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THE HOLOMORPHIC SECTIONAL CURVATURE OF GENERAL NATURAL KÄHLER STRUCTURES ON COTANGENT BUNDLES

BY

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Dedicated to the memory of Professor Neculai Papaghiuc

Abstract. We study the conditions under which a Kählerian structure (G, J) of general natural lift type on the cotangent bundle T^*M of a Riemannian manifold (M, g) has constant holomorphic sectional curvature. We obtain that a certain parameter involved in the condition for (T^*M, G, J) to be a Kählerian manifold, is expressed as a rational function of the other two parameters, their derivatives, the constant sectional curvature of the base manifold (M, g), and the constant holomorphic sectional curvature of the general natural Kählerian structure (G, J).

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1. Introduction.

The natural lifts introduced on the cotangent bundle of a Riemannian manifold (M, g) led to some geometric structures studied in the last years in a few papers like [9], [15]-[22]. A part of the results obtained in these works are similar to some results from the geometry of the tangent bundle TM, the dual of the cotangent bundle T^*M (see [1], [2] [7],[10]-[12]). The differences which appear are related to the construction of the lifts on the cotangent bundle, the technics being different from those used in the geometry of the tangent bundle (see [23]).

In the paper [13], Oproiu introduced the general expression for the natural 1-st order almost complex structure J on the tangent bundle TM

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and the notion of general natural lifted metric G on TM, defined by the Riemannian metric g from the base manifold, just like the natural lifts are obtained in [5] and [6]. With respect to the new metric the horizontal and vertical distributions are no more orthogonal to each other, contrary to the diagonal case, treated in [14]. The author obtained that the family of Kählerian structures (G, J) of general natural lift type on TM depends on three essential parameters (two of them are involved in the expression of the integrable almost complex structure J, and the third one is a certain proportionality factor, from the condition for (G, J) to be almost Hermitian).

The present author defined in the paper [3], an almost complex structure of general natural lifted type on the cotangent bundle T^*M, and a general natural lifted metric to T^*M, obtained from the Riemannian metric g of the base manifold M. The main result is that the family of general natural Kähler structures on T^*M depends on three essential parameters (one is a certain proportionality factor obtained from the condition for the structure to be almost Hermitian and the other two are some coefficients involved in the definition of the integrable almost complex structure J on T^*M).

In the joint work [4], Oproiu and the present author studied the conditions under which the Kählerian manifold (TM, G, J) of general natural lift type has constant holomorphic sectional curvature. They obtained that the proportionality factor involved in the condition for (TM, G, J) to be Kählerian is expressed as a rational function of the two essential parameters involved in the expression of J (integrable almost complex structure on TM), their derivatives, the constant sectional curvature of (M, g) and the constant holomorphic sectional curvature of (TM, G, J).

In the present paper we are interested in finding some properties of the curvature tensor field K of the general natural Kähler structure (G, J) on the cotangent bundle T^*M. Namely, we find the conditions under which the Kählerian structure considered on T^*M has constant holomorphic sectional curvature. By doing some quite long computations with the RICCI package from Mathematica, we get the expressions of the components of the curvature tensor field of the manifold (T^*M, G) and those of the curvature tensor field K_0 of the Kählerian manifold (T^*M, G, J) having constant holomorphic sectional curvature k. The vanishing conditions for the components of the difference $K - K_0$ lead to the conclusion that (T^*M, G, J) has constant holomorphic sectional curvature k, if and only if the proportionality factor involved in the condition for (T^*M, G, J) to be Kählerian is a rational function depending on the two essential parameters involved in the expression of the integrable almost complex structure J, their derivatives, the constant sectional curvature of (M, g) and k.

The manifolds, tensor fields and other geometric objects considered in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). The Einstein summation convention is used throughout this paper, the range of the indices h, i, j, k, l, m, r being always $\{1, \ldots, n\}$.

2. Preliminary results. If (M, g) is a smooth Riemannian manifold of the dimension n, and $\pi : T^*M \to M$ its cotangent bundle, then the total space T^*M may be endowed with a structure of $2n$-dimensional smooth manifold, induced from the structure of the base manifold, as follows: from every local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$, it is induced a local chart, $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n,\ p_1, \ldots, p_n)$ on T^*M, such that for a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first n local coordinates $q^1, \ldots, q^n$ are the local coordinates of its base point $x = \pi(p)$ in the local chart (U, φ) (in fact we have $q^i = \pi^* x^i = x^i \circ \pi,\ i = 1, \ldots n$); the last n local coordinates $p_1, \ldots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of p with respect to the natural basis $(dx^1_{\pi(p)}, \ldots, dx^n_{\pi(p)})$, defined by the local chart (U, φ), i.e. $p = p_i dx^i_{\pi(p)}$.

The notion of M-tensor field on the tangent bundle was introduced in the paper [8] and it is called also d-tensor field (e.g. see [2]). On the cotangent bundle T^*M, an M-tensor field of type (r, s) is defined by sets of n^{r+s} components (functions depending on q^i and p_i), with r upper indices and s lower indices, assigned to the induced local charts $(\pi^{-1}(U), \Phi)$ on T^*M, such that the local coordinate change rule is that of the local coordinate components of a tensor field of type (r, s) on the base manifold M. An usual tensor field of type (r, s) on M may be thought as an M-tensor field of type (r, s) on T^*M. If the considered tensor field on M is covariant only, the corresponding M-tensor field on T^*M may be identified with the induced (pullback by π) tensor field on T^*M.

Some useful M-tensor fields on T^*M may be obtained as follows. Let $v, w : [0, \infty) \to \mathbf{R}$ be smooth functions and let $\|p\|^2 = g^{-1}_{\pi(p)}(p, p)$ be the square of the norm of the cotangent vector $p \in \pi^{-1}(U)$ (g^{-1} is the tensor field of type (2,0) having the components $(g^{kl}(x))$ which are the entries of the inverse of the matrix $(g_{ij}(x))$ defined by the components of g in the local chart (U, φ)). The components $v(\|p\|^2) g_{ij}(\pi(p))$, p_i, $w(\|p\|^2) p_i p_j$ define respective M-tensor fields of types $(0, 2)$, $(0, 1)$, $(0, 2)$ on T^*M. Similarly, the components $v(\|p\|^2)g^{kl}(\pi(p))$, $g^{0i} = p_h g^{hi}$, $w(\|p\|^2)g^{0k}g^{0l}$ define respective M-tensor fields of type $(2,0)$, $(1,0)$, $(2,0)$ on T^*M. Of course, all the components considered above are in the induced local chart $(\pi^{-1}(U), \Phi)$.

We recall the splitting of the tangent bundle to T^*M into the vertical distribution $VT^*M = \text{Ker }\pi_*$ and the horizontal one determined by the Levi Civita connection $\dot\nabla$ of g:

$$TT^*M = VT^*M \oplus HT^*M. \tag{1}$$

If $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$ is a local chart on T^*M, induced from the local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$, the local vector fields $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}$ define a local frame for HT^*M over $\pi^{-1}(U)$, where

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^0_{ih}\frac{\partial}{\partial p_h}, \quad \Gamma^0_{ih} = p_k\Gamma^k_{ih}$$

and $\Gamma^k_{ih}(\pi(p))$ are the Christoffel symbols of g.

The set of vector fields $\{\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}\}$ defines a local frame on T^*M, adapted to the direct sum decomposition (1).

We consider

$$t = \frac{1}{2}\|p\|^2 = \frac{1}{2}g^{-1}_{\pi(p)}(p,p) = \frac{1}{2}g^{ik}(x)p_ip_k, \quad p \in \pi^{-1}(U) \tag{2}$$

the energy density defined by g in the cotangent vector p. We have $t \in [0, \infty)$ for all $p \in T^*M$.

The computations will be done in local coordinates, using a local chart (U, φ) on M and the induced local chart $(\pi^{-1}(U), \Phi)$ on T^*M.

We shall use the following lemma, which may be proved easily.

Lemma 2.1. *If $n > 1$ and u, v are smooth functions on T^*M such that*

$$ug_{ij} + vp_ip_j = 0, \quad ug^{ij} + vg^{0i}g^{0j} = 0, \quad \text{or} \quad u\delta^i_j + vg^{0i}p_j = 0,$$

on the domain of any induced local chart on T^*M, then $u = 0,\ v = 0$.

In the paper [3], the present author considered the real valued smooth functions $a_1,\ a_2,\ a_3,\ a_4,\ b_1,\ b_2,\ b_3,\ b_4$ on $[0, \infty) \subset \mathbf{R}$ and studied a general natural tensor of type $(1,1)$ on T^*M, defined by the relations

$$\begin{cases} JX^H_p = a_1(t)(g_X)^V_p + b_1(t)p(X)p^V_p + a_4(t)X^H_p + b_4(t)p(X)(p^\sharp)^H_p, \\ J\theta^V_p = a_3(t)\theta^V_p + b_3(t)g^{-1}_{\pi(p)}(p,\theta)p^V_p - a_2(t)(\theta^\sharp)^H_p \\ \qquad\quad -b_2(t)g^{-1}_{\pi(p)}(p,\theta)(p^\sharp)^H_p, \end{cases} \tag{3}$$

in every point p of the induced local card $(\pi^{-1}(U), \Phi)$ on T^*M, $\forall\ X \in \mathcal{X}(M)$, $\forall \theta \in \Lambda^1(M)$, where g_X is the 1-form on M defined by $g_X(Y) = g(X,Y)$, $\forall Y \in \mathcal{X}(M)$, $\theta^\sharp = g_\theta^{-1}$ is a vector field on M defined by $g(\theta^\sharp, Y) = \theta(Y)$, $\forall\ Y \in \mathcal{X}(M)$, the vector $p^\sharp$ is tangent to M in $\pi(p)$, p^V is the Liouville vector field on T^*M , and $(p^\sharp)^H$ is the similar horizontal vector field on T^*M.

With respect to the adapted frame $\{\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\}_{i,j=1,\dots,n}$ on T^*M, the expression (3) becomes

$$
(4)\qquad \begin{cases} J\dfrac{\delta}{\delta q^i} = a_1(t) g_{ij}\dfrac{\partial}{\partial p_j} + b_1(t) p_i C + a_4(t)\dfrac{\delta}{\delta q^i} + b_4(t) p_i \widetilde{C}, \\[2ex] J\dfrac{\partial}{\partial p_i} = a_3(t)\dfrac{\partial}{\partial p_i} + b_3(t) g^{0i} C - a_2(t) g^{ij}\dfrac{\delta}{\delta q^j} - b_2(t) g^{0i}\widetilde{C}, \end{cases}
$$

where $C = p^V$ is the Liouville vector-field on T^*M and $\widetilde{C} = (p^\sharp)^H$ is the corresponding horizontal vector field on T^*M.

We may write also

$$
(5)\qquad \begin{cases} J\dfrac{\delta}{\delta q^i} = J^{(1)}_{ij}\dfrac{\partial}{\partial p_j} + J4^j_i\dfrac{\delta}{\delta q^j}, \\[2ex] J\dfrac{\partial}{\partial p_i} = J3^i_j\dfrac{\partial}{\partial p_j} - J^{ij}_{(2)}\dfrac{\delta}{\delta q^j}, \end{cases}
$$

where

$$
J^{(1)}_{ij} = a_1(t) g_{ij} + b_1(t) p_i p_j, \quad J4^j_i = a_4(t)\delta^j_i + b_4(t) g^{0j} p_i
$$

$$
J3^i_j = a_3(t)\delta^i_j + b_3(t) g^{0i} p_j, \quad J^{ij}_{(2)} = a_2(t) g^{ij} + b_2(t) g^{0i} g^{0j}.
$$

Theorem 2.2 ([3])**.** *A natural tensor field J of type $(1,1)$ on T^*M given by (4) or (5) defines an almost complex structure on T^*M, if and only if $a_4 = -a_3, b_4 = -b_3$ and the coefficients a_1, a_2, a_3, b_1, b_2 and b_3 are related by*

$$
(6)\qquad a_1 a_2 = 1 + a_3^2\ , \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1 + (a_3 + 2tb_3)^2.
$$

Remark. From the conditions (6) we have that the coefficients $a_1, a_2, a_1 + 2tb_1$, $a_2 + 2tb_2$ have the same sign and maynot vanish. We assume that $a_1 > 0, a_2 > 0, a_1 + 2tb_1 > 0, a_2 + 2tb_2 > 0$ for all $t \geq 0$.

Remark. The relations (6) allow us to express two of the coefficients $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3$ as functions of the other four; e.g. we have:

$$a_2=\frac{1+a_3^2}{a_1},\quad b_2=\frac{2a_3b_3-a_2b_1+2tb_3^2}{a_1+2tb_1}. \tag{7}$$

The integrability condition for the above almost complex structure J on a manifold M is characterized by the vanishing of its Nijenhuis tensor field N_J, defined by

$$N_J(X,Y)=[JX,JY]-J[JX,Y]-J[X,JY]-[X,Y],$$

for all the vector fields X and Y on M.

Theorem 2.3 ([3]). *Let (M,g) be an $n(>2)$-dimensional connected Riemannian manifold. The almost complex structure J defined by* (4) *on T^*M is integrable if and only if (M,g) has constant sectional curvature c and the coefficients b_1,b_2,b_3 are given by:*

$$\begin{cases} b_1=\dfrac{2c^2ta_2^2+2cta_1a_2'+a_1a_1'-c+3ca_3^2}{a_1-2ta_1'-2cta_2-4ct^2a_2'},\\[2ex] b_2=\dfrac{2ta_3'^2-2ta_1'a_2'+ca_2^2+2cta_2a_2'+a_1a_2'}{a_1-2ta_1'-2cta_2-4ct^2a_2'},\\[2ex] b_3=\dfrac{a_1a_3'+2ca_2a_3+4cta_2'a_3-2cta_2a_3'}{a_1-2ta_1'-2cta_2-4ct^2a_2'}. \end{cases} \tag{8}$$

Remark. In the diagonal case, where $a_3=0$ it follows $b_3=0$ too, and we have:

$$a_2=\frac{1}{a_1},\ b_1=\frac{a_1a_1'-c}{a_1-2ta_1'},\ b_2=\frac{c-a_1a_1'}{a_1(a_1^2-2ct)}.$$

In the paper cited bellow, the present author introduced a Riemannian metric G of general natural lift type on T^*M, defined by the relations

$$\begin{cases} G_p(X^H,Y^H)=c_1(t)g_{\pi(p)}(X,Y)+d_1(t)p(X)p(Y),\\[1ex] G_p(\theta^V,\omega^V)=c_2(t)g^{-1}_{\pi(p)}(\theta,\omega)+d_2(t)g^{-1}_{\pi(p)}(p,\theta)g^{-1}_{\pi(p)}(p,\omega),\\[1ex] G_p(X^H,\theta^V)=G_p(\theta^V,X^H)=c_3(t)\theta(X)+d_3(t)p(X)g^{-1}_{\pi(p)}(p,\theta), \end{cases} \tag{9}$$

$\forall\, X, Y \in \mathcal{X}(M), \forall\, \theta, \omega \in \Lambda^1(M), \forall\, p \in T^*M$.

Using the adapted frame $\{\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\}_{i,j=1,\dots,n}$ on T^*M, we may write the expression (9) in the next form

$$\begin{cases} G\left(\dfrac{\delta}{\delta q^i}, \dfrac{\delta}{\delta q^j}\right) = c_1(t)g_{ij} + d_1(t)p_i p_j = G^{(1)}_{ij}, \\ G\left(\dfrac{\partial}{\partial p_i}, \dfrac{\partial}{\partial p_j}\right) = c_2(t)g^{ij} + d_2(t)g^{0i}g^{0j} = G^{ij}_{(2)}, \\ G\left(\dfrac{\partial}{\partial p_i}, \dfrac{\delta}{\delta q^j}\right) = G\left(\dfrac{\delta}{\delta q^i}, \dfrac{\partial}{\partial p_j}\right) = c_3(t)\delta^j_i + d_3(t)p_i g^{0j} = G3^j_i, \end{cases} \tag{10}$$

where $c_1, c_2, c_3, d_1, d_2, d_3$ are six smooth functions of the density energy on T^*M.

The conditions for G to be positive definite are assured if

$$c_1 + 2td_1 > 0,\ c_2 + 2td_2 > 0,\ (c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 > 0. \tag{11}$$

The metric G is almost Hermitian with respect to the general almost complex structure J, if $G(JX, JY) = G(X, Y)$, for all the vector fields X, Y on T^*M.

The author proved the following result

Theorem 2.4 ([3])**.** *The family of general natural Riemannian metrics G on T^*M such that (T^*M, G, J) is an almost Hermitian manifold, is given by* (10)*, provided that the coefficients c_1, c_2, c_3, d_1, d_2, and d_3 are related to the coefficients a_1, a_2, a_3, b_1, b_2, and b_3 by the following proportionality relations*

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \frac{c_3}{a_3} = \lambda \tag{12}$$

$$\frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \frac{c_3 + 2td_3}{a_3 + 2tb_3} = \lambda + 2t\mu, \tag{13}$$

where the proportionality coefficients $\lambda > 0$ and $\lambda + 2t\mu > 0$ are some functions depending on t.

Remark. In the case where $a_3 = 0$, it follows that $c_3 = d_3 = 0$ and we obtain the almost Hermitian structure considered in [18], [20]. Moreover, if

$\lambda = 1$ and $\mu = 0$, we obtain the almost Kählerian structure considered in the mentioned papers.

Considering the two-form Ω defined by the almost Hermitian structure (G, J) on T^*M

$$\Omega(X, Y) = G(X, JY),$$

for all the vector fields X, Y on T^*M, we obtain the following result from [3]:

Proposition 2.5 ([3]). *The expression of the* 2*-form* Ω *in the local adapted frame* $\{\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\}_{i,j=1,\ldots,n}$ *on* T^*M*, is given by*

$$\Omega\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = 0,\ \Omega\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right) = 0,\ \Omega\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right) = \lambda\delta^i_j + \mu g^{0i} p_j$$

or, equivalently

$$\Omega = (\lambda\delta^i_j + \mu g^{0i} p_j) Dp_i \wedge dq^j, \tag{14}$$

where $Dp_i = dp_i - \Gamma^0_{ih} dq^h$ *is the absolute differential of* p_i*.*

Next, by calculating the exterior differential of Ω, we may state:

Theorem 2.6 ([3]). *The almost Hermitian structure* (T^*M, G, J) *is almost Kählerian if and only if* $\mu = \lambda'$*.*

Remark. The family of general natural almost Kählerian structures on T^*M depends on five essential coefficients $a_1,\ a_3,\ b_1,\ b_3,\ \lambda$, which must satisfy the supplementary conditions $a_1 > 0,\ a_1 + 2tb_1 > 0,\ \lambda > 0,\ \lambda + 2t\mu > 0$.

The main result obtained in [3] is the next one:

Theorem 2.7 ([3]). *A general natural lifted almost Hermitian structure* (G, J) *on* T^*M *is Kählerian if and only if the almost complex structure* J *is integrable* (*see Theorem* 2.3) *and* $\mu = \lambda'$*.*

Remark. The family of general natural Kählerian structures on T^*M depends on three essential coefficients $a_1,\ a_3,\ \lambda$, which must satisfy the supplementary conditions $a_1 > 0,\ a_1 + 2tb_1 > 0,\ \lambda > 0,\ \lambda + 2t\lambda' > 0$, where b_1 is is given by (8).

Examples of such structures may be found in [18], [20].

3. General natural Kähler structures of constant holomorphic sectional curvature on cotangent bundles. The Levi-Civita connection ∇ of the Riemannian manifold (T^*M, G) is obtained from the formula

$$2G(\nabla_X Y, Z) = X(G(Y,Z)) + Y(G(X,Z)) - Z(G(X,Y)) \\ + G([X,Y],Z) - G([X,Z],Y) - G([Y,Z],X); \forall X,Y,Z \in \chi(M)$$

and it is characterized by the conditions $\nabla G = 0,\ T = 0$, where T is the torsion tensor of ∇.

In the case of the cotangent bundle T^*M we may obtain the explicit expression of ∇. The symmetric $2n \times 2n$ matrix

$$\begin{pmatrix} G^{(1)}_{ij} & G3^j_i \\ G3^i_j & G^{ij}_{(2)} \end{pmatrix}$$

associated to the metric G in the base $\{\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}\}_{i,j=1,\dots,n}$ has the inverse

$$\begin{pmatrix} H^{ij}_{(1)} & H3^j_i \\ H3^i_j & H^{(2)}_{ij} \end{pmatrix}$$

where the entries are the blocks

$$(15)\quad H^{kl}_{(1)} = e_1 g^{kl} + f_1 g^{0k} g^{0l},\ H^{(2)}_{kl} = e_2 g_{kl} + f_2 p_k p_l,\ H3^k_l = e_3 \delta^k_l + f_3 g^{0k} p_l.$$

Here g^{kl} are the components of the inverse of the matrix (g_{ij}), $g^{0k} = p_i g^{ik}$, and $e_1, f_1, e_2, f_2, e_3,\ f_3 : [0,\infty) \to \mathbb{R}$, some real smooth functions. Their expressions are obtained by solving the system:

$$(16)\quad \begin{cases} G^{(1)}_{ih} H^{hk}_{(1)} + G3^h_i H3^k_h = \delta^k_i \\ G^{(1)}_{ih} H3^h_k + G3^h_i H^{(2)}_{hk} = 0 \\ G3^i_h H^{hk}_{(1)} + G^{ih}_{(2)} H3^k_h = 0 \\ G3^i_h H3^h_k + G^{ih}_{(2)} H^{(2)}_{hk} = \delta^i_k, \end{cases}$$

in which we substitute the relations (10) and (15). By using lemma 2.1, we get $e_1,\ e_2,\ e_3$ as functions of $c_1,\ c_2,\ c_3$

$$(17)\qquad e_1 = \frac{c_2}{c_1 c_2 - c_3^2},\quad e_2 = \frac{c_1}{c_1 c_2 - c_3^2},\quad e_3 = -\frac{c_3}{c_1 c_2 - c_3^2}$$

and f_1, f_2, f_3 as functions of $c_1, c_2, c_3,\ d_1, d_2, d_3,\ e_1, e_2, e_3$

$$
\begin{aligned}
f_1 &= -\frac{c_2d_1e_1 - c_3d_3e_1 - c_3d_2e_3 + c_2d_3e_3 + 2d_1d_2e_1t - 2d_3^2e_1t}{c_1c_2 - c_3^2 + 2c_2d_1t + 2c_1d_2t - 4c_3d_3t + 4d_1d_2t^2 - 4d_3^2t^2},\\
f_2 &= \frac{(c_3+2d_3t)[(d_3e_1+d_2e_3)(c_1+2d_1t) - (d_1e_1+d_3e_3)(c_3+2d_3t)]}{(c_2+2d_2t)[(c_1+2d_1t)(c_2+2d_2t) - (c_3+2d_3t)^2]}\\
&\quad -\frac{d_2e_2+d_3e_3}{c_2+2d_2t},\\
f_3 &= -\frac{(d_3e_1+d_2e_3)(c_1+2d_1t) - (d_1e_1+d_3e_3)(c_3+2d_3t)}{(c_1+2d_1t)(c_2+2d_2t) - (c_3+2d_3t)^2}.
\end{aligned}
\tag{18}
$$

Next we may obtain the expression of the Levi Civita connection of the Riemannian metric G on T^*M.

Theorem 3.1. *The Levi-Civita connection ∇ of G has the following expression in the local adapted frame $\{\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}\}_{i,j=1,\dots,n}$*

$$
\begin{cases}
\nabla_{\frac{\partial}{\partial p_i}} \dfrac{\partial}{\partial p_j} = Q^{ij}{}_h \dfrac{\partial}{\partial p_h} + \widetilde{Q}^{ijh} \dfrac{\delta}{\delta q^h},\\
\nabla_{\frac{\delta}{\delta q^i}} \dfrac{\partial}{\partial p_j} = (-\Gamma^j_{ih} + \widetilde{P}_i{}^j{}_h) \dfrac{\partial}{\partial p_h} + P_i{}^{jh} \dfrac{\delta}{\delta q^h}\\
\nabla_{\frac{\partial}{\partial p_i}} \dfrac{\delta}{\delta q^j} = P_j{}^{ih} \dfrac{\delta}{\delta q^h} + \widetilde{P}_j{}^i{}_h \dfrac{\partial}{\partial p_h},\\
\nabla_{\frac{\delta}{\delta q^i}} \dfrac{\delta}{\delta q^j} = (\Gamma^h_{ij} + \widetilde{S}_{ij}{}^h) \dfrac{\delta}{\delta p_h} + S_{ijh} \dfrac{\partial}{\partial p_h},
\end{cases}
$$

where Γ^h_{ij} are the Christoffel symbols of the connection $\dot\nabla$ and the M-tensor fields appearing as coefficients in the above expressions are given by

$$
\begin{cases}
Q^{ij}{}_h = \frac{1}{2}(\partial^i G^{jk}_{(2)} + \partial^j G^{ik}_{(2)} - \partial^k G^{ij}_{(2)}) H^{(2)}_{kh} + \frac{1}{2}(\partial^i G3^j_k + \partial^j G3^i_k) H3^k_h,\\
\widetilde{Q}^{ijh} = \frac{1}{2}(\partial^i G^{jk}_{(2)} + \partial^j G^{ik}_{(2)} - \partial^k G^{ij}_{(2)}) H3^h_k + \frac{1}{2}(\partial^i G3^j_k + \partial^j G3^i_k) H^{kh}_{(1)},\\
P_j{}^{ih} = \frac{1}{2}(\partial^i G3^k_j - \partial^k G3^i_j) H3^h_k + \frac{1}{2}(\partial^i G^{(1)}_{jk} - R^0_{ljk} G^{li}_{(2)}) H^{kh}_{(1)},\\
\widetilde{P}_j{}^i{}_h = \frac{1}{2}(\partial^i G3^k_j - \partial^k G3^i_j) H^{(2)}_{kh} + \frac{1}{2}(\partial^i G^{(1)}_{jk} - R^0_{ljk} G^{li}_{(2)}) H3^k_h,\\
S_{ijh} = \frac{1}{2}(R^0_{lij} G^{lk}_{(2)} - \partial^k G^{(1)}_{ij}) H^{(2)}_{kh} - c_3 R^0_{ijk} H3^k_h,\\
\widetilde{S}_{ij}{}^h = \frac{1}{2}(R^0_{lij} G^{lk}_{(2)} - \partial^k G^{(1)}_{ij}) H3^h_k - c_3 R^0_{ijk} H^{kh}_{(1)},
\end{cases}
\tag{19}
$$

where R^h_{kij} are the components of the curvature tensor field of the Levi Civita connection $\dot\nabla$ of the base manifold (M, g).

If we replace in (19) the relations (10) which define the metric G, the expressions (15) for the inverse matrix H of G, and the formulas (17), (18) we obtain the detailed expressions of $P_i{}^{jh}$, $Q^{ij}{}_h$, S_{ijh}, $\widetilde{P}_j{}^i{}_h$, $\widetilde{Q}^{ijh}$, $\widetilde{S}_{ij}{}^h$.

The curvature tensor field K of the connection ∇ is defined by

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad X,Y,Z \in \mathcal{X}(TM).$$

By using the local adapted frame $\{\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}\}_{i,j=1,\dots,n} = \{\delta_i, \partial^j\}_{i,j=1,\dots,n}$ we obtain the horizontal and vertical components of the curvature tensor field:

$$\begin{aligned}
K(\delta_i,\delta_j)\delta_k &= QQQQ_{ijk}{}^h\delta_h + QQQP_{ijkh}\partial^h,\\
K(\delta_i,\delta_j)\partial^k &= QQPQ_{ij}{}^{kh}\delta_h + QQPP_{ij}{}^k{}_h\partial^h,\\
K(\partial^i,\partial^j)\delta_k &= PPQQ^{ij}{}_k{}^h\delta_h + PPQP^{ij}{}_{kh}\partial^h,\\
K(\partial^i,\partial^j)\partial^k &= PPPQ^{ijkh}\delta_h + PPPP^{ijk}{}_h\partial^h,\\
K(\partial^i,\delta_j)\delta_k &= PQQQ^i{}_{jk}{}^h\delta_h + PQQP^i{}_{jkh}\partial^h,\\
K(\partial^i,\delta_j)\partial^k &= PQPQ^i{}_j{}^{kh}\delta_h + PQPP^i{}_j{}^k{}_h\partial^h,
\end{aligned}$$

where the coefficients are the M-tensor fields given by

$$\begin{aligned}
QQQQ_{ijk}{}^h &= \widetilde{S}_{jk}{}^l\widetilde{S}_{il}{}^h + P_i{}^{lh}S_{jkl} - \widetilde{S}_{jl}{}^h\widetilde{S}_{ik}{}^l - P_j{}^{lh}S_{ikl}\\
&\quad - R^0_{lij}P_k{}^{lh} + R^h_{kij}\\
QQQP_{ijkh} &= \widetilde{S}_{jk}{}^l S_{ilh} + \widetilde{P}_i{}^l{}_h S_{jkl} - \widetilde{S}_{ik}{}^l S_{jlh} - \widetilde{P}_j{}^l{}_h S_{ikl} - \widetilde{P}_k{}^l{}_h R^0_{lij}\\
QQPQ_{ij}{}^{kh} &= \widetilde{P}_j{}^k{}_l P_i{}^{lh} + P_j{}^{kl}\widetilde{S}_{il}{}^h - \widetilde{P}_i{}^k{}_l P_j{}^{lh} - P_i{}^{kl}\widetilde{S}_{jl}{}^h\\
&\quad - R^0_{lij}\widetilde{Q}^{lkh},\\
QQPP_{ij}{}^k{}_h\partial^h &= \widetilde{P}_j{}^k{}_l\widetilde{P}_i{}^l{}_h + P_j{}^{kl}S_{ilh} - \widetilde{P}_i{}^k{}_l\widetilde{P}_j{}^l{}_h - P_i{}^{kl}S_{jlh}\\
&\quad - R^0_{lij}Q^{lk}{}_h - R^k_{lij},\\
PPQQ^{ij}{}_k{}^h\delta_h &= \partial^i P_k{}^{jh} - \partial^j P_k{}^{ih} + \widetilde{P}_k{}^j{}_l\widetilde{Q}^{ilh} + P_k{}^{jl}P_l{}^{ih}\\
&\quad - \widetilde{P}_k{}^i{}_l\widetilde{Q}^{jlh} - P_k{}^{il}P_l{}^{jh},\\
PPQP^{ij}{}_{kh} &= \partial^i\widetilde{P}_k{}^j{}_h - \partial^j\widetilde{P}_k{}^i{}_h + \widetilde{P}_k{}^j{}_l Q^{il}{}_h + P_k{}^{jl}\widetilde{P}_l{}^i{}_h\\
&\quad - \widetilde{P}_k{}^i{}_l Q^{jl}{}_h - P_k{}^{il}\widetilde{P}_l{}^j{}_h,\\
PPPQ^{ijkh} &= \partial^i\widetilde{Q}^{jkh} - \partial^j\widetilde{Q}^{ikh} + Q^{jk}{}_l\widetilde{Q}^{ilh} + \widetilde{Q}^{jkl}P_l{}^{ih}\\
&\quad - Q^{ik}{}_l\widetilde{Q}^{jlh} - \widetilde{Q}^{ikl}P_l{}^{jh},
\end{aligned}$$

$$
\begin{aligned}
PPPP^{ijk}{}_h &= \partial^i Q^{jk}{}_h - \partial^j Q^{ik}{}_h + Q^{jk}{}_l Q^{il}{}_h + \widetilde{Q}^{jkl}\widetilde{P}_l{}^i{}_h \\
&\quad - Q^{ik}{}_l Q^{jl}{}_h - \widetilde{Q}^{ikl}\widetilde{P}_l{}^j{}_h, \\
PQQQ^i{}_{jk}{}^h\delta_h &= \partial^i \widetilde{S}_{jk}{}^h + S_{jkl}\widetilde{Q}^{ilh} + \widetilde{S}_{jk}{}^l P_l{}^{ih} - \widetilde{P}_k{}^i{}_l P_j{}^{lh} - P_k{}^{il}\widetilde{S}_{jl}{}^h \\
PQQP^i{}_{jkh} &= \partial^i S_{jkh} + \widetilde{S}_{jk}{}^l Q^{il}{}_h + \widetilde{S}_{jk}{}^l \widetilde{P}_l{}^i{}_h - \widetilde{P}_k{}^i{}_l \widetilde{P}_j{}^l{}_h - P_k{}^{il} S_{jlh} \\
PQPQ^i{}_j{}^{kh} &= \partial^i P_j{}^{kh} + \widetilde{P}_j{}^k{}_l \widetilde{Q}^{ilh} + P_j{}^{kl} P_l{}^{ih} - Q^{ik}{}_l P_j{}^{lh} - \widetilde{Q}^{ikl}\widetilde{S}_{jl}{}^h, \\
PQPP^i{}_j{}^k{}_h &= \partial^i \widetilde{P}_j{}^k{}_h + \widetilde{P}_j{}^k{}_l Q^{il}{}_h + P_j{}^{kl}\widetilde{P}_l{}^i{}_h - Q^{ik}{}_l \widetilde{P}_j{}^l{}_h - \widetilde{Q}^{ikl} S_{jlh}.
\end{aligned}
$$

In order to get the final expressions of the above M-tensor fields, we have to compute the first and second order partial derivatives with respect to the cotangential coordinates p_i of the usual tensor fields involved in the definition of the Riemannian metric G.

$$
\begin{aligned}
\partial^i G^{(1)}_{jk} &= c_1' g^{0i} g_{jk} + d_1' g^{0i} p_j p_k + d_1 \delta^i_j p_k + d_1 p_j \delta^i_k \\
\partial^i G^{jk}_{(2)} &= c_2' g^{0i} g^{jk} + d_2' g^{0i} g^{0j} g^{0k} + d_2 g^{ij} g^{0k} + d_2 g^{0j} g^{ik} \\
\partial^i G3^j_k &= c_3' g^{0i} \delta^j_k + d_3' g^{0i} g^{0j} p_k + d_3 g^{ij} p_k + d_3 g^{0j} \delta^i_k \\
\partial^i \partial^j G^{(1)}_{kl} &= c_1'' g^{0i} g^{0j} g_{kl} + c_1' g^{ij} g_{kl} + d_1'' g^{0i} g^{0j} p_k p_l + d_1' g^{ij} p_k p_l \\
&\quad + d_1' g^{0j} \delta^i_k p_l + d_1' g^{0j} p_k \delta^i_l + d_1' g^{0i} \delta^j_k p_l + d_1 \delta^j_k \delta^i_l \\
&\quad + d_1' g^{0i} p_k \delta^j_l + d_1 \delta^i_k \delta^j_l \\
\partial^i \partial^j G^{kl}_{(2)} &= c_2'' g^{0i} g^{0j} g^{kl} + c_2' g^{ij} g^{kl} + d_2'' g^{0i} g^{0j} g^{0k} g^{0l} + d_2' g^{ij} g^{0k} g^{0l} \\
&\quad + d_2' g^{0j} g^{ik} g^{0l} + d_2' g^{0j} g^{0k} g^{il} + d_2' g^{0i} g^{jk} g^{0l} + d_2 g^{jk} g^{il} \\
&\quad + d_2' g^{0i} g^{0k} g^{jl} + d_2 g^{ik} g^{jl} \\
\partial^i \partial^j G3^k_l &= c_3'' g^{0i} g^{0j} \delta^k_l + c_3' g^{ij} \delta^k_l + d_3'' g^{0i} g^{0j} g^{0k} p_l + d_3' g^{ij} g^{0k} p_l \\
&\quad + d_3' g^{0j} g^{ik} p_l + d_3' g^{0j} g^{0k} \delta^i_l + d_3' g^{0i} g^{jk} p_l + d_3 g^{jk} \delta^i_l \\
&\quad + d_3' g^{0i} g^{0k} \delta^j_l + d_3 g^{ik} \delta^j_l \\
\partial^i H^{jk}_{(1)} &= e_1' g^{0i} g^{jk} + f_1' g^{0i} g^{0j} g^{0k} + f_1 g^{ij} g^{0k} + f_1 g^{0j} g^{ik} \\
\partial^i H^{(2)}_{jk} &= e_2' g^{0i} g_{jk} + f_2' g^{0i} p_j p_k + f_2 \delta^i_j p_k + f_2 p_j \delta^i_k \\
\partial^i H3^j_k &= e_3' g^{0i} \delta^j_k + f_3' g^{0i} g^{0j} p_k + f_3 g^{ij} p_k + f_3 g^{0j} \delta^i_k
\end{aligned}
$$

We get the first order partial derivatives of the M-tensor fields $P_i{}^{jh}$, $Q^{ij}{}_h$, S_{ijh}, $\widetilde{P}_j{}^i{}_h$, $\widetilde{Q}^{ijh}$, $\widetilde{S}_{ij}{}^h$ with respect to the cotangential coordinates p_i and we replace these derivatives, and the expressions (17), (18) of the functions $e_1,\ e_2,\ e_3,\ f_1,\ f_2,\ f_3$ and of their derivatives in order to obtain the components of the curvature tensor as functions of $a_1,\ a_2,\ a_3$ and their derivatives of first, second and third order only. The detailed expressions may be obtained by using the Mathematica package RICCI.

$$\begin{aligned}
\partial^i Q^{jk}{}_h &= \frac{1}{2}\partial^i H^{(2)}_{lh}(\partial^j G^{(2)}_{kl} + \frac{1}{2}H^{(2)}_{lh}(\partial^i\partial^j G^{(2)}_{kl} + \partial^i\partial^k G^{jl}_{(2)} - \partial^i\partial^l G^{jk}_{(2)})\\
&+ \frac{1}{2}\partial^i H3^l_h(\partial^j G3^k_l + \partial^k G3^k_l) + \frac{1}{2}H3^l_h(\partial^i\partial^j G3^k_l + \partial^i\partial^k G3^k_l),\\
\partial_i \widetilde{Q}^{jkh} &= \frac{1}{2}\partial^i H3^h_l(\partial^j G^{kl}_{(2)} + \partial^k G^{jl}_{(2)} - \partial^l G^{jk}_{(2)})\\
&+ \frac{1}{2}H3^h_l(\partial^i\partial^j G^{kl}_{(2)} + \partial^i\partial^k G^{jl}_{(2)} - \partial^i\partial^l G^{jk}_{(2)})\\
&+ \frac{1}{2}\partial^i H^{lh}_{(1)}(\partial^j G3^k_l + \partial^k G3^j_l) + \frac{1}{2}H^{lh}_{(1)}(\partial^i\partial^j G3^k_l + \partial^i\partial^k G3^j_l),\\
\partial^i \widetilde{P}_j{}^k{}_h &= \frac{1}{2}\partial^i H^{(2)}_{lh}(\partial^k G3^l_j - \partial^l G3^k_j) + \frac{1}{2}H^{(2)}_{lh}(\partial^i\partial^k G3^l_j - \partial^i\partial^l G3^k_j)\\
&+ \frac{1}{2}\partial^i H3^l_h(\partial^k G^{(1)}_{jl} - R^0_{mjl}G^{mk}_{(2)})\\
&+ \frac{1}{2}H3^l_h(\partial^i\partial^k G^{(1)}_{jl} - R^i_{mjl}G^{mk}_{(2)} - R^0_{mjl}\partial^i G^{mk}_{(2)}),\\
\partial^i P_j{}^{kh} &= \frac{1}{2}\partial^i H3^h_l(\partial^k G3^l_j - \partial^l G3^k_j) + \frac{1}{2}H3^h_l(\partial^i\partial^k G3^l_j - \partial^i\partial^l G3^k_j)\\
&+ \frac{1}{2}\partial^i H^{hl}_{(1)}(\partial^k G^{(1)}_{jl} - R^0_{mjl}G^{mk}_{(2)})\\
&+ \frac{1}{2}H^{hl}_{(1)}(\partial^i\partial^k G^{(1)}_{jl} - R^i_{mjl}G^{mk}_{(2)} - R^0_{mjl}\partial^i G^{mk}_{(2)}),\\
\partial^i S_{jkh} &= \frac{1}{2}[(c_2' g^{0i}R^0_{mjk} + c_2 R^i_{mjk} - \partial^i\partial^l G^{(1)}_{jk})H^{(2)}_{lh} + (c_2 R^0_{mjk} - \partial^l G^{(1)}_{jk})\partial^i H^{(2)}_{lh}]\\
&- c_3' g^{0i}R^0_{jkl}H3^l_h - c_3(R^i_{jkl}H3^l_h + R^0_{jkl}\partial^i H3^l_h),\\
\partial^i \widetilde{S}_{jk}{}^h &= \frac{1}{2}[(c_2' g^{0i}R^0_{mjk} + c_2 R^i_{mjk} - \partial^i\partial^l G^{(1)}_{jk})H3^h_l + (c_2 R^0_{mjk} - \partial^l G^{(1)}_{jk})\partial^i H3^h_l]\\
&- c_3' g^{0i}R^0_{jkl}H^{lh}_{(1)} - c_3(R^i_{jkl}H^{lh}_{(1)} + R^0_{jkl}\partial^i H^{lh}_{(1)}).
\end{aligned}$$

The tensor field corresponding to the curvature tensor field of a Kälerian manifold (T^*M, G, J) having constant holomorphic sectional curvature k is given by the formula:

$$K_0(X,Y)Z = \frac{k}{4}[G(Y,Z)X - G(X,Z)Y + G(JY,Z)JX$$

$$-G(JX,Z)JY + 2G(X,JY)JZ].$$

With respect to the adapted frame $\{\delta_i, \partial^j\}_{i,j=1,\dots,n}$, the expressions are

$$\begin{aligned}
K_0(\delta_i,\delta_j)\delta_k &= QQQQ_{0ijk}{}^h\delta_h + QQQP_{0ijkh}\partial^h,\\
K_0(\delta_i,\delta_j)\partial^k &= QQPQ_{0ij}{}^{kh}\delta_h + QQPP_{0ij}{}^k{}_h\partial^h,\\
K_0(\partial^i,\partial^j)\delta_k &= PPQQ_0{}^{ij}{}_k{}^h\delta_h + PPQP_0{}^{ij}{}_{kh}\partial^h,\\
K_0(\partial^i,\partial^j)\partial^k &= PPPP_0{}^{ijk}{}_h\partial^h + PPPQ_0{}^{ijkh}\delta_h,\\
K_0(\partial^i,\delta_j)\delta_k &= PQQQ_0{}^i{}_{jk}{}^h\delta_h + PQQP_0{}^i{}_{jkh}\partial^h,\\
K_0(\partial^i,\delta_j)\partial^k &= PQPP_0{}^i{}_j{}^k{}_h\partial^h + PQPQ_0{}^i{}_j{}^{k\,h}\delta_h,
\end{aligned}$$

where

$$\begin{aligned}
QQQQ_{0ijk}{}^h &= \frac{k}{4}[G^{(1)}_{jk}\delta^h_i - G^{(1)}_{ik}\delta^h_j - J3^h_i(J^{(1)}_{jl}G3^l_k - J3^l_jG^{(1)}_{lk})\\
&\quad + J3^h_j(J^{(1)}_{il}G3^l_k - J3^l_iG^{(1)}_{lk}) - 2J3^h_k(J^{(1)}_{jl}G3^l_i - J3^l_jG^{(1)}_{il})],\\
QQQP_{0ijkh} &= \frac{k}{4}[J^{(1)}_{ih}(J^{(1)}_{jl}G3^l_k - J3^l_jG^{(1)}_{lk}) - J^{(1)}_{jh}(J^{(1)}_{il}G3^l_k - J3^l_iG^{(1)}_{lk})\\
&\quad + 2J^{(1)}_{kh}(J^{(1)}_{jl}G3^l_i - J3^l_jG^{(1)}_{il})],\\
QQPQ_{0ij}{}^{kh} &= \frac{k}{4}[G3^k_j\delta^h_i - G3^k_i\delta^h_j - J3^h_i(J^{(1)}_{jl}G^{lk}_{(2)} - J3^l_jG3^k_l)\\
&\quad + J3^h_j(J^{(1)}_{il}G^{lk}_{(2)} - J3^l_iG3^k_l) - 2J^{kh}_{(2)}(J^{(1)}_{jl}G3^l_i - J3^l_jG^{(1)}_{il})],\\
QQPP_{0ij}{}^k{}_h &= \frac{k}{4}[J^{(1)}_{ih}(J^{(1)}_{jl}G^{lk}_{(2)} - J3^l_jG3^k_l) - J^{(1)}_{jh}(J^{(1)}_{il}G^{lk}_{(2)} - J3^l_iG3^k_l)\\
&\quad + 2J3^k_h(J^{(1)}_{jl}G3^l_i - J3^l_jG^{(1)}_{il})],\\
PPQQ_0{}^{ij}{}_k{}^h &= \frac{k}{4}[-J^{ih}_{(2)}(J3^j_lG3^l_k - J^{jl}_{(2)}G^{(1)}_{lk}) + J^{jh}_{(2)}(J3^i_lG3^l_k - J^{il}_{(2)}G^{(1)}_{lk})\\
&\quad - 2J3^h_k(J3^j_lG^{il}_{(2)} - J^{jl}_{(2)}G3^i_l)],\\
PPQP_0{}^{ij}{}_{kh} &= \frac{k}{4}[G3^j_k\delta^i_hh - G3^i_k\delta^j_h + J3^i_h(J3^j_lG3^l_k - J^{jl}_{(2)}G^{(1)}_{lk})\\
&\quad - J3^j_h(J3^i_lG3^l_k - J^{il}_{(2)}G^{(1)}_{lk}) + 2J^{(1)}_{kh}(J3^j_lG^{il}_{(2)} - J^{jl}_{(2)}G3^i_l)],\\
PPPQ_0{}^{ijkh} &= \frac{k}{4}[-J^{ih}_{(2)}(J3^j_lG^{lk}_{(2)} - J^{jl}_{(2)}G3^k_l) + J^{jh}_{(2)}(J3^i_lG^{lk}_{(2)} - J^{il}_{(2)}G3^k_l)\\
&\quad - 2J^{kh}_{(2)}(J3^j_lG^{il}_{(2)} - J^{jl}_{(2)}G3^i_l)],
\end{aligned}$$

$$PPPP_0{}^{ijk}{}_h = \frac{k}{4}[G^{jk}_{(2)}\delta^i_h - G^{ik}_{(2)}\delta^j_h + J3^i_h(J3^j_l G^{lk}_{(2)} - J^{jl}_{(2)}G3^k_l)$$
$$- J3^j_h(J3^i_l G^{lk}_{(2)} - J^{il}_{(2)}G3^k_l) + 2J3^k_h(J3^j_l G^{il}_{(2)} - J^{jl}_{(2)}G3^i_l)],$$
$$PQQQ_0{}^i{}_{jk}{}^h = \frac{k}{4}[-G3^i_k\delta^h_j - J^{ih}_{(2)}(J^{(1)}_{jl}G3^l_k - J3^l_j G^{(1)}_{lk})$$
$$+ J3^h_j(J3^i_l G3^l_k - J^{il}_{(2)}G^{(1)}_{lk}) - 2J3^h_k(J^{(1)}_{jl}G^{il}_{(2)} - J3^l_j G3^i_l)],$$
$$PQQP_0{}^i{}_{jkh} = \frac{k}{4}[G^{(1)}_{jk}\delta^i_h + J3^i_h(J^{(1)}_{jl}G3^l_k - J3^l_j G^{(1)}_{lk})$$
$$- J^{(1)}_{jh}(J3^i_l G3^l_k - J^{il}_{(2)}G^{(1)}_{lk}) + 2J^{(1)}_{kh}(J^{(1)}_{jl}G^{il}_{(2)} - J3^l_j G3^i_l)],$$
$$PQPQ_0{}^i{}_j{}^{kh} = \frac{k}{4}[-G^{ik}_{(2)}\delta^h_j - J^{ih}_{(2)}(J^{(1)}_{jl}G^{lk}_{(2)} - J3^l_j G3^k_l)$$
$$+ J3^h_j(J3^i_l G^{lk}_{(2)} - J^{il}_{(2)}G3^k_l) - 2J^{kh}_{(2)}(J^{(1)}_{jl}G^{il}_{(2)} - J3^l_j G3^i_l)],$$
$$PQPP_0{}^i{}_j{}^k{}_h = \frac{k}{4}[G3^k_j\delta^i_h + J3^i_h(J^{(1)}_{jl}G^{lk}_{(2)} - J3^l_j G3^k_l)$$
$$- J^{(1)}_{jh}(J3^i_l G^{lk}_{(2)} - J^{il}_{(2)}G3^k_l) + 2J3^k_h(J^{(1)}_{jl}G^{il}_{(2)} - J3^l_j G3^i_l)].$$

The Kählerian manifold (T^*M, G, J) is of constant holomorphic sectional curvature if and only if all the components of the difference $K - K_0$ vanish. In the study of the vanishing conditions for the components of $K - K_0$ we use the following result similar to the lemma 2.1.

Lemma 3.2. *If $\alpha_1, \ldots, \alpha_{10}$ are smooth functions on T^*M such that*

$$\alpha_1 g_{hj}g^{ik} + \alpha_2\delta^i_h\delta^k_j + \alpha_3\delta^k_h\delta^i_j + \alpha_4 g^{ik}p_h p_j + \alpha_5\delta^k_j p_h g^{0i} + \alpha_6\delta^k_h p_j g^{0i}$$
$$+\alpha_7\delta^i_j p_h g^{0k} + \alpha_8 g_{hj}g^{0i}g^{0k} + \alpha_9\delta^i_h p_j g^{0k} + \alpha_{10}p_h p_j g^{0i}g^{0k} = 0, \qquad (20)$$

then $\alpha_1 = \cdots = \alpha_{10} = 0$.

Proof. If we multiply the expression (20) by $g^{hj}g_{ik}$, we have

$$\alpha_1 n^2 + \alpha_2 n + \alpha_3 n + 2\alpha_4 nt + 2\alpha_5 t + 2\alpha_6 t + 2\alpha_7 t + 2\alpha_8 nt + 2\alpha_9 t + 4\alpha_{10}t^2 = 0.$$

Since the expression does not depend on the dimension n of the base manifold, we obtain that

$$\alpha_1 = 0,\ \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_8)t = 0,\ (\alpha_5 + \alpha_6 + \alpha_7 + \alpha_9)t + 2\alpha_{10}t^2 = 0.$$

Similarly, we get that α_2 and α_3 are also zero, if we multiply the expression (20), respectively by $\delta^h_i\delta^j_k$ and $\delta^h_k\delta^j_i$.

The product between (20) and $g_{ik}g^{0h}g^{0j}$, $\delta_k^j g^{0h}p_i$, $\delta_k^h g^{0j}p_i$, $\delta_i^j g^{0h}p_k$, $g^{hj}p_ip_k$, or $\delta_i^h g^{0j}p_k$, leads to some expressions in which the coefficients of n are, respectively $2(\alpha_1 t+2\alpha_4 t^2)$, $2(\alpha_2 t+2\alpha_5 t^2)$, $2(\alpha_3 t+2\alpha_6 t^2)$, $2(\alpha_1 t+2\alpha_8 t^2)$, $2(\alpha_3 t+2\alpha_7 t^2)$, $2(\alpha_1 t+2\alpha_8 t^2)$, $2(\alpha_2 t+2\alpha_9 t^2)$. This expressions must vanish for all $t \geq 0$. Since $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we obtain that $\alpha_4 = \cdots = \alpha_9 = 0$ too.

Multiplying by $g^{0h}g^{0j}p_ip_k$, the relation (20) becomes

$$4[(\alpha_1+\alpha_2+\alpha_3)t^2 + 2(\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9)t^3 + 4\alpha_{10}t^4] = 0. \tag{21}$$

Taking into account that $\alpha_i = 0$, $\forall i = 1, \ldots, 9$, it follows from (21) that $\alpha_{10} = 0$. □

The final theorem gives the condition under which the Kählerian manifold of general natural lift type has constant holomorphic sectional curvature

Theorem 3.3. *The Kählerian manifold (T^*M, G, J) with G and J obtained as general natural lifts of the metric g from the Riemannian manifold (M, g), has constant holomorphic sectional curvature k if and only if the parameter λ is given by*

$$\lambda = \frac{4a_1c}{k(a_1^2 + 2ct + 2a_3^2ct)} \tag{22}$$

Proof. The expressions of the differences that we study are quite long, but in $PQPP^i{}_j{}^k{}_h - PQPP0^i{}_j{}^k{}_h$ two coefficients have shorter expressions. From the first term, which contains $g_{hj}g^{ik}$, by imposing the vanishing condition for the coefficient, we get

$$\lambda' = -\lambda\frac{a_1'(a_1^2 - 2ct - 2a_3^2ct) + 2a_1c(1 + a_3^2 + 2a_3a_3't)}{a_1(a_1^2 + 2ct + 2a_3^2ct)}, \tag{23}$$

If we substitute this expression in the second term (which contains $\delta_h^i\delta_j^k$) we have that λ is given by (22).

The value of λ' obtained by differentiating the relation (22), coincides with that obtained by replacing λ in (23). Using RICCI, we prove that all the components of the difference $K - K_0$ are zero, when the obtained values of λ', λ'' and λ''' are replaced in these components. The computation of some differences, such as $PQPP^i{}_j{}^k{}_h - PQPP0^i{}_j{}^k{}_h$,$PQPQ^i{}_j{}^{kh} - PQPQ0^i{}_j{}^{kh}$, $PQQP^i{}_{jkh} - PQQP0^i{}_{jkh}$, and $PQQQ^i{}_{jk}{}^h - PQQQ0^i{}_{jk}{}^h$ is quite hard, since after imposing the integrability conditions for the almost complex structure J, the expressions become very long, and the command TensorSimplify did not work on a PC with a RAM memory of 2GB. Thus I had to impose the integrability conditions in every coefficient appearing in the above differences, and to sum the expressions afterwards. □

Remark. If $a_3 = 0$ we obtain the condition for (T^*M, G, J) to have constant holomorphic sectional curvature in the case where G, J are natural lifts of diagonal type (see [18]).

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