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THE J_k^p SPACES

BY

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Dedicated to Professors Radu Miron and Mihai Anastasiei

Abstract. In the last time lot of generalization of Riemann spaces are given, as Finsler spaces, Lagrange spaces, generalized Lagrange spaces, Lagrange spaces of higher order, $Osc^k M$ spaces, generalized Hamilton spaces, multitime Lagrange spaces, J_k^2 spaces etc. Some of them are listed in References. Here it is shown that all these spaces can be considered as special cases of J_k^p spaces. For the mentioned spaces the group of transformation is given and some connections between them are obtained. The most important new result is Theorem 4.1, which gives the explicite expression for y^{li} in $Osc^k M$ and its subspaces.

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1. The J_k^p spaces.

Some k-dimensional subspace of R^n is given by

$$x^i = x^i(t_1, t_2, \dots, t_k), \quad i = \overline{1,n}(k < n) \tag{1.1}$$

where the functions x^i are C^∞. We shall use the notation

$$y^{\alpha i} = \frac{\partial x^i}{\partial t_\alpha}, \quad y^{\alpha\beta i} = \frac{\partial^2 x^i}{\partial t_\alpha \partial t_\beta}, \dots, \tag{1.2}$$
$$y^{\alpha\beta\dots\rho_i} = \frac{\partial^p x^i}{\underbrace{\partial t_\alpha \partial t_\beta \dots \partial t_\rho}_{p}}, \ \ \alpha, \beta, \rho = \overline{1,k}.$$

The Latin indices take value $\overline{1,n}$ and the Greek $\overline{1,k}$. From (1.2) it is clear that $y^{\alpha\beta i}, y^{\alpha\beta\gamma i}, \dots, y^{\alpha\beta\dots\rho_i}$ are symmetric in all Greek indices.

It exists no abstacle to take p arbitrary natural number, but the number of Greek indices is restricted. The problem can be solved partially by indexing the Greek letters as $\alpha_1, \alpha_2, \ldots$, but as we can see later on, this notation will not solve all problems.

We shall consider the following allowable transformations

$$
\begin{aligned}
x^{i'} &= x^{i'}(x^1, x^2, \ldots, x^n) \quad rank\left(\frac{\partial x^{i'}}{\partial x^i}\right) = n \\
y^{\alpha i'} &= \frac{\partial x^{i'}}{\partial x^i} y^{\alpha i} = y^{\alpha i'}(x^i, y^{\gamma i}) \\
y^{\alpha\beta i'} &= \frac{\partial y^{\alpha i'}}{\partial x^i} y^{\beta i} + \frac{\partial y^{\alpha i'}}{\partial y^{\gamma i}} y^{\gamma\beta i} = y^{\alpha\beta i'}(x^i, y^{\delta i}, y^{\delta\kappa i}), \\
y^{\alpha\beta\gamma i'} &= \frac{\partial y^{\alpha\beta i'}}{\partial x^i} y^{\gamma i} + \frac{\partial y^{\alpha\beta i'}}{\partial y^{\delta i}} y^{\delta\gamma i} + \frac{\partial y^{\alpha\beta i'}}{\partial y^{\delta\kappa i}} y^{\delta\kappa\gamma i} \\
&= y^{\alpha\beta\gamma i'}(x^i, y^{\kappa i}, y^{\kappa\lambda i}, y^{\kappa\lambda\rho i}) \\
y^{\alpha\beta\gamma\delta i'} &= \frac{\partial y^{\alpha\beta\gamma i'}}{\partial x^i} y^{\delta i} + \frac{\partial y^{\alpha\beta\gamma i'}}{\partial y^{\kappa i}} y^{\kappa\delta i} \\
&\quad + \frac{\partial y^{\alpha\beta\gamma i'}}{\partial y^{\kappa\lambda i}} y^{\kappa\lambda\delta i} + \frac{\partial y^{\alpha\beta\gamma i'}}{\partial y^{\kappa\lambda\rho i}} y^{\kappa\lambda\rho\delta i}, \ldots.
\end{aligned} \tag{1.3}
$$

From the above it is obvious that in (1.3) the parameters $(t_1, t_2, \ldots, t_k)$ are fixed, they do not transform.

Theorem 1.1. *The transformations of type* (1.3) *form a pseudo-group.*

Definition 1.1. *The space J_k^p is a C^∞ manifold of dimension m*

$$m = n\left[1 + \binom{k}{1} + \binom{k+1}{2} + \binom{k+2}{3} + \cdots + \binom{k+p-1}{p}\right]$$

whose elements have coordinates

$$(x^i, y^{\alpha i}, y^{\alpha\beta i}, \ldots, y^{\alpha\beta\ldots\rho i})$$

given by (1.1) *and* (1.2) *and the transformation group is given by* (1.3).

In J_k^p k denotes the number of parameters and p the order of higherest partial derivatives which appeare.

2. Special cases of J_k^p. *Case 1.* For $p = 1$ some element u of J_k^1 has coordinates

$$u = (x^i, y^{1i}, y^{2i}, \dots, y^{ki}), \quad i = \overline{1,n}, \tag{2.1}$$

where

$$x^i = x^i(t_1, t_2, \dots, t_k), \quad y^{\alpha i} = \frac{\partial x^i}{\partial t_\alpha}, \quad \alpha = \overline{1,k} \tag{2.2}$$

are C^∞ functiones and the structure group is given by

$$\text{(a)} \quad x^{i'} = x^{i'}(x^1, x^2, \dots, x^n), \quad rank\left(\frac{\partial x^{i'}}{\partial x^i}\right) = n \tag{2.3}$$

$$\text{(b)} \quad y^{\alpha i'} = \frac{\partial x^{i'}}{\partial x^i} y^{\alpha i}, \quad \alpha = \overline{1,k}.$$

Here we have k coordinate invariant parameters $t = (t_1, t_2, \dots, t_k)$ and only first partial derivatives of x^i with respect to t_α appeare ($p = 1$).

Such spaces are called k-Lagrange spaces or multi-time Lagrange spaces and are studied by Miron, Atanasiu, Anastasiei, Munteanu, Balan, Kirkovits, Čomić and many others (see, for example, [2], [4], [7], [8]).

The equation (2.1) and (2.2) mean, that elements of J_k^1 are points $(x) = (x^1, x^2, \dots, x^n)$ of the k-dimensional subspace in R^n, because $x^i = x^i(t_1, t_2, \dots, t_k)$, $i = \overline{1,n}$, together with the k tangent vectors $y^\alpha = (y^{\alpha i}) = (y^{\alpha 1}, y^{\alpha 2}, \dots, y^{\alpha n})$, $\alpha = \overline{1,k}$ to the coordinate curves $x(t_\alpha) = (x^1(t_\alpha), x^2(t_\alpha), \dots, x^n(t_\alpha))$ $\alpha = \overline{1,k}$. $x(t_\alpha)$ means, that all parameters in (2.2) are fixed, only t_α is a variable.

Theorem 2.1. *The transformations of form* (2.3) *form a pseudo-group.*

Some generalization of these spaces can be obtained if instead of (2.3)(b) we put

$$\text{(c)} \quad y^{\alpha i'} = M_i^{i'}(x) y^{\alpha i}, \quad rank(M_i^{i'}) = n. \tag{2.3}$$

Case 2. For $k = 1$, $p = k$ some element u of J_1^k has coordinates

$$u = (x^i, y^{1i}, y^{2i}, \dots, y^{ki}), \tag{2.4}$$

where

$$x^i = x^i(t), \quad y^{\alpha i} = \frac{d^\alpha x^i}{dt^\alpha} \left(\text{or } \frac{1}{\alpha!}\frac{d^\alpha x^i}{dt^\alpha}\right), \quad i = \overline{1,n}, \quad \alpha = \overline{1,k}. \tag{2.5}$$

Here we have only one coordinate invariant parameter t and the order of highest derivative of x^i with respect to t is $k(p=k)$.

We shall use sometimes the notation $y^{0i} = x^i$.

As here we have only one parameter, the general notations in (1.2) for this case take the form

$$y^{0i} = x^i, \quad y^{\alpha i} = y^{1i}, \quad y^{\alpha\beta i} = y^{2i}, \quad y^{\alpha\beta\gamma i} = y^{3i}, \quad y^{\alpha\beta\gamma\delta i} = y^{4i}, \dots .$$

The allowable transformations (1.3) for this special case given by (2.4) and (2.5) are

$$\begin{aligned}
&y^{0i'} = y^{0i'}(y^{01}, y^{02}, \dots, y^{0n}) \left(x^{i'} = x^{i'}(x^i), rank \frac{\partial x^{i'}}{\partial x^i} = n \right) \\
&y^{1i'} = \frac{\partial x^{i'}}{\partial x^i} y^{1i} = y^{1i'}(y^{0i}, y^{1i}) \\
&y^{2i'} = \frac{\partial y^{1i'}}{\partial y^{0i}} y^{1i} + \frac{\partial y^{1i'}}{\partial y^{1i}} y^{2i} = y^{2i'}(y^{0i}, y^{1i}, y^{2i}) \\
&y^{3i'} = \frac{\partial y^{2i'}}{\partial y^{0i}} y^{1i} + \frac{\partial y^{2i'}}{\partial y^{1i}} y^{2i} + \frac{\partial y^{2i'}}{\partial y^{2i}} y^{3i} = y^{3i'}(y^{0i}, y^{1i}, y^{2i}, y^{3i}) \\
&y^{4i'} = \frac{\partial y^{3i'}}{\partial y^{0i}} y^{1i} + \frac{\partial y^{3i'}}{\partial y^{1i}} y^{2i} + \frac{\partial y^{3i'}}{\partial y^{2i}} y^{3i} + \frac{\partial y^{3i'}}{\partial y^{3i}} y^{4i}, \dots , \\
&y^{ki'} = \frac{\partial y^{(k-1)i'}}{\partial y^{0i}} y^{1i} + \frac{\partial y^{(k-1)i'}}{\partial y^{1i}} y^{2i} + \dots + \frac{\partial y^{(k-1)i'}}{\partial y^{(k-1)i}} y^{ki}.
\end{aligned} \tag{2.6}$$

Theorem 2.2. *The transformations of form* (2.6) *form a pseudo group.*

The spaces J_1^k are introduced by Miron and Atanasiu in [18], [19] and they called them $Osc^k M$, or k-th order Lagrange spaces. There are several books (see, for example, [3], [6], [17], [20], [21]) and many papers (see, for example, [4], [5], [9]–[13], [22], [23]) in which such spaces are examined.

Case 3. For $k = 1$, $p = 1$ some point $u \in J_1^1$ has coordinates

$$u = (x^i, y^i), \quad x^i = x^i(t), \quad y^i = \frac{dx^i}{dt}, \quad i = \overline{1,n}. \tag{2.7}$$

Here we have only one coordinate invariant parameter t and only the first derivative of $x^i(t)$ appeare. The group of transformation for (2.7) is given by

$$x^{i'} = x^{i'}(x^i), \quad y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i, \quad rank \left(\frac{\partial x^{i'}}{\partial x^i} \right) = n. \tag{2.8}$$

The Lagrange spaces have the transformation group given by (2.8). Special cases of Lagrange spaces are Finsler spaces in which the homogenity condition is valid [6], [15], [16], [24].

Case 4. For $p = 0$, $k = 0$ some element $x \in J_0^0$ has coordinates

$$x = (x^i) = (x^1, x^2, \ldots, x^n).$$

The number of parameters is equal to zero, also the order of derivatives is zero. The group of transformation is given by

$$x^{i'} = x^{i'}(x^i). \tag{2.9}$$

The Riemannian spaces have the transformation group given by (2.9).

3. The space J_k^4. The only reason that we restrict our attention on $p = 4$ is to obtain shorter formulae, but long enough to see the property we want to be extracted.

In J_k^4 space the number of parameter is k and the highest order of derivative is 4. Using the notation $x^i = y^i$ some point $u \in J_k^4$ has coordinates:

$$u = (y^i, y^{\alpha i}, y^{\alpha\beta i}, y^{\alpha\beta\gamma i}, y^{\alpha\beta\gamma\delta i})$$

and

$$\dim J_k^4 = n\left[1 + \binom{k}{1} + \binom{k+1}{2} + \binom{k+2}{3} + \binom{k+3}{4}\right],$$

because the coordinates in all Greek indices are symmetric. The transformation group is given by:

$$\begin{aligned}
y^{i'} &= y^{i'}(y^1, y^2, \ldots, y^n)\\
y^{\alpha i'} &= \frac{\partial y^{i'}}{\partial y^i} y^{\alpha i}\\
y^{\alpha\beta i'} &= \frac{\partial^2 y^{i'}}{\partial y^i \partial y^j} y^{\alpha i} y^{\beta j} + \frac{\partial y^{i'}}{\partial y^i} y^{\alpha\beta i}\\
y^{\alpha\beta\gamma i'} &= \frac{\partial^3 y^{i'}}{\partial y^i \partial y^j \partial y^k} y^{\alpha i} y^{\beta j} y^{\gamma k} + \frac{\partial^2 y^{i'}}{\partial y^i \partial y^j}[y^{\alpha\gamma i} y^{\beta j}\\
&\quad + y^{\alpha i} y^{\beta\gamma j} + y^{\gamma j} y^{\alpha\beta i}] + \frac{\partial y^{i'}}{\partial y^i} y^{\alpha\beta\gamma i}\\
y^{\alpha\beta\gamma\delta i'} &= \frac{\partial^4 y^{i'}}{\partial y^i \partial y^j \partial y^h \partial y^k} y^{\alpha i} y^{\beta j} y^{\gamma h} y^{\delta k}
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&+\frac{\partial^3 y^{i'}}{\partial y^i \partial y^j \partial y^h}[y^{\alpha\delta i}y^{\beta j}y^{\gamma h}+y^{\alpha i}y^{\beta\delta j}y^{\gamma h}+y^{\alpha i}y^{\beta j}y^{\gamma\delta h}\\
&+y^{\delta h}y^{\alpha\gamma i}y^{\beta j}+y^{\delta h}y^{\alpha i}y^{\beta\gamma j}+y^{\delta h}y^{\gamma j}y^{\alpha\beta i}]\\
&+\frac{\partial^2 y^{i'}}{\partial y^i \partial y^j}[y^{\alpha\gamma\delta i}y^{\beta j}+y^{\alpha\gamma i}y^{\beta\delta j}+y^{\alpha\delta i}y^{\beta\gamma j}\\
&+y^{\alpha i}y^{\beta\gamma\delta j}+y^{\gamma\delta j}y^{\alpha\beta i}+y^{\gamma j}y^{\alpha\beta\delta i}+y^{\alpha\beta\gamma i}y^{\delta j}]+\frac{\partial y^{i'}}{\partial y^i}y^{\alpha\beta\gamma\delta i}.
\end{aligned}
$$

Theorem 3.1. *The following relations are valid:*

$$
\begin{aligned}
&\frac{\partial y^{i'}}{\partial y^i}=\frac{\partial y^{\alpha i'}}{\partial y^{\alpha i}}=\frac{\partial y^{\alpha\beta i'}}{\partial y^{\alpha\beta i}}=\frac{\partial y^{\alpha\beta\gamma i'}}{\partial y^{\alpha\beta\gamma i}}=\frac{\partial y^{\alpha\beta\gamma\delta i'}}{\partial y^{\alpha\beta\gamma\delta i}}\\
&\frac{\partial y^{\alpha i'}}{\partial y^i}=\frac{\partial y^{\alpha\beta i'}}{\partial y^{\beta i}}=\frac{\partial y^{\alpha\beta\gamma i'}}{\partial y^{\beta\gamma i}}=\frac{\partial y^{\alpha\beta\gamma\delta i'}}{\partial y^{\beta\gamma\delta i}}\\
&\frac{\partial y^{\alpha\beta i'}}{\partial y^i}=\frac{\partial y^{\alpha\beta\gamma i'}}{\partial y^{\gamma i}}=\frac{\partial y^{\alpha\beta\gamma\delta i'}}{\partial y^{\gamma\delta i}}\\
&\frac{\partial y^{\alpha\beta\gamma i'}}{\partial y^i}=\frac{\partial y^{\alpha\beta\gamma\delta i'}}{\partial y^{\delta i}}.
\end{aligned}
\tag{3.2}
$$

Special case of J^4_k is $J^4_1 = Osc^4 M$ i.e. the space, where the number of parameter is equal to 1 and the highest order of derivatives is equal to 4. Some point $u \in Osc^4 M$ has coordinates:

$$u = (y^i, y^{1i}, y^{2i}, y^{3i}, y^{4i})$$

and the dimension of space is

$$m = \left[1 + \binom{1}{1} + \binom{2}{2} + \binom{3}{3} + \binom{4}{4}\right] = 5n.$$

The group of transformations given by (3.1) are reduced to

$$
\begin{aligned}
y^{i'} &= y^{i'}(y^1, y^2, \dots, y^n)\\
y^{1i'} &= \frac{\partial y^{i'}}{\partial y^i}y^{1i}\\
y^{2i'} &= \frac{\partial^2 y^{i'}}{\partial y^i \partial y^j}y^{1i}y^{1j}+\frac{\partial y^{i'}}{\partial y^i}y^{2i}\\
y^{3i'} &= \frac{\partial^3 y^{i'}}{\partial y^i \partial y^j \partial y^h}y^{1i}y^{1j}y^{1h}+\frac{\partial^2 y^{i'}}{\partial y^i \partial y^j}3y^{1i}y^{2j}+\frac{\partial y^{i'}}{\partial y^i}y^{3i}
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
y^{4i'} \quad = \quad & \frac{\partial^4 y^{i'}}{\partial y^i \partial y^j \partial y^h \partial y^k} y^{1i} y^{1j} y^{1h} y^{1k} + \frac{\partial^3 y^{i'}}{\partial y^i \partial y^j \partial y^h} 6 y^{2i} y^{1j} y^{1h} \\
& + \frac{\partial^2 y^{i'}}{\partial y^i \partial y^j} (4 y^{3i} y^{1j} + 3 y^{2i} y^{2j}) + \frac{\partial y^{i'}}{\partial y^i} y^{4i}.
\end{aligned}
$$

From (3.2) and (3.3) we obtain

Theorem 3.2. *In $Osc^4 M$ the following relations are valid*

$$
\begin{aligned}
\frac{\partial y^{i'}}{\partial y^i} &= \frac{\partial y^{1i'}}{\partial y^{1i}} = \frac{\partial y^{2i'}}{\partial y^{2i}} = \frac{\partial y^{3i'}}{\partial y^{3i}} = \frac{\partial y^{4i'}}{\partial y^{4i}} \\
\frac{\partial y^{1i'}}{\partial y^i} &= \frac{1}{\binom{2}{1}} \frac{\partial y^{2i'}}{\partial y^{1i}} = \frac{1}{\binom{3}{2}} \frac{\partial y^{3i'}}{\partial y^{2i}} = \frac{1}{\binom{4}{3}} \frac{\partial y^{4i'}}{\partial y^{3i}} \\
\frac{\partial y^{2i'}}{\partial y^i} &= \frac{1}{\binom{3}{1}} \frac{\partial y^{3i'}}{\partial y^{1i}} = \frac{1}{\binom{4}{2}} \frac{\partial y^{4i'}}{\partial y^{2i}} \\
\frac{\partial y^{3i'}}{\partial y^i} &= \frac{1}{\binom{4}{1}} \frac{\partial y^{4i'}}{\partial y^{1i}}.
\end{aligned} \tag{3.4}
$$

The relations (3.4) are wellknown formulae in $Osc^4 M$ (see [18], [19]).

It is obvious that the transformation group in J_k^4 is complicated, it is difficult to obtain the adapted bases of $T(J_k^4)$ or $T^*(J_k^4)$, whose elements are transforming as tensors.

In [1], [2], [14], [22], [23] the spaces J_k^2 are studied and the curvature theory is given.

4. The special transformation group in $\boldsymbol{Osc^k M}$. In this section we want to introduce the subspaces of $Osc^k M$, to obtain the connections between the coordinates of subspaces and surrounding space and as a special case, to obtain the general formula for $y^{ki'}$ of type (3.3).

We shall consider the special group of transformation $x^{i'} = x^{i'}(x^i)$, namely, when

$$
\begin{aligned}
& x^i = x^i(u^1, u^2, \dots, u^m, v^{m+1}, \dots, v^n) = (u^\alpha, v^{\hat\alpha}) \\
& a, b, c, \dots = \overline{1, n}, \quad \alpha, \beta, \gamma, \dots = \overline{1, m}, \quad \hat\alpha, \hat\beta, \hat\gamma, \dots = \overline{m+1, n}.
\end{aligned} \tag{4.1}
$$

For $v^{\hat\alpha} = C^{\hat\alpha}$, $\hat\alpha = \overline{m+1, n}$ we obtain:

$$
x^i = x^i(u^1, u^2, \dots, u^m, C^{\widehat{m+1}}, \dots, C^{\hat n}), \quad i = \overline{1, n},
$$

the family of m-dimensional subspaces M_m of M_n and for $u^\alpha = C^\alpha$, $\alpha = \overline{1,m}$ we get

$$x^i = x^i(C^1, C^2, \dots, C^m, v^{m+1}, \dots, v^n), \quad i = \overline{1,n},$$

the family of $n-m$ dimensional subspaces M_{n-m} of M_n. In the new coordinate system the point (x^i) has coordinates:

$$x^{i'} = x^{i'}(u^{1'}, \dots, u^{m'}, v^{(m+1)'}, \dots, v^{n'}),$$

where

$$u^{\alpha'} = u^{\alpha'}(u^\alpha), \quad v^{\hat\alpha'} = v^{\hat\alpha'}(v^{\hat\alpha}).$$

The curves $u^\alpha = u^\alpha(t)$ in M_m and $v^{\hat\alpha} = v^{\hat\alpha}(t)$ in M_{n-m} induces a curve $x^i = x^i(t) = x^i(u^1(t), \dots, u^m(t), v^{m+1}(t), \dots, v^n(t)) = x^i(u^\alpha(t), v^{\hat\alpha}(t))$ in M_n.

Let us introduce the notations

$$u^{A\alpha}(t) = \frac{d^A u^\alpha(t)}{dt^A}, \quad v^{A\hat\alpha} = \frac{d^A v^{\hat\alpha}(t)}{dt^A}, \tag{4.2}$$
$$y^{Ai}(t) = \frac{d^A x^i(t)}{dt^A}, A = \overline{0,k}.$$

In $E = Osc^k M_n$, which is $(k+1)n$ dimension C^∞ manifold, two family of subspaces $E_1 = Osc^k M_m$ $(dim E_1 = (k+1)m)$ and $E_2 = Osc^k M_{(n-m)}$ $(dim E_2 = (k+1)(n-m))$ can be introduced. We have

$$(u^\alpha = u^{0\alpha}, u^{1\alpha}, \dots, u^{k\alpha}) \in E_1 \quad \alpha = \overline{1,m}$$
$$(v^{\hat\alpha} = v^{0\hat\alpha}, v^{1\hat\alpha}, \dots, v^{k\hat\alpha}) \in E_2 \quad \hat\alpha = \overline{m+1,n}$$
$$(x^i = y^{0i}, y^{1i}, \dots, y^{ki}) \in E \quad i = \overline{1,n}.$$

We shall use the notations:

$$\nabla = \left(\frac{\partial}{\partial u^\alpha} + \frac{\partial}{\partial v^{\hat\alpha}}\right), \quad U^A = u^{A\alpha} + v^{A\hat\alpha} \tag{4.3}$$

to define

$$Y^A = U^A \nabla = u^{A\alpha}\frac{\partial}{\partial u^\alpha} + v^{A\hat\alpha}\frac{\partial}{\partial v^{\hat\alpha}}, \quad A = \overline{1,k}. \tag{4.4}$$

From (4.3) and (4.4) we have for instance

$$\begin{aligned}(Y^1)^2 &= \left(u^{1\alpha}\frac{\partial}{\partial u^\alpha} + v^{1\hat{\alpha}}\frac{\partial}{\partial v^{\hat{\alpha}}}\right)^2 \\ &= u^{1\alpha}u^{1\beta}\frac{\partial^2}{\partial u^\alpha \partial u^\beta} + 2u^{1\alpha}v^{1\hat{\beta}}\frac{\partial^2}{\partial u^\alpha \partial v^{\hat{\beta}}} + v^{1\hat{\alpha}}v^{1\hat{\beta}}\frac{\partial^2}{\partial v^{\hat{\alpha}} \partial v^{\hat{\beta}}}.\end{aligned}$$

Proposition 4.1. *The following relations are valid (if* $()' = \frac{d()}{dt}$*):*

$$(U^1)' = U^2, (U^2)' = U^3, \ldots, (U^{k-1})' = U^k \tag{4.5}$$

$$\begin{aligned}&\nabla' = Y^1\nabla = (U^1\nabla)\nabla = u^{1\beta}\frac{\partial^2}{\partial u^\alpha \partial u^\beta} + v^{1\hat{\beta}}\frac{\partial^2}{\partial v^{\hat{\alpha}} \partial v^{\hat{\beta}}} \\ &+u^{1\beta}\frac{\partial^2}{\partial v^{\hat{\alpha}} \partial u^\beta} + v^{1\hat{\beta}}\frac{\partial^2}{\partial u^\alpha \partial v^{\hat{\beta}}} \\ &(\nabla^2)' = Y^1\nabla^2 = (U^1\nabla)\nabla^2 = U^1\nabla^3, \ldots, \\ &(\nabla^{k-1})' = Y^1\nabla^{k-1} = U^1\nabla^k.\end{aligned} \tag{4.6}$$

For the further calculation it is useful to know:

$$\begin{aligned}&(U^A \cdot U^B)' = (U^A)' \cdot U^B + U^A(U^B)' = U^{A+1} \cdot U^B + U^A U^{B+1}, \\ &U^A \cdot U^B \neq U^{A+B}.\end{aligned}$$

As ∇ is a differential operator, the product rule is not valid, i.e.

$$\begin{aligned}&(\nabla^s \cdot \nabla^l)' = (\nabla^{s+l})' = U^1\nabla^{s+l+1}, \\ &(\nabla^s \cdot \nabla^l)' \neq (\nabla^s)'\nabla^l + \nabla^s(\nabla^l)' = 2U^1\nabla^{s+l+1}.\end{aligned}$$

We have

$$(U^1\nabla^s)' = (U^1\nabla^{s-1})'\nabla \tag{4.7}$$

because

$$\begin{aligned}&(U^1\nabla^s)' = U^2\nabla^s + U^1 \cdot U^1\nabla^{s+1} \\ &(U^1\nabla^{s-1})'\nabla = (U^2\nabla^{s-1} + U^1 \cdot U^1\nabla^s)\nabla.\end{aligned}$$

The relation (4.7) will be used in

Proposition 4.2. *The derivatives of* ∇ *are given by*

$$\begin{aligned}
\nabla' &= Y^1\nabla = (U^1\nabla)\nabla\\
\nabla'' &= (Y^1)'\nabla = (U^1\nabla)'\nabla = (U^2\nabla + (U^1)^2\nabla^2)\nabla\\
\nabla''' &= (Y^1)''\nabla = (U^1\nabla)''\nabla = (U^3\nabla + 2U^2\nabla' + U^1\nabla'')\\
\nabla'^v &= (Y^1)'''\nabla = (U^1\nabla)'''\nabla = (U^4\nabla + 3U^3\nabla' + 3U^2\nabla'' + U^1\nabla''')\nabla\\
\nabla^v &= (Y^1)'^v\nabla = (U^1\nabla)'^v\nabla = (U^5\nabla + 4U^4\nabla'\\
&\quad +6U^3\nabla'' + 4U^2\nabla''' + U^1\nabla'^v)\nabla, \dots,\\
\nabla^{(l+1)} &= (Y^1)^{(l)}\nabla = (U^1\nabla)^{(l)}\nabla = [\binom{l}{0}(U^1)^{(l)}\nabla + \binom{l}{1}(U^1)^{(l-1)}\nabla'\\
&\quad +\binom{l}{2}(U^1)^{(l-2)}\nabla'' + \dots + \binom{l}{l-1}(U^1)'\nabla^{(l-1)} + \binom{l}{l}U^1\nabla^{(l)}]\nabla.
\end{aligned}$$

If we in the above formula substitute $\nabla', \nabla'', \dots$, we obtain

Proposition 4.3. *The explicite form of derivatives of* ∇ *are given by*

$$\begin{aligned}
\nabla' &= (U^1\nabla)\nabla\\
\nabla'' &= (\nabla')' = (U^1\nabla)'\nabla = [U^2\nabla + (U^1)^2\nabla^2]\nabla\\
\nabla''' &= (\nabla'')' = [U^3\nabla + 3U^1U^2\nabla^2 + (U^1)^3\nabla^3]\nabla\\
\nabla'^v &= (\nabla''')' = [U^4\nabla + (4U^1U^3 + 3(U^2)^2)\nabla^2\\
&\quad +6(U^1)^2U^2\nabla^3 + (U^1)^4\nabla^4]\nabla\\
\nabla^v &= (\nabla'^v)' = [U^5\nabla + (5U^1U^4 + 10U^2U^3)\nabla^2 + (10(U^1)^2U^3\\
&\quad +15U^1(U^2)^2)\nabla^3 + 10(U^1)^3U^2\nabla^4 + (U^1)^5\nabla^5]\nabla, \dots.
\end{aligned}$$

If we compare formulae from Proposition 4.2 and 4.3 we can easely obtain $(Y^1)^{(n)} = (U'\nabla)^{(n)}$ for $n = 1, 2, 3, 4, 5, \dots$.

Proposition 4.4. *The following relations are valid:*

$$\begin{aligned}
y^{1i} &= \frac{dy^{0i}}{dt} = Y^1|y^{0i} = (U^1\nabla)|y^{0i} = B^i_\alpha u^{1\alpha} + B^i_{\hat\alpha}v^{1\hat\alpha}\\
y^{2i} &= \frac{dy^{1i}}{dt} = (U^1\nabla)'|y^{0i} = [(U^1)^2\nabla^2 + U^2\nabla]|y^{0i}\\
y^{3i} &= \frac{dy^{2i}}{dt} = (U^1\nabla)''|y^{0i} = [(U^1)^3\nabla^3 + 3U^1U^2\nabla^2 + U^3\nabla]|y^{0i}\\
y^{4i} &= \frac{dy^{3i}}{dt} = (U^1\nabla)'''|y^{0i} = [(U^1)^4\nabla^4 + 6(U^1)^2U^2\nabla^3 + (4U^1U^3\\
&\quad +3(U^2)^2)\nabla^2 + U^4\nabla]|y^{0i}
\end{aligned}$$

$$y^{5i} = \frac{dy^{4i}}{dt} = (U^1\nabla)'^v|y^{0i} = [(U^1)^5\nabla^5 + 10(U^1)^3U^2\nabla^4 + (15U^1(U^2)^2 +10(U^1)^2U^3)\nabla^3 + (10U^2U^3 + 5U^1U^4)\nabla^2 + U^5\nabla]|y^{0i} \dots .$$

The above formulae are very complicated but they are written in the shortest form. The explicite form of some expressions are for instance:

$$\begin{aligned}(U^1)^2U^2\nabla^3|y^{0i} &= 2B^i_{\alpha\beta\gamma}u^{1\alpha}u^{1\beta}u^{2\gamma} + B^i_{\alpha\beta\hat{\gamma}}u^{1\alpha}u^{1\beta}v^{2\hat{\gamma}} \\ &+2B^i_{\alpha\beta\hat{\gamma}}u^{1\alpha}u^{2\beta}v^{1\hat{\gamma}} + B^i_{\alpha\hat{\beta}\hat{\gamma}}u^{2\alpha}v^{1\hat{\beta}}v^{1\hat{\gamma}} + B^i_{\alpha\hat{\beta}\hat{\gamma}}u^{1\alpha}v^{1\hat{\beta}}v^{2\hat{\beta}} \\ &+B^i_{\hat{\alpha}\hat{\beta}\hat{\gamma}}v^{1\hat{\alpha}}v^{1\hat{\beta}}v^{2\hat{\gamma}},\end{aligned}$$

or

$$U^1U^4\nabla^2|y^{0i} = B^i_{\alpha\beta}u^{1\alpha}u^{4\beta} + B^i_{\alpha\hat{\beta}}u^{1\alpha}v^{4\hat{\beta}} + B^i_{\hat{\alpha}\beta}v^{1\hat{\alpha}}u^{4\beta} + B^i_{\hat{\alpha}\hat{\beta}}v^{1\hat{\alpha}}v^{4\hat{\beta}},$$

or

$$U^3\nabla|y^{0i} = B^i_{\alpha}u^{3\alpha} + B^i_{\hat{\alpha}}v^{3\hat{\alpha}}.$$

Proposition 4.4 gives relations between y^{Ai} and $(u^{1\alpha}, u^{2\alpha}, \dots, u^{A\alpha}, v^{1\hat{\alpha}}, v^{2\hat{\alpha}}, \dots, v^{A\hat{\alpha}})$, $\forall A = 1, 2, \dots, k$. These relations are very complicated. To obtain some simpler formulae we restrict our attention for the case when $B^i_{\alpha\hat{\beta}} = 0$, i.e. when

$$x^i = x^i(u^1, u^2, \dots, u^m) + x^i(v^{m+1}, \dots, v^n), \quad i = \overline{1, n}. \tag{4.8}$$

Proposition 4.5. *For the transformations of form* (4.8) *we have*

$$\begin{aligned} y^{1i} &= B^i_{\alpha}u^{1\alpha} + B^i_{\hat{\alpha}}v^{1\hat{\alpha}} = B^i_{\alpha}u^{1\alpha} + (u/v) \\ y^{2i} &= (B^i_{\alpha\beta}u^{1\alpha}u^{1\beta} + B^i_{\alpha}u^{2\alpha}) + (u/v) \\ y^{3i} &= (B^i_{\alpha\beta\gamma}u^{1\alpha}u^{1\beta}u^{1\gamma} + 3B^i_{\alpha\beta}u^{1\alpha}u^{2\beta} + B^i_{\alpha}u^{3\alpha}) + (u/v) \\ y^{4i} &= (B^i_{\alpha\beta\gamma\delta}u^{1\alpha}u^{1\beta}u^{1\gamma}u^{1\delta} + 6B^i_{\alpha\beta\gamma}u^{2\alpha}u^{1\beta}u^{1\gamma} \\ &+B^i_{\alpha\beta}(3u^{2\alpha}u^{2\beta} + 4u^{3\alpha}u^{1\beta}) + B^i_{\alpha}u^{4\alpha}) + (u/v) \\ y^{5i} &= (B^i_{\alpha\beta\gamma\delta\theta}u^{1\alpha}u^{1\beta}u^{1\gamma}u^{1\delta}u^{1\theta} + 10B^i_{\alpha\beta\gamma\delta}u^{1\alpha}u^{1\beta}u^{1\gamma}u^{2\delta} \\ &+B^i_{\alpha\beta\gamma}(15u^{1\alpha}u^{2\beta}u^{2\gamma} + 10u^{1\alpha}u^{1\beta}u^{3\gamma}) \\ &+B^i_{\alpha\beta}(10u^{2\alpha}u^{3\beta} + 5u^{1\alpha}u^{4\beta}) + B^i_{\alpha}u^{5\alpha}) + (u/v), \dots ,\end{aligned}$$

where (u/v) *in every formula means the expression in the former bracket, when* u *is substituted by* v *and the Greek indices obtain* $\wedge$.

If we instead of transformation of form (4.8) take

$$y^{i'} = y^{i'}(y^1, y^2, \dots, y^n) \quad i = \overline{1,n} \quad (x^i \to y^i, x^{i'} \to y^{i'}),$$

then we in Proposition 4.5 have the following changes in notation:

$$y^{Ai} \to y^{Ai'}, \quad B^i_\alpha = \frac{\partial y^{i'}}{\partial y^i}, \quad B^i_{\alpha\beta} \to \frac{\partial^2 y^{i'}}{\partial y^i \partial y^j}, \quad u^{A\alpha} \to y^{Ai}.$$

From Proposition 4.5 we get $y^{1i'}$, $y^{2i'}$, $y^{3i'}$, $y^{4i'}$ as it was given by (3.3) and for $y^{5i'}$ we obtain

$$\begin{aligned} y^{5i'} &= \frac{\partial^5 y^{i'}}{\partial y^i \partial y^j \partial y^h \partial y^k \partial y^l} y^{1i} y^{1j} y^{1h} y^{1k} y^{1l} + \frac{\partial^4 y^{i'}}{\partial y^i \partial y^j \partial y^h \partial y^k} 10 y^{1i} y^{1j} y^{1h} y^{2k} \\ &+ \frac{\partial^3 y^{i'}}{\partial y^i \partial y^j \partial y^h} (15 y^{1i} y^{2j} y^{2h} + 10 y^{1i} y^{1j} y^{3h}) \\ &+ \frac{\partial^2 y^{i'}}{\partial y^i \partial y^j} (10 y^{2i} y^{3j} + 5 y^{1i} y^{4j}) + \frac{\partial y^{i'}}{\partial y^i} y^{5i}. \end{aligned}$$

Now we can prove the following important theorem, which brings order in the former complicated expressions.

Theorem 4.1. *Under the transformations of form* (4.1), (4.2) *we have*

$$\begin{aligned} y^{(l+1)i} &= \binom{l}{0} Y^1 y^{li} + \binom{l}{1} Y^2 y^{(l-1)i} + \binom{l}{2} Y^3 y^{(l-2)i} \\ &+ \dots + \binom{l}{l-1} Y^l y^{1i} + \binom{l}{l} Y^{(l+1)} y^{0i}. \end{aligned} \tag{4.9}$$

Proof. From Proposition 4.4 we have

$$\begin{aligned} y^{(l+1)i} &= \frac{d^{l+1} x^i}{dt^{l+1}} = (Y^1)^{(l)} | y^{0i} = (U^1 \nabla)^{(l)} | y^{0i} \\ &= \left[\binom{l}{0} U^1 \nabla^{(l)} + \binom{l}{1} (U^1)' \nabla^{(l-1)} + \binom{l}{2} (U^1)'' \nabla^{(l-2)} \right. \\ &\quad \left. + \dots + \binom{l}{l-1} (U^1)^{(l-1)} \nabla' + \binom{l}{l} (U^1)^{(l)} \nabla \right] | u^{0i}. \end{aligned}$$

Using (4.5), (4.6), (4.7) and Proposition 4.2 we have

$$\nabla^{(s)} = (Y^1)^{(s-1)} \nabla = (U^1 \nabla)^{(s-1)} \nabla = \nabla (U^1 \nabla)^{(s-1)}.$$

Now we can write

$$\begin{aligned} y^{(l+1)i} &= \Big[\binom{l}{0}(U^1\nabla)(U^1\nabla)^{(l-1)} + \binom{l}{1}(U^2\nabla)(U^1\nabla)^{(l-2)} \\ &+ \binom{l}{2}(U^3\nabla)(U^1\nabla)^{(l-3)} + \cdots + \binom{l}{l-1}(U^l\nabla)(U^1\nabla) \\ &+ \binom{l}{l}U^{l+1}\nabla\Big] |y^{0i} == \binom{l}{0}Y^1y^{li} + \binom{l}{1}Y^2y^{(l-1)i} \\ &+ \binom{l}{2}Y^3y^{(l-2)i} + \cdots + \binom{l}{l-1}Y^ly^{1i} + Y^{l+1}y^{0i}. \end{aligned}$$

If we have the transformations of form

$$y^{i'} = y^{i'}(y^i), \quad y^{li} = \frac{d^{(l)}y^i}{dt^l}, \quad y^{li'} = \frac{d^ly^{i'}}{dt^l}$$

then (4.9) has the form

$$\begin{aligned} y^{(l+1)i'} &= \binom{l}{0}Y^1y^{li'} + \binom{l}{1}Y^2y^{(l-1)i'} \\ &+ \cdots + \binom{l}{l-1}Y^ly^{1i'} + \binom{l}{l}Y^{l+1}y^{0i'}, \end{aligned} \tag{4.10}$$

where

$$Y^1 = y^{1j}\frac{\partial}{\partial y^j}, \quad Y^2 = y^{2j}\frac{\partial}{\partial y^j}, \dots, Y^{l+1} = y^{(l+1)j}\frac{\partial}{\partial y^j}.$$

For $l = 0$, (4.10) gives

$$y^{1i'} = Y^1y^{0i'} = Y^1y^{i'} = y^{1j}\frac{\partial y^{i'}}{\partial y^j}.$$

For $l = 1$ (4.10) gives

$$\begin{aligned} y^{2i'} &= \binom{1}{0}Y^1y^{1i'} + \binom{1}{1}Y^2y^{0i'} = y^{1h}\frac{\partial}{\partial y^h}\left(y^{1j}\frac{\partial y^{i'}}{\partial y^j}\right) + y^{2j}\frac{\partial y^{i'}}{\partial y^j} \\ &= \frac{\partial^2 y^{i'}}{\partial y^j \partial y^h}y^{1h}y^{1j} + \frac{\partial y^{i'}}{\partial y^j}y^{2j}. \end{aligned}$$

It is easy to see, that $y^{(l+1)i'}$ for $l = 2, 3, \dots$ obtained from (4.10) are the same as those in (3.3). □

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