_j + \lambda\beta_j, \beta_j; B_j)_{m+1,q} \end{matrix}\right],$$

(7)

$$\left(M\overline{H}_{r,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j, B_j)_{m+1,q} \end{matrix}\right]\right)(s) = \overline{\varphi}(s),$$

$$\left(\min_{1 \le i \le n}\left[\frac{Re(1 - a_i)}{\alpha_i}\right]\right) < Re(s) < \left(\min_{1 \le j \le n}\left[\frac{Re(b_j)}{\beta_j}\right]\right)$$

where $\overline{\varphi}(s)$ is defined by (3) and M is the Mellin transform

$$\left(M\psi\right)(s) = \int_0^{\infty} \psi(x) x^{s-1} dx.$$

Further asymptotic behavior of $\overline{H}^{m,n}_{r,q}(z)$ near zero is given in (4). Now we give definition and further we state some theory about $F_{p,\mu}$ and $F'_{p,\mu}$ developed by McBride [15], [16]. For $1 \leq p < \infty$ and $\mu \in C$ we denote by $F_{p,\mu}$ the space of functions

$$F_{p,\mu} = \left\{ \phi \in C^{\infty}(R_+) {:} x^k \frac{d^k}{dx^k} \left(x^{-\mu}\phi\right) \in L^p(R_+) \text{ for k} \in \mathrm{N}_0 \right\} \tag{8}$$

for $1 \leq p < \infty$ and

$$F_{\infty,\mu} = \left\{ \phi \in C^{\infty}(R_+) : x^k \frac{d^k}{dx^k} \left(x^{-\mu}\phi\right) \right. \\ \left. \to 0 \text{ as } x \to +0 \text{ and } x \to \infty \text{ for } k \in N_0 \right\} \tag{9}$$

for $p = \infty$,where $N_0 = 0, 1, 2, \ldots$. For each p and μ, $F_{p},_{\mu}$, is a complete countable multi-normed space (Frechet space) equipped with the topology generated by the family of semi norms $\{\gamma_k^{p,\mu}\}_{k=0}^{\infty}$ with

$$\gamma_k^{p,\mu}(\phi) = \left\| x^k \frac{d^k}{dx^k}\left(x^{-\mu}\phi\right) \right\|_p (k \in N_0) \tag{10}$$

$F'_{p,\mu}$ is the space of continuous linear functionals on $F_{p,\mu}$ equipped with the weak topology. We shall denote by f and element of $F'_{p,\mu}$ and by $< f, \varphi >$ the value of f at a test function $\varphi \in F_{p,\mu}$. For any $f \in F'_{p,\mu}$ we denote $x^{\lambda}f$ the functional defined by

$$\left\langle x^{\lambda}f, \varphi \right\rangle = \left\langle f, x^{\lambda}\varphi \right\rangle, \qquad (\varphi \in F_{p,\mu-\lambda}).$$

We must know that the space $F'_{p,\mu}$ is always a complete space.

Now we will need some Lemmas to prove our theorems.

Lemma 1 ([15, p. 18, Corollary 2.7.]). *The space $C_0^{\infty}(R_+)$ of functions $\varphi \in C^{\infty}(R_+)$ which have compact support is dense in $F_{p,\mu}$ for $1 \leq p < \infty$ and $\mu \in C$.*

Lemma 2 ([15, p. 21, Corollary 2.11.]). *The operator x^{λ} with $\lambda \in C$ defined by*

$$(x^{\lambda}\varphi) = x^{\lambda}\varphi(x) \tag{11}$$

is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+\lambda}$ with the inverse $x^{-\lambda}$.

Lemma 3 ([16, p. 533, Lemma 6.1.]). *Let the function $k(x)$ be defined almost everywhere on R_+ and*

$$\int_0^\infty x^{1/p-Re(\mu)-1}\,|k(x)|\,dx<\infty. \tag{12}$$

If T is an integral transform defined by

$$(T\varphi)(x)=(k*\varphi)(x)=\int_0^\infty k\left(\frac{x}{t}\right)\frac{1}{t}\varphi(t)dt \tag{13}$$

then T is a continuous linear mapping from $F_{p,\mu}$ into itself.

Lemma 4 ([15, p. 32, Theorem 2.22.]). *For $\lambda\in C$ the operator x^λ is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-\gamma}$.*

Existence of generalized fractional calculus operator on $\boldsymbol{F_{p,\mu}}$. We denote $p'=\frac{p}{p-1}$ for $1\le p\le\infty$.

Theorem 1. *Let ρ be defined by (4). Then*

I. If $Re(\mu)>-\rho-\rho'-\frac{1}{p'}$ then the operator $R^{(m,M),(n,N)}_{(r,R),(q,Q)}$ is a continuous linear mapping from $F_{p,\mu}$ into itself, and for $\varphi\in F_{p,\mu}$ and $\lambda\in C$ there hold the relation

$$\left(x^\lambda R^{(m,M),(n,N)}_{(r,R),(q,Q)}\left[\begin{array}{l}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,r}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{array}\right]\right.$$
$$\left.\cdot\left[\begin{array}{l}\left(a'_j,\alpha'_j;A'_j\right)_{1,N},\left(a'_j,\alpha'_j\right)_{N+1,R}\\\left(b'_j,\beta'_j\right)_{1,M},\left(b'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{array}\right]\varphi\right)(x)$$

$$=\left(R^{(m,M),(n,N)}_{(r,R),(q,Q)}\left[\begin{array}{l}\left(a_j-\frac{\lambda}{2}\alpha_j,\alpha_j;A_j\right)_{1,n},\left(a_j-\frac{\lambda}{2}\alpha_j,\alpha_j\right)_{n+1,r}\\\left(b_j-\frac{\lambda}{2}\beta_j,\beta_j\right)_{1,m},\left(b_j-\frac{\lambda}{2}\beta_j,\beta_j;B_j\right)_{m+1,q}\end{array}\right]\right.$$

$$\left.\cdot\left[\begin{array}{l}\left(a'_j-\frac{\lambda}{2}\alpha'_j,\alpha'_j;A'_j\right)_{1,N},\left(a'_j-\frac{\lambda}{2}\alpha'_j,\alpha'_j\right)_{N+1,R}\\\left(b'_j-\frac{\lambda}{2}\beta'_j,\beta'_j\right)_{1,M},\left(b'_j-\frac{\lambda}{2}\beta'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{array}\right]x^\lambda\varphi\right)(x) \tag{14}$$

II. If $Re(\mu) < \rho + \rho' - \frac{1}{p'}$*, then the operator* $A^{(m,M),(n,N)}_{(r,R),(q,Q)}$ *is a continuous linear mapping from* $F_{p,\mu}$ *into itself, and for* $\varphi \in F_{p,\mu}$ *and* $\lambda \in C$ *there hold the relation*

(15)
$$\left(x^{\lambda} A^{(m,M),(n,N)}_{(r,R),(q,Q)} \left[\begin{matrix} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{matrix} \right] \cdot \left[\begin{matrix} \left(a'_j, \alpha'_j, A'_j\right)_{1,N}, \left(a'_j, \alpha'_j\right)_{N+1,R} \\ \left(b'_j, \beta'_j\right)_{1,M}, \left(b'_j, \beta'_j; B'_j\right)_{M+1,Q} \end{matrix} \right] \varphi \right)(x)$$
$$= \left(A^{(m,M),(n,N)}_{(r,R),(q,Q)} \left[\begin{matrix} \left(a_j + \frac{\lambda}{2}\alpha_j, \alpha_j; A_j\right)_{1,n}, \left(a_j + \frac{\lambda}{2}\alpha_j, \alpha_j\right)_{n+1,r} \\ \left(b_j + \frac{\lambda}{2}\beta_j, \beta_j\right)_{1,m}, \left(b_j + \frac{\lambda}{2}\beta_j, \beta_j; \beta_j\right)_{m+1,q} \end{matrix} \right] \cdot \left[\begin{matrix} \left(a'_j + \frac{\lambda}{2}\alpha'_j, \alpha'_j; A'_j\right)_{1,N}, \left(a'_j + \frac{\lambda}{2}\alpha'_j, \alpha'_j\right)_{N+1,R} \\ \left(b'_j + \frac{\lambda}{2}\beta'_j, \beta'_j\right)_{1,M}, \left(b'_j + \frac{\lambda}{2}\beta'_j, \beta'_j, B'_j\right)_{M+1,Q} \end{matrix} \right] x^{\lambda}\varphi \right)(x).$$

Proof. To prove the statement we will use the Lemma 3. Here we know that kernel of $R^{(m,M),(n,N)}_{(r,R),(q,Q)}$ is

$$k(u) = \begin{cases} 0, & \text{if } 0 < u < 1, \\ \frac{1}{u}\overline{H}^{m,n}_{r,q}\left(\frac{1}{u}\right)\overline{H}^{M,N,}_{R,Q}\left(\frac{1}{u}\right), & \text{if } u \geq 1. \end{cases}$$

and therefore,

$$\int_0^{\infty} u^{1/p - Re(\mu) - 1} |k(u)|\, du = \int_0^1 0 du$$
$$+ \int_1^{\infty} u^{1/p - Re(\mu) - 1} \frac{1}{u} \left| \overline{H}^{m,n}_{r,q}\left(\frac{1}{u}\right) \overline{H}^{M,N}_{R,Q}\left(\frac{1}{u}\right) \right| du$$
$$= \int_1^{\infty} u^{1/p - Re(\mu) - 2} \left| \overline{H}^{m,n}_{r,q}\left(\frac{1}{u}\right) \overline{H}^{M,N}_{R,Q}\left(\frac{1}{u}\right) \right| du.$$

Now, let $\frac{1}{u} = x$ so that $-\frac{1}{u^2}\, du = dx$ so

$$= -\int_1^0 x^{-1/p' + Re(\mu)} \left| \overline{H}^{m,n}_{r,q}(x) \overline{H}^{M,N}_{R,Q}(x) \right| dx$$

(16)
$$= \int_0^1 x^{Re(\mu) - 1/p'} \left| \overline{H}^{m,n}_{r,q}(x)\, \overline{H}^{M,N}_{R,Q}(x) \right| dx.$$

Here integral is finite always in interval $(0,1]$ but for $x = 0$, we have to check asymptotic behavior of integral $x^{Re(\mu)-\frac{1}{p}}|\overline{H}^{m,n}_{r,q}(x)\overline{H}^{M,N}_{R,Q}(x)| \sim x^{Re(\mu)+\rho+\rho'-\frac{1}{p}}$ and so we obtain,

$$\int_0^1 x^{Re(\mu)-\frac{1}{p}}\left|\overline{H}^{m,n}_{r,q}(x)\ \overline{H}^{M,N}_{R,Q}(x)\right|dx = \int_0^1 x^{\left[Re(\mu)+\rho+\rho'-\frac{1}{p}+1\right]-1}$$

and we know that integral $\int_0^1 x^{n-1}dx$ exists iff $n > 0$. Similarly we can obtain

$$Re(\mu)+\rho+\rho'-\frac{1}{p}+1>0$$

i.e.

$$Re(\mu) > -\rho-\rho'+\frac{1}{p}-1$$

or

$$Re(\mu) > -\rho-\rho'-\left(\frac{p-1}{p}\right)$$

or $Re(\mu) > -\rho-\rho'-\frac{1}{p'}$ where $p' = \frac{p}{p-1}$, and we see that (16) will converge iff $Re(\mu) > -\rho-\rho^{'}-\frac{1}{p'}$. Now to prove the equality of (14), we take L.H.S. of (14)

$$\left(x^{\lambda}R^{(m,M),(n,N)}_{(r,R),(q,Q)}\left[\begin{array}{l}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,r}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{array}\right]\right.$$

$$\left.\cdot\left[\begin{array}{l}\left(a'_j,\alpha'_j;A'_j\right)_{1,N},\left(a'_j,\alpha'_j\right)_{N+1,R}\\\left(b'_j,\beta'_j\right)_{1,M},\left(b'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{array}\right]\varphi\right)(x)$$

$$=\frac{1}{x}\int_0^x x^{\lambda}\overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\right]\overline{H}^{M,N}_{R,Q}\left[\frac{t}{x}\right]\varphi(t)dt$$

$$=\frac{1}{x}\int_0^x\left(\frac{t}{x}\right)^{-\frac{\lambda}{2}}\overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\right]\left(\frac{t}{x}\right)^{-\frac{\lambda}{2}}\overline{H}^{M,N}_{R,Q}\left[\frac{t}{x}\right]t^{\lambda}\varphi(t)dt$$

(multiplying and dividing by t^{λ}). Further using property of $\overline{H}$-function, we have

$$=\frac{1}{x}\int_0^x\overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\left|\begin{array}{l}\left(a_j-\frac{\lambda}{2}\alpha_j,\alpha_j;A_j\right)_{1,n},\left(a_j-\frac{\lambda}{2}\alpha_j,\alpha_j\right)_{n+1,r}\\\left(b_j-\frac{\lambda}{2}\beta_j,\beta_j\right)_{1,m},\left(b_j-\frac{\lambda}{2}\beta_j,\beta_j;B_j\right)_{m+1,q}\end{array}\right.\right]$$

$$\cdot \overline{H}_{R,Q}^{M,N}\left[\frac{t}{x}\left|\begin{array}{l}\left(a_j^i-\frac{\lambda}{2}\alpha_j',\alpha_j';A_j'\right)_{1,N},\left(a_j'-\frac{\lambda}{2}\alpha_j',\alpha_j'\right)_{N+1,R}\\ \left(b_j'-\frac{\lambda}{2}\beta_j',\beta_j'\right)_{1,M},\left(b_j'-\frac{\lambda}{2}\beta_j',\beta_j';B_j'\right)_{M+1,Q}\end{array}\right.\right]t^{\lambda}\varphi(t)dt$$

$$=\left(R_{(r,R),(q,Q)}^{(m,M),(n,N)}\left[\begin{array}{l}\left(a_j-\frac{\lambda}{2}\alpha_j,\alpha_j;A_j\right)_{1,n},\left(a_j-\frac{\lambda}{2}\alpha_j,\alpha_j\right)_{n+1,r}\\ \left(b_j-\frac{\lambda}{2}\beta_j,\beta_j\right)_{1,m},\left(b_j-\frac{\lambda}{2}\beta_j,\beta_j;B_j\right)_{m+1,q}\end{array}\right]\right.$$

$$\left.\cdot\left[\begin{array}{l}\left(a_j'-\frac{\lambda}{2}\alpha_j',\alpha_j';A_j'\right)_{1,N},\left(a_j'-\frac{\lambda}{2}\alpha_j',\alpha_j'\right)_{N+1,R}\\ \left(b_j'-\frac{\lambda}{2}\beta_j',\beta_j'\right)_{1,M},\left(b_j'-\frac{\lambda}{2}\beta_j',\beta_j';B_j'\right)_{M+1,Q}\end{array}\right]t^{\lambda}\varphi\right)(x)$$

which prove the required result. On the similar lines we can prove part (II) of Theorem 1. Equations (14) and (15) are defined when $Re(\mu) > -\rho-\rho'-\frac{1}{p'}$ and $Re(\mu) < \rho+\rho'-\frac{1}{p'}$ respectively.

Theorem 2. I. *Let* $\varphi \in F_{p,\mu}$, $Re(\mu) > -\rho-\rho'-\frac{1}{p'}$ *and*

$$-\min_{1\le j\le m}\left[\frac{Re\,(b_j)}{\beta_j}\right] < Re(s)-1 < \min_{1\le i\le m}\left[\frac{1-Re(a_i)}{\alpha_i}\right],$$

$$-\min_{1\le j\le M}\left[\frac{Re(b_j')}{\beta_j'}\right] < Re(s)-1 < \min_{1\le i\le M}\left[\frac{1-Re(a_i')}{\alpha_i'}\right]$$

$$\left(MR_{(r,R),(q,Q)}^{(m,M),(n,N)}\left[\begin{array}{l}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,r}\\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{array}\right]\right.$$

$$\left.\cdot\left[\begin{array}{l}\left(a_j',\alpha_j';A_j'\right)_{1,N},\left(a_j',\alpha_j'\right)_{N+1,R}\\ \left(b_j',\beta_j'\right)_{1,M},\left(b_j',\beta_j';B_j'\right)_{M+1,Q}\end{array}\right]\varphi\right)(x)$$

$$=\sum_{h=1}^{M}\sum_{\sigma=0}^{\infty}\frac{\prod_{j=1}^{m}\Gamma\left(b_j'-\beta_j'\xi_{h,\sigma}\right)\prod_{j=1}^{m}\Gamma\left[(b_j-\beta_j+\beta_js)-\beta_j\xi_{h,\sigma}\right]}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b_j'+\beta_j'\xi_{h,\sigma}\right)\right\}^{B_j'}}$$

$$\cdot\frac{\prod_{j=1}^{N}\left\{\Gamma\left(1-a_j'+\alpha_j'\xi_{h,\sigma}\right)\right\}^{A_j'}}{\prod_{j=m+1}^{q}\left\{\Gamma\left[1-(b_j-\beta_j+\beta_is)+\beta_j\xi_{h,\sigma}\right]\right\}^{B_j}}$$

(17)

$$\frac{\prod_{j=1}^{m}\left\{\Gamma\left[1-(a_j-\alpha_j+\alpha_j s)\right]+\alpha_j\xi_{h,\sigma}\right\}^{A_j}}{\prod_{j=N+1}^{R}\Gamma\left(a'_j-\alpha'_j\xi_{h,\sigma}\right)\prod_{j=n+1}^{r}\Gamma[(a_j-\alpha_j+\alpha_j s)-\alpha_j\xi_{h,\sigma}]\sigma!\eta_h}(M\varphi)(s).$$

II. If $Re(\mu)<\rho+\rho'-\frac{1}{p'}$, $\varphi\in\ F_{p,\mu}$ and

$$-\min_{1\le i\le m}\left[\frac{1-Re\left(a_i\right)}{\alpha_i}\right]<Re(s)-1<\min_{1\le j\le m}\left[\frac{Re(b_j)}{\beta_j}\right],$$

$$-\min_{1\le i\le M}\left[\frac{1-Re(a'_i)}{\alpha'_i}\right]<Re(s)-1<\min_{1\le j\le M}\left[\frac{Re(b'_j)}{\beta'_j}\right]$$

$$\left(MA^{(m,M),(n,N)}_{(r,R),(q,Q)}\left[\begin{array}{l}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,r}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{array}\right]\right.$$
$$\left.\cdot\left[\begin{array}{l}\left(a'_j,\alpha'_j;A'_j\right)_{1,N},\left(a'_j,\alpha'_j\right)_{N+1,R}\\\left(b'_j,\beta'_j\right)_{1,M},\left(b'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{array}\right]\phi\right)(x)$$

(18)
$$=\sum_{h=1}^{M}\sum_{\sigma=0}^{\infty}\frac{\prod_{j=1}^{M}\Gamma\left(b'_j-\beta'_j\xi_{h,\sigma}\right)}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b'_j+\beta'_j\xi_{h,\sigma}\right)\right\}^{B'_j}}$$
$$\cdot\frac{\prod_{j=1}^{m}\Gamma\left[(b_j+\beta_j-\beta_j s)+\beta_j\xi_{h,\sigma}\right]}{\prod_{j=m+1}^{q}\left\{\Gamma\left[1-(b_j+\beta_j-\beta_j s)-\beta_j\xi_{h,\sigma}\right]\right\}^{B_j}}$$

$$=\sum_{h=1}^{M}\sum_{\sigma=0}^{\infty}\frac{\prod_{j=1}^{M}\Gamma\left(b'_j-\beta'_j\xi_{h,\sigma}\right)}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b'_j+\beta'_j\xi_{h,\sigma}\right)\right\}^{B'_j}}$$
$$\cdot\frac{\prod_{j=1}^{m}\Gamma\left[(b_j+\beta_j-\beta_j s)+\beta_j\xi_{h,\sigma}\right]}{\prod_{j=m+1}^{q}\left\{\Gamma\left[1-(b_j+\beta_j-\beta_j s)-\beta_j\xi_{h,\sigma}\right]\right\}^{B_j}}$$

$$\cdot\frac{\prod_{j=1}^{N}\{\Gamma(1-a'_j-\alpha'_j\xi_{h,r})\}^{A'_j}\prod_{j=1}^{n}\{\Gamma[1-(a_j+\alpha_j-\alpha_j s)-\alpha_j\xi_{h,r}]\}^{A_j}}{\prod_{j=N+1}^{R}\Gamma(a'_j-\alpha'_j\xi_{h,r})\prod_{j=n+1}^{r}\Gamma[(a_j+\alpha_j-\alpha_j s)+\alpha_j\xi_{h,r}]\sigma!\eta_h}(M\varphi)(s).$$

Proof. If $\varphi \in C_0^\infty(R_+)$, the relation (17) and (18) are proved by the Mellin Convolution relation

$$(M(k*\mu))(s) = (Mk)(s)(M\mu)(s),$$

the properties of the Mellin transform

$$(Mx^\lambda\varphi)(s) = (M\varphi)(s+\lambda) \quad (\lambda \in C)$$

$$(M\,\phi\,(x^{-1}))(s) \,=\, (M\,\phi)\,(-s)$$

and by the formulas stated in section 2. Taking L.H.S. of (17), we obtain

$$\left(MR^{(m,M),(n,N)}_{(r,R),(q,Q)}\varphi\right)(s)$$
$$=\left(M\left\{\frac{1}{x}\int_0^x \overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n}\,,\,(a_j,\alpha_j)_{n+1,r}\\(b_j,\beta_j)_{1,m}\,,\,(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]\right.\right.$$
$$\left.\left.\cdot\,\overline{H}^{M,N}_{R,Q}\left[\frac{t}{x}\left|\begin{matrix}\left(a'_j,\alpha'_j;A'_j\right)_{1,N}\,,\,\left(a'_j,\alpha'_j\right)_{N+1,R}\\\left(b'_j,\beta'_j\right)_{1,M}\,,\,\left(b'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{matrix}\right.\right]\varphi(t)dt\right\}(x)\right)(s)$$
$$=\left(M\;\left\{\frac{1}{x}\int_0^x \overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N}\,,\,(a_j,\alpha_j)_{n+1,r}\\(b_j,\beta_j)_{1,M}\,,\,(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]\right\}\right)$$
$$\cdot\left(\sum_{h=1}^{M}\sum_{\sigma=0}^{\infty}\overline{\phi}^{M,N}_{R,Q}\left(\frac{t}{x}\right)^{\xi_{h,\sigma}}\right)\varphi(t)dt\Bigg\}(x)\Bigg)(s)$$

where

$$\overline{\phi}^{M,N}_{R,Q} = \frac{\prod_{j=1}^{M}\Gamma\left(b'_j-\beta'_j\xi_{h,\sigma}\right)\prod_{j=1}^{m}\left\{\Gamma\left(1-a'_j+\alpha'_j\xi_{h,\sigma}\right)\right\}^{A'_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b'_j+\beta'_j\xi_{h,\sigma}\right)\right\}^{B'_j}\prod_{j=N+1}^{R}\Gamma\left(a'_j-\alpha'_j\xi_{h,\sigma}\right)\sigma!\eta_h}$$

and

$$\xi_{h},\sigma = \frac{b'_h + \sigma}{\beta'_h} = \sum_{h=1}^{M} \sum_{\sigma=0}^{\infty} \overline{\phi}_{R,Q}^{M,N} \left(M \left\{ \int_0^x \frac{1}{x} \left(\frac{t}{x} \right)^{\xi_{h,\sigma}} \right.\right.$$

$$\left.\left. \cdot \overline{H}_{r,q}^{m,n} \left[\frac{t}{x} \middle| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{array} \right) \varphi(t) dt \right\} (x) \right) (s)$$

$$= \sum_{h=1}^{M} \sum_{\sigma=0}^{\infty} \overline{\phi}_{R,Q}^{M,N} \left(M \left\{ \int_0^x \left(\frac{t}{x} \right)^{\xi_{h,\sigma}+1} \right.\right.$$

$$\left.\left. \cdot \overline{H}_{r,q}^{m,n} \left[\frac{t}{x} \middle| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{array} \right] \frac{\varphi(t)}{t} dt \right\} (x) \right) (s)$$

$$= \sum_{h=1}^{M} \sum_{\sigma=0}^{\infty} \overline{\phi}_{R,Q}^{M,N} \left(M \left\{ \left(\frac{1}{x} \right)^{\xi_{h,\sigma}+1} \right.\right.$$

$$\left.\left. \cdot \overline{H}_{r,q}^{m,n} \left[\frac{1}{x} \middle| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{array} \right] \right\} \right) (s)(M\varphi)(s)$$

$$= \sum_{h=1}^{M} \sum_{\sigma=0}^{\infty} \overline{\phi}_{R,Q}^{M,N} \left(M \left\{ x^{\xi_{h,\sigma}+1} \right.\right.$$

$$\left.\left. \cdot \overline{H}_{r,q}^{m,n} \left[x \middle| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{array} \right] \right\} \right) (-s)(M\varphi)(s)$$

$$= \sum_{h=1}^{M} \sum_{\sigma=0}^{\infty} \overline{\phi}_{R,Q}^{M,N} \left(M \left\{ \overline{H}_{r,q}^{m,n} \right.\right.$$

$$\left.\left. \cdot \left[x \middle| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_j; B_j)_{m+1,q} \end{array} \right] \right\} \right) (\xi_{h,\sigma} + 1 - s)(M\varphi)(s)$$

$$= \sum_{h=1}^{M} \sum_{\sigma=0}^{\infty} \overline{\phi}_{R,Q}^{M,N} \frac{\prod_{j=1}^{m} \Gamma(b_j - \xi_{h,\sigma}\beta_j + \beta_j s - \beta_j)}{\prod_{j=n+1}^{r} \Gamma(a_j - \alpha_j - \alpha_j \xi_{h,\sigma} + \alpha_j s)}$$

$$\cdot \frac{\prod_{j=1}^{n} \{\Gamma(1 - a_j + \alpha_j + \xi_{h,\sigma}\alpha_j - \alpha_j s)\}^{A_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \beta_j + \beta_j \xi_{h,\sigma} + \beta_j s)\}^{B_j}} (M\varphi)(s)$$

$$
= \sum_{h=1}^{M} \sum_{\sigma=0}^{\infty} \frac{\prod_{j=1}^{m} \Gamma\left(b'_j - \beta'_j \xi_{h,\sigma}\right) \prod_{j=1}^{m} \Gamma\left[(b_j - \beta_j + \beta_j s) - \beta_j \xi_{h,\sigma}\right]}{\prod_{j=M+1}^{Q} \left\{\Gamma\left(1 - b'_j + \beta'_j \xi_{h,\sigma}\right)\right\}^{B'_j}}
$$

$$
\cdot \frac{\prod_{j=1}^{N} \left\{\Gamma\left(1 - a'_j + \alpha'_j \xi_{h,\sigma}\right)\right\}^{A_j}}{\prod_{j=m+1}^{q} \left\{\Gamma\left[1 - (b_j - \beta_j + \beta_j s) + \beta_j \xi_{h,\sigma}\right]\right\}^{B_j}}
$$

$$
\cdot \frac{\prod_{j=1}^{N} \left\{\Gamma\left[1 - (a_j - \alpha_j + \alpha_j s) + \alpha_j \xi_{h,\sigma}\right]\right\}^{A_j}}{\prod_{j=N+1}^{R} \Gamma\left(a'_j - \alpha'_j \xi_{h,\sigma}\right) \prod_{j=n+1}^{r} \Gamma\left[(a_j - \alpha_j + \alpha_j s) - \alpha_j \xi_{h,\sigma}\right] \sigma! \eta_h} (M\varphi)(s),
$$

which proves (17). The relation (18) is proved similarly. By Lemma 1, (17) and (18) hold for $\varphi \in F_{p,\mu}$.

Theorem 3. *For $\varphi \in F_{p,\mu}$, $\psi \in F'_{p,\mu}$ and $Re(\mu) < \rho + \rho' - \frac{1}{p'}$ then there holds the formula of integration by parts,*

$$
\int_0^\infty \left(\frac{1}{x} R^{(m,M),(n,N)}_{(r,R),(q,Q)} \psi\right)(x) \varphi(x) dx
$$

$$
= \int_0^\infty \left(\frac{1}{x} A^{(m,M),(n,N)}_{(r,R),(q,Q)} \left[\begin{array}{l} (a_j + \alpha_j, \alpha_j; A_j)_{1,n}, (a_j + \alpha_j, \alpha_j)_{n+1,r} \\ (b_j + \beta_j, \beta_j)_{1,m}, (b_j + \beta_j, \beta_j; B_j)_{m+1,q} \end{array}\right]\right.
$$

$$
(19) \qquad \left. \cdot \left[\begin{array}{l} \left(a'_j + \alpha'_j, \alpha'_j; A'_j\right)_{1,N}, \left(a'_j + \alpha'_j, \alpha'_j\right)_{N+1,R} \\ \left(b'_j + \beta'_j, \beta'_j\right)_{1,M}, \left(b'_j + \beta'_j, \beta'_j; B'_j\right)_{M+1,Q} \end{array}\right] \varphi\right)(x) \psi(x) dx.
$$

Proof. Taking L.H.S.

$$
\int_0^\infty \left(\frac{1}{x} R^{(m,M),(n,N)}_{(r,R),(q,Q)} \psi\right)(x) \varphi(x) dx
$$

$$
= \int_0^\infty \frac{1}{x} \left\{\frac{1}{x} \int_0^x \overline{H}^{m,n}_{r,q} \left[\frac{t}{x} \middle| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,n}, (a_j, \alpha_j)_{n+1,r} \\ (b_j, \beta_j)_{1,m}, (b_j, \beta_J; B_j)_{m+1,q} \end{array}\right]\right.
$$

$$\cdot \overline{H}^{M,N}_{R,Q}\left[\frac{t}{x}\left|\begin{matrix}\left(a'_j,\alpha'_j;A'_j\right)_{1,N},\left(a'_j,\alpha'_j\right)_{N+1,R}\\ \left(b'_j,\beta'_j\right)_{1,M},\left(b'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{matrix}\right.\right]\psi(t)dt\Bigg\}\varphi(x)dx$$

$$=\int_0^\infty\int_0^x \overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\right]\overline{H}^{M,N}_{R,Q}\left[\frac{t}{x}\right]\psi(t)\frac{\varphi(x)}{x^2}dt\,dx.$$

In this situation strip of integration is parallel to t-axis when we convert order of integration we obtain new strip parallel to x-axis and therefore the above integral reduces to

$$=\int_0^\infty\int_t^\infty \overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\right]\overline{H}^{M,N}_{R,Q}\left[\frac{t}{x}\right]\psi(t)\frac{\varphi(x)}{x^2}dxdt$$

multiplying and dividing with t^2 and we will have

$$=\int_0^\infty \frac{\psi(t)}{t^2}\left\{\int_t^\infty \frac{t^2}{x^2}\overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\right]\overline{H}^{M,N}_{R,Q}\left[\frac{t}{x}\right]\varphi(x)dx\right\}dt$$

$$=\int_0^\infty \frac{\psi(t)}{t^2}\left\{\int_t^\infty \frac{t}{x}\overline{H}^{m,n}_{r,q}\left[\frac{t}{x}\right]\frac{t}{x}\overline{H}^{M,N}_{R,Q}\left[\frac{t}{x}\right]\varphi(x)dx\right\}dt$$

interchanging x and t mutually and we will have

$$=\int_0^\infty \frac{\psi(x)}{x}\left\{\frac{1}{x}\int_x^\infty \overline{H}^{m,n}_{r,q}\left[\frac{x}{t}\left|\begin{matrix}(a_j+\alpha_j,\alpha_j;A_j)_{1,n},(a_j+\alpha_j,\alpha_j)_{n+1,r}\\ (b_j+\beta_j,\beta_j)_{1,m},(b_j+\beta_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]\right.$$

$$.\overline{H}^{M,N}_{R,Q}\left[\frac{x}{t}\left|\begin{matrix}\left(a'_j+\alpha'_j,\alpha'_j;A'_j\right)_{1,N},\left(a'_j+\alpha'_j,\alpha'_j\right)_{N+1,R}\\ \left(b'_j+\beta'_j,\beta'_j\right)_{1,M},\left(b'_j+\beta'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{matrix}\right.\right]\varphi(t)dt\Bigg\}dx$$

$$=\int_0^\infty\left(\frac{1}{x}A^{(m,M),(n,N)}_{(r,R),(q,Q)}\left[\begin{matrix}(a_j+\alpha_j,\alpha_j;A_j)_{1,n},(a_j+\alpha_j,\alpha_j)_{n+1,r}\\ (b_j+\beta_j,\beta_j)_{1,m},(b_j+\beta_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right]\right.$$

$$.\left\{\left[\begin{matrix}\left(a'_j+\alpha'_j,\alpha'_j,A'_j\right)_{1,N},\left(a'_j+\alpha'_j,\alpha'_j\right)_{N+1,R}\\ \left(b'_j+\beta'_j,\beta'_j\right)_{1,M},\left(b'_j+\beta'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{matrix}\right]\varphi\right\}(x)\psi(x)dx.$$

To show that (19) is true for $\varphi\in F_{p,\mu}$ and $\psi\in F_{p',-\mu}$, it is sufficient to prove that both sides of (19) denotes bounded bilinear functions on $L^p_{\mu-1}\times L^{p'}_{1-\mu}$, where $L^p_\mu=\{\varphi : x^{-\mu}\varphi(x)\in L_p(R_+)\}$ with the defined norm

$$\|\varphi\|_{p,\mu}=\left(\int_0^\infty\left|x^{-\mu}\varphi(x)\right|^p dx\right)^{\frac{1}{p}}$$

(see [15] and (16).

From Holder's inequality and Theorem 1 we have

$$\left|\int_0^\infty \left(\frac{1}{x}R^{(m,M),(n,N)}_{(r,R),(q,Q)}\psi\right)(x)\varphi(x)dx\right|$$
$$= \left|\int_0^\infty x^{\mu-1}\left(R^{(m,M),(n,N)}_{(r,R),(q,Q)}\psi\right)(x)\left(x^{1-\mu}\varphi\right)(x)\,dx\right|$$
$$\leq \left\|R^{(m,M),(n,N)}_{(r,R),(q,Q)}\psi\right\|_{p',1-\mu}\|\varphi\|_{p,\mu-1} \leq K\,\|\psi\|_{p',1-\mu}\,\|\varphi\|_{p,\mu-1}$$

with K being a positive constant. Therefore the left hand side of (19) denotes a bounded linear functional on $\lambda^p_{\mu-1}\times\lambda^{p'}_{1-\mu}$, as similarly, does the right hand of (19). Thus the theorem is proved. □

5. Generalized fractional operators on $F'_{p,\mu}$. Using the formula of integration by parts (19), we may define $R^{(m,M),(n,N)}_{(r,R),(q,Q)}f$ and $A^{(m,M),(n,N)}_{(r,R),(q,Q)}f$ on $F'_{p,\mu}$ as following

I. For $f\in F'_{p,\mu}$ and $Re(\mu)<\rho+\rho'+\frac{1}{p'}$, we define $R^{(m,M),(n,N)}_{(r,R),(q,Q)}f\in F'_{p,\mu}$ by

$$\left\langle\frac{1}{x}R^{(m,M),(n,N)}_{(r,R),(q,Q)}f,\varphi\right\rangle = \left\langle f,\frac{1}{x}A^{(m,M),(n,N)}_{(r,R),(q,Q)}\right.$$
$$\cdot\left[\begin{array}{l}(a_j+\alpha_j,\alpha_j;A_j)_{1,n},\,(a_j+\alpha_j,\alpha_j)_{n+1,r}\\ (b_j+\beta_j,\beta_j)_{1,m},\,(b_j+\beta_j,\beta_j;B_j)_{m+1,q}\end{array}\right]$$
$$\left.\cdot\left[\begin{array}{l}\left(a'_j+\alpha'_j,\alpha'_j;A'_j\right)_{1,N},\,\left(a'_j+\alpha'_j,\alpha'_j\right)_{N+1,R}\\ \left(b'_j+\beta'_j,\beta'_j\right)_{1,M},\,\left(b'_j+\beta'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{array}\right]\varphi\right\rangle$$

for $\phi\in\ F_{p,\mu}$.

II. For $f\in F'_{p,\mu}$ and $Re(\mu)<-\rho-\rho'+\frac{1}{p'}$, we define $A^{(m,M),(n,N)}_{(r,R),(q,Q)}f\in F'_{p,\mu}$ by

$$\left\langle\frac{1}{x}A^{(m,M),(n,N)}_{(r,R),(q,Q)}f,\varphi\right\rangle = \left\langle f,\frac{1}{x}R^{(m,M),(n,N)}_{(r,R),(q,Q)}\right.$$
$$\cdot\left[\begin{array}{l}(a_j-\alpha_j,\alpha_j;A_j)_{1,n},\,(a_j-\alpha_j,\alpha_j)_{n+1,r}\\ (b_j-\beta_j,\beta_j)_{1,m},\,(b_j-\beta_j,\beta_j;B_j)_{m+1,q}\end{array}\right]$$
$$\left.\cdot\left[\begin{array}{l}\left(a'_j-\alpha'_j,\alpha'_j;A'_j\right)_{1,N},\,\left(a'_j-\alpha'_j,\alpha'_j\right)_{N+1,R}\\ \left(b'_j-\beta'_j,\beta'_j\right)_{1,M},\,\left(b'_j-\beta'_j,\beta'_j;B'_j\right)_{M+1,Q}\end{array}\right]\varphi\right\rangle$$

for $\phi \in F_{p,\mu}$.

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