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TENSOR EQUATIONS OF DISCRETE DYNAMICALLY DEFINED AND UNDEFINED SYSTEMS WITH HEREDITARY AND CREEP LIGHT ELEMENTS

BY

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Dedicated to Acad. Prof. Dr. Radu Miron at his 80th anniversary

Abstract. A survey as review of author's research results in area of dynamics of hybrid systems as well as a new construction of analytical dynamics of hereditary discrete systems (coauthored with O. A. Goroshko) are presented. In this paper we shall present basic structures of a series hybrid systems as well as tensor equations of discrete dynamically defined and undefined discrete systems with hereditary and creep light elements.

Equations of dynamics of a discrete hereditary system with finite number of the constraints and standard creep elements in covariant form are composed.

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1. Introduction. Analytical dynamics as general science of mechanical system motions was founded by Lagrange (Joseph Luis Lagrange (1735-1813)) in the period of his work at Berlin Academy of Sciences. The Lagrange's books "Mècanique Analytique" (see [5-7]) contain basic analytical methods of mechanics and was published in France in 1788. Introduced analytical methods in Mechanics by Lagrange are main and first base of analytical mechanics in general. Lagrange's equations of second kind and Lagrange's equations of first kind with unknown Lagrange's multipliplicators of constraints are main fundament of the Analytical Dynamics.

After first period of the Analytical Mechanics foundation, canonical equations obtained by Hamilton (Wiliam Rowan Hamilton (1805–1865)) was main advanced results of fundamental progress of analytical methods in mechanics.

In the Lagrange's opinion his equations first and second kind are universal and applicable to the all mechanical systems. In 1985, S.A. Chapligin analysis of the paper written by E. Landenkof follows to the conclusions that Lagrage's equations aren't applicable to the nonholonimic mechanical systems (see [5-7]). Than, S.A. Chapligin proposed beginning of the research new area of Analytical mechanics under the name Nonholonomic Mechanics – Mechanics of the nonholonomic mechanical systems.

Appearance and distribution of the new material for construction on the basis of synthetics with clear rheological properties are source and inspiration for develop of the new area of mechanics – *Mechanics of hereditary systems.* Mechanics of hereditary continuum is presented by series of fundamental publications and monograph and is applied for estimations of the construction build by new material. New, area – Analytical dynamics of the discrete hereditary system was proposed and opened in the monograph [1] by Gorošhko and Hedrih.

In the name "*hereditary*" properties of rheological body "*to remember*" history of loading is fully described for the both cases: for first case of the short time loading period when rheological body posses property to obtain quickly previous unloaded body form after unloaded and second case of the long time loading when rheological body posses property to obtain previous unloaded body form after unloaded for long period when there material property "*to remember*" is present as history of loading, then the name "hereditary elasticity" is corresponding name (see [1-23]).

The papers by Goroshko and Puchko (see [3-4]) are concerned with the multibodies systems in which the interaction between the bodies are described by the standard or weakly singular hereditary models. Starting from the general dynamic equation - the Lagrange's equations supplemented by the generalized relaxation are constructed. In the monograph (see ref. [1]) a class of dynamically determinate systems is separated for which the three forms of motion equations are presented: differential Lagrange's equations of the 3rd or higher order and two integro-differential forms of equations with rheological and relaxational kernels. The typical examples are given.

Varied constructive methods using exact Galerkin-type methods for a reduction of the systems with an infinite number of freedom degrees have been developed for an investigations in the dynamics of multibodies discrete hereditary systems and for going from a complex medium to the discrete models. In the present work the methods of the analytical dynamics are developing for the discrete mechanical systems with the hereditary interactions between their particles. Lagrange's equations forms are determined for these systems.

2. Standard light elements. Basic elements of discrete hereditary systems are (see [13], [14], [15], [21] and [26]).

1°. *Material particles* with mass m_ν, $\nu = 1, 2, 3, ..., N$ with each particle having three degrees of motion freedom, defined by the position vectors of which are $\vec{r}_\nu = y^i_\nu \vec{e}_i$, $i = 1, 2, 3$; $\nu = 1, 2, 3, ..., N$.

2°. *Light standard coupling hereditary element* of negligible mass in the form of axially stressed rod without bending, and which has the ability to resist deformation under static and dynamic conditions; Constitutive relation between rheological restitution force $\mathbf{P}$ and elongation y can be written down in the form $f_{psr}(\mathbf{P}, \dot{\mathbf{P}}, y, \dot{y}, D, J, n, c, \tilde{c}, \mu, c_\alpha, T, U, \ldots)$=0, where D and J are differential and integral operators (see [1-25]) which find their justification in experimental verifications of material behavior, while $n, c, \tilde{c}, \mu, c_\alpha \ldots$ are material constants, which are also determined experimentally.

For every single light standard coupling element of negligible mass, we shall define a specific law of dynamics. This means that we will define dynamics constitutive relation as determinants of rheological reaction forces and/or change of forces with distances and changes of distances in time, with accuracy up to constants which depend on the accuracy of their determination through experiment.

The accuracy of those constant in the laws and with them the constitutive equations of rheological reaction forces and elongations will depend not only on knowing the nature of object, but also on our having the knowledge necessary for dealing with very complex stress-strain relations (see [13], [14], [15], [21] and [26]). In this paper we shall use three such light standard constraint elements, and they will be:

1°. *Light standard ideally elastic element* for which the stress-strain relation for the restitution force as the function of element axial elongation is given by a linear relation of the form

$$\mathbf{P} = -cy, \tag{1}$$

where c is a rigidity coefficient or an elasticity coefficient (see [10-12], [16] and [25]). In natural, non-stressed force and deformation of such element are equal to zero.

2°. *Light standard hereditary element* (see [1-7], [9], [14-161] and [20-24]) for which the stress-strain relation for the rheological restitution force as the function of element elongation is given by a stress-strain constitutive relation:

2. $a*$ in *differential form*

$$D\mathbf{P} = Cy \ \ or \ \ n\dot{P}(t) + P(t) = ncy(t) + \tilde{c}y(t) \tag{2}$$

where, the following differential operators are introduced:

$$D = n\frac{d}{dt} + 1 \ \text{ and } \ C = nc\frac{d}{dt} + \tilde{c} \tag{3}$$

and n is a relaxation time and $c, \tilde{c}$ are rigidity coefficients – momentary and prolonged one.

2. $b*$ in *integral form*

$$P(t) = c\left[y(t) - \int_0^t R(t-\tau)\, y(\tau)\, d\tau\right], \tag{4}$$

where $R(t-\tau) = \frac{c-\tilde{c}}{nc} e^{-\frac{1}{n}(t-\tau)}$ is *relaxation kernel* (or resolvente).

2. $c*$ in *integral form*

$$y(t) = \frac{1}{c}\left[P(t) + \int_0^t K(t-\tau)\, P(\tau)\, d\tau\right], \tag{5}$$

where $K(t-\tau) = \frac{c-\tilde{c}}{nc} e^{-\frac{\tilde{c}}{nc}(t-\tau)}$ is *kernel of rheology* (or retardation).

3. *Light standard creep element* (see [13], [17-18] and [25-28]) for which the stress-strain relation for the restitution force as the function of element elongation is given by fractional order derivatives in the form

$$P(t) = -\left\{c_0 x(t) + c_\alpha D_t^\sigma [x(t)]\right\} \tag{6}$$

where $D_t^\sigma[\bullet]$ is operator of the σ^{th} derivative with respect to time t in the following form:

$$D_t^\sigma [x(t)] = \frac{d^\sigma x(t)}{dt^\alpha} = x^{(\varsigma)}(t) = \frac{1}{\Gamma(1-\sigma)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\sigma} d\tau \tag{7}$$

where c, c_σ are rigidity coefficients– momentary and prolonged one, and σ a rational number between 0 and 1, $0 < \sigma < 1$.

3. Equations of dynamics of a discrete systems with finite holonomic stationary constraints and standard light hereditary and creep elements. We investigate discrete dynamical system (see figures 1, 2 and 3) of N material particles with masses m_ν, ν=1, 2, 3, ..., N, the position vectors of which are $\vec{r}_\nu = y^i_\nu \vec{e}_i$, i=1, 2, 3; ν=1, 2, 3, ..., N. Material particles are constrained by S finite number of the holonomic stationary constraints:

$$\bar{f}_\mu \left(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N\right) = f_\mu \left(y^1, y^2, ..., y^{3N}\right) \quad \mu = 1, 2, 3, ..., S \tag{8}$$

and where we introduce the following notations: $y^k_\nu = : y^{3\nu-(3-k)}$, k=1, 2, 3; $m_{3\nu-k} = m_{3\nu}$, k=1, 2, 3, ν=1, 2, 3, . . ., N; as well as by K standard hereditary elements neglected mass and material properties parameters of which are: $n_{(\nu,\nu+1)k}$, k=1, 2, 3, . . ., K, are times of relaxation, and $c_{(\nu,\nu+1)k}$ and $\tilde{c}_{(\nu,\nu+1)k}$. are an instantaneous rigid stiffness modulus as prolonged ones; and as well as by C standard creep elements neglected mass and material properties parameters of which are: $n_{(\nu,\nu+1)k}$, k=1, 2, 3, . . ., K, are proper material constants of the characteristic creep law of creep elements materials, $E_{0(\nu,\nu+1)c}$, c=1, 2, 3, . . ., C_ν and $E_{\sigma(\nu,\nu+1)c}$, c=1, 2, 3, . . ., C_ν are modulus of elasticity and creeping properties of standard creep elements. Relations between reactions and deformations of the hereditary light element in the discrete system can be defined in the relaxational forms by using integral stress strain state relations (4) and (5), and

$$\begin{aligned} \rho_{(\nu,\nu+1)k} &= \left|\vec{\rho}_{(\nu,\nu+1)k}\right| = \left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|, \\ \bar{\rho}_{(\nu,\nu+1)k} &= \left|\vec{\rho}_{(\nu,\nu+1)k}\right| - \rho_{(\nu,\nu+1)k0} = \left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right| - \rho_{(\nu,\nu+1)k0} \end{aligned} \tag{9}$$

and $\rho_{(\nu,\nu+1)k0}$ is natural length of a hereditary element in natural stress-strain state, when the strain and stress in the element are equal to zero.

Relations between reactions and deformations of the creep light element in the discrete system (see [18]) can be defined in the form by using stress-strain constitutive relations (6) and (7) expressed by fractional order derivative. By using velocity conditions we can write ortogonality conditions for the corresponding material particle velocity in the form: $(grad_\nu\, f_\mu, \vec{v}^\nu) = 0$, $\nu = 1, 2, 3, ..., N$, $\mu = 1, 2, 3, \ldots, S$ as an orthogonality between material particle velocities and gradients of the finite constraints. For ideal constraints reactions we can write the following:

$$\overrightarrow{R}_\nu = \sum_{\mu=1}^{\mu=S} \lambda_\mu grad_\nu\, f_\mu \left(\vec{r}_1, ..., \vec{r}_N\right), \quad \nu = 1, 2, 3, ..., N \tag{10}$$

in which the λ_μ, $\mu = 1, 2, 3, \dots, S$ are Lagrange's multiplicators of the S finite number of the holonomic stationary constraints.

The resulting reactions of the K standard hereditary elements into ν-rd ($\nu+1$ with opposite direction) mass material particle is:

$$\begin{aligned} \overrightarrow{P}_{(\nu,\nu+j)K}(t) &= \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+j)k}(t) \frac{\vec{\rho}_{(\nu,\nu+j)k}}{\left|\vec{\rho}_{(\nu,\nu+j)k}\right|} \\ &= \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+j)k}(t) \frac{\vec{r}_{(\nu+j)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+j)k} - \vec{r}_{(\nu)k}\right|}. \end{aligned} \tag{11}$$

The resulting reactions of the K standard creep elements into ν-rd ($\nu+1$ with opposite direction) mass material particle is:

$$\begin{aligned} \overrightarrow{P}_{(\nu,\nu+j)C}(t) &= \sum_{c=1}^{c=C_\nu} P_{(\nu,\nu+j)c}(t) \frac{\vec{\rho}_{(\nu,\nu+j)c}}{\left|\vec{\rho}_{(\nu,\nu+j)c}\right|} \\ &= \sum_{c=1}^{c=C_\nu} P_{(\nu,\nu+j)c}(t) \frac{\vec{r}_{(\nu+j)c} - \vec{r}_{(\nu)c}}{\left|\vec{r}_{(\nu+j)c} - \vec{r}_{(\nu)c}\right|}. \end{aligned} \tag{12}$$

Resulting reaction forces of finite constraints, standard hereditary and creep light elements applied to the ν-th material particle, in the observed system are:

$$\begin{aligned} \overrightarrow{R}_\nu = & \sum_{\mu=1}^{\mu=S} \lambda_\mu grad_\nu f_\mu(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \\ & + \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+1)k}(t) \frac{\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+1)k} - \vec{r}_{(\nu)k}\right|} + \overrightarrow{R}_{\nu\mathbf{T}} \\ & + \sum_{j=1}^{j=N} \left\{ \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+j)k}(t) \frac{\vec{r}_{(\nu+j)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+j)k} - \vec{r}_{(\nu)k}\right|} \right. \\ & \left. + \sum_{c=1}^{c=C_\nu} P_{(\nu,\nu+j)c}(t) \frac{\vec{r}_{(\nu+j)c} - \vec{r}_{(\nu)c}}{\left|\vec{r}_{(\nu+j)c} - \vec{r}_{(\nu)c}\right|} \right\}. \end{aligned} \tag{13}$$

From a principle of the work on the virtual system displacements can be written in the following form:

$$\sum_{\nu=1}^{\nu=N} \{\vec{I}_\nu + \vec{F}_\nu + \vec{R}_\nu + \vec{P}_\nu^H + \vec{P}_\nu^C + \vec{R}_{\nu T}\} \cdot \delta\vec{r}_\nu = 0. \tag{14}$$

Lagrange's equations first kind of the discrete system dynamic arise from previous equation in the following form

$$
\begin{aligned}
m_\nu \ddot{\vec{r}}_\nu &= \vec{F}_\nu(t) + \sum_{\mu=1}^{\mu=S} \lambda_\mu grad_\nu f_\mu(\vec{r}_1,...,\vec{r}_N) + \vec{R}_{\nu T} \\
&+ \sum_{j=1}^{N} \left[\sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+j)k}(t) \frac{\vec{r}_{(\nu+j)k} - \vec{r}_{(\nu)k}}{\left|\vec{r}_{(\nu+j)k} - \vec{r}_{(\nu)k}\right|} \right. \\
&\left. + \sum_{c=1}^{c=C_\nu} P_{(\nu,\nu+j)c}(t) \frac{\vec{r}_{(\nu+j)c} - \vec{r}_{(\nu)c}}{\left|\vec{r}_{(\nu+j)c} - \vec{r}_{(\nu)c}\right|} \right], \quad \nu = 1,2,3,...,N,
\end{aligned} \tag{15}
$$

where rheological reaction forces defined by constitutive stress strain relations are in the following forms:

* differential

$$
P_{(\nu,\nu+j)k} + n\dot{P}_{(\nu,\nu+j)k} = n_k c_k \dot{\rho}_{(\nu,\nu+j)k}(q) + \tilde{c}\rho_{(\nu,\nu+j)k}(q) \tag{16}
$$

* integral

$$
\begin{aligned}
P_{(\nu,\nu+j)k}(t) &= -c_{(\nu,\nu+j)k}[\rho_{(\nu,\nu+j)k}(t) \\
&\quad - \int_{-\infty}^{t} R_{(\nu,\nu+j)k}(t-\tau)\,\rho_{(\nu,\nu+j)k}(\tau)\,d\tau]
\end{aligned} \tag{17}
$$

where $R(t-\tau) = \frac{c-\tilde{c}}{nc} e^{-\frac{1}{n}(t-\tau)}$ is *relaxation kernels* (or resolvents).

Fractional order constitutive relation for standard light creep element

$$
P_{(\nu,\nu+j)k}(t) = -\left\{ c_{0(\nu,\nu+j)k}\rho_{(\nu,\nu+j)k}(t) + c_{\sigma(\nu,\nu+j)k} D_t^\sigma \left[\rho_{(\nu,\nu+j)k}(t)\right] \right\} \tag{18}
$$

where $D_t^\sigma[\bullet]$ is operator of the α^{th} derivative with respect to time t in the following form:

$$
\begin{aligned}
D_t^\sigma\left[\rho_{(\nu,\nu+j)k}(t)\right] &= \frac{d^\sigma \rho_{(\nu,\nu+j)k}(t)}{dt^\alpha} = \rho^{(\sigma)}_{(\nu,\nu+j)k}(t) \\
&= \frac{1}{\Gamma(1-\sigma)} \frac{d}{dt} \int_0^t \frac{\rho_{(\nu,\nu+j)k}(\tau)}{(t-\tau)^\sigma} d\tau,
\end{aligned} \tag{19}
$$

where $c_{0(\nu,\nu+j)k}, c_{\sigma(\nu,\nu+j)k}$ are rigidity coefficients– momentary and prolonged one, and σ a rational number between 0 and 1, $0 < \sigma < 1$.

4. Dynamically defined hereditary discrete systems. From the all previous considered theory of the hereditary discrete systems and numerous examples as well as presented in the main key points hereditary discrete systems it must to separate two groups of the hereditary discrete systems on the basis of the possibilities to solve governing equations (15) with respect to the rheological reactions P_k. Solvability of the governing equations (15) with respect to the rheological reactions P_k firstly it is studied by GOROSHKO and his coauthors (see [3] and [4]) and is possible under the following two conditions:

1. Number $K$of the rheological hereditary elements must be less or equal to the number n of the degrees of the hereditary discrete system freedom, $K \leq n$;

2. Structure of the mechanical hereditary discrete system must be like that, that there are possibility of the chouses of the generalized coordinate that is possible to obtain inequities with zero defined by $|b_{kj}(q)| = |\sum_{i=1}^{3N} e_{ik}(q)\frac{\partial x_i}{\partial q_j}| \neq 0$ for $K \leq n$.

These conditions are generalization of the known conditions of the static defined mechanical system, applied large for the solving problems in the strength of materials.

5. Covariant tensor basic equations of the discrete hereditary systems. By using Principle of the work on the virtual displacement in the previous chapter we obtain system basic equations (14) and (15) with corresponding constitutive equations (stress-strain relations) described by (16), (17), (20), (18) and (19).

Now, the virtual displacement can be expressed by using generalized coordinates q^α, $\alpha = 1, 2, 3, \ldots, n(= 3N - s)$ in the form: $\delta\vec{r}_\nu = \sum_{\alpha=1}^{\alpha=n} \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\delta q^\alpha$ and introduced into the previous vector equation (14) for Principle of the work of the active and reactive forces on the virtual displacements, and we obtained the following:

$$\sum_{\alpha=1}^{\alpha=n} [I_\alpha + Q_\alpha + P_\alpha + Q^*_\alpha]\,\delta q^\alpha = 0. \tag{20}$$

By analyzing the members from previous expression we have the following fictive, active and reactive forces:

$$I_\alpha = -\sum_{\nu=1}^{\nu=N} m_\nu \left(\ddot{\vec{r}}_\nu, \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right) = -\sum_{\nu=1}^{\nu=N} m_\nu \left(\frac{d\vec{v}_\nu}{dt}, \frac{\partial\vec{r}_\nu}{\partial q^\alpha}\right)$$

$$= \quad -\sum_{\nu=1}^{\nu=N} m_\nu \left(\frac{d}{dt} \sum_{\beta=0}^{\beta=n} \frac{\partial \vec{r}_\nu}{\partial q^\beta} \dot{q}^\beta, \frac{\partial \vec{r}_\nu}{\partial q^\alpha} \right)$$

$$(21) \qquad = \quad -\sum_{\nu=1}^{\nu=N} m_\nu \left(\sum_{\gamma=1}^{\gamma=n} \sum_{\beta=1}^{\beta=n} \frac{\partial^2 \vec{r}_\nu}{\partial q^\beta \partial q^\gamma} \dot{q}^\beta \dot{q}^\gamma + \sum_{\beta=1}^{\beta=n} \frac{\partial \vec{r}_\nu}{\partial q^\beta} \ddot{q}^\beta, \frac{\partial \vec{r}_\nu}{\partial q^\alpha} \right)$$

$$= \quad -[a_{\alpha\beta}(\ddot{q}^\beta + \Gamma^\alpha_{\gamma\delta} \dot{q}^\gamma \dot{q}^\delta)] = -a_{\alpha\beta} \frac{D\dot{q}^\beta}{dt},$$

$$\alpha = 1, 2, 3, ..., n; n = 3N - s$$

$$(22) \qquad Q_\alpha = \sum_{\nu=1}^{\nu=N} \left(\vec{F}_\nu\,(t)\,, \frac{\partial \vec{r}_\nu}{\partial q^\alpha} \right)$$

$$(23) \qquad Q^f_\alpha = \sum_{\nu=1}^{\nu=N} \sum_{\mu=1}^{\mu=S} \lambda_\mu \left(grad_\nu\, f_\mu\,(\vec{r}_1, ..., \vec{r}_N)\,, \frac{\partial \vec{r}_\nu}{\partial q^\alpha} \right) = 0$$

$$(24) \qquad \mathbf{P}^H_\alpha = \sum_{\nu=1}^{\nu=N} \left\{ \sum_{j=1}^{j=N} \sum_{k=1}^{k=K_\nu} P_{(\nu,\nu+j)k}\,(t)\, \frac{(\vec{\rho}_{(\nu,\nu+j)k}, \frac{\partial \vec{r}_\nu}{\partial q^\alpha})}{|\vec{\rho}_{(\nu,\nu+j)k}|} \right\}$$

$$(25) \qquad \mathbf{P}^C_\alpha = \sum_{\nu=1}^{\nu=N} \left\{ \sum_{j=1}^{j=N} \sum_{c=1}^{c=C_\nu} P_{(\nu,\nu+j)c}\,(t)\, \frac{(\vec{\rho}_{(\nu,\nu+j)k}, \frac{\partial \vec{r}_\nu}{\partial q^\alpha})}{|\vec{\rho}_{(\nu,\nu+j)k}|} \right\}$$

$$(26) \qquad Q^*_\alpha = \sum_{\nu=1}^{\nu=N} \left(\vec{R}_{\nu T}\,(t)\,, \frac{\partial \vec{r}_\nu}{\partial q^\alpha} \right).$$

System of Lagrange's equations of the dynamics of discrete hereditary system with standard light hereditary and creep elements and properties and with n degrees of freedom in the covariant coordinates can write in following form:

$$(27) \qquad a_{\alpha\beta} \frac{D\dot{q}^\beta}{dt} = Q_\alpha + Q^*_\alpha + P_\alpha \quad \alpha = 1, 2, 3, ..., n; \quad n = 3N - S$$

And taking into account additional system of the constitutive rheological reactions of the hereditary and creep elements in the forms described by (16), (17), (20), (18) and (19) through expressions of the corresponding generalized rheological reactive forces (24) and (25).

For next step in consideration and for solving dynamics of the discrete hereditary system by using previous system (27), with corresponding additional and necessary equations, there are three possibilities. These three possibilities for obtaining these equations are: in two integro-differential forms and also in differential forms. Derivations of these form are not possible in every case. Reason is that there are possibilities depend of the discrete hereditary system definiteness or no-definiteness. This is connected with relations between number of the degrees of freedom and number of the discrete standard light hereditary elements in the discrete hereditary system, and also of the types of the coupling with material particles and massless nodes un the discrete hereditary system.

5.1. Integro-differential form of the tensor equations. System of Lagrange's equations of the dynamics of discrete hereditary system with standard light hereditary and creep elements and properties and with n degrees of freedom in the covariant coordinates can write in the form (27) and taking into account additional system of the constitutive rheological reactions of the hereditary and creep elements in the forms described by (16), (17), (20), (18) and (19) through expressions of the corresponding generalized rheological reactive forces (24) and (25) we can write for:

a* corresponding rheological reaction forces of the standard light hereditary elements in the form:

$$\mathbf{P}_{\alpha}^{H} = -\sum_{\nu=1}^{\nu=N}\left\{\sum_{j=1}^{j=N}\sum_{k=1}^{k=K_\nu} c_{(\nu,\nu+j)k} b_{(\nu,\nu+j)k,\alpha}[\rho_{(\nu,\nu+j)k}(t) - \int_{-\infty}^{t} R_{(\nu,\nu+j)k}(t-\tau)\,\rho_{(\nu,\nu+j)k}(\tau)\,d\tau]\right\} \tag{28}$$

where $\rho_{(\nu,\nu+j)k}(t)$ are the rheological coordinates:

$$\rho_{(\nu,\nu+j)k}(t) = |\vec{r}_{\nu+1}(q^0(t), q^1(t), \dots, q^\beta(t), ., q^n(t)) - \vec{r}_{\nu}(q^0(t), q^1(t), \dots, q^\beta(t), ., q^n(t))|_k \tag{29}$$

and

$$b_{(\nu,\nu+j)k,\alpha} = \frac{(\vec{\rho}_{(\nu,\nu+j)k}, \frac{\partial \vec{r}_\nu}{\partial q^\alpha})}{|\vec{\rho}_{(\nu,\nu+j)k}|} \tag{30}$$

b* corresponding rheological reaction forces of the standard light creep elements in the form:

$$\begin{aligned} \mathbf{P}_\alpha^C \quad = \quad & -\sum_{\nu=1}^{\nu=N} \left\{ \sum_{j=1}^{j=N} \sum_{k=1}^{k=K_\nu} \left\{ c_{0(\nu,\nu+j)k} \rho_{(\nu,\nu+j)k}(t) \right. \right. \\ & \left. \left. + c_{\sigma(\nu,\nu+j)k} D_t^\sigma \left[\rho_{(\nu,\nu+j)k}(t) \right] \right\} b_{(\nu,\nu+j)k,\alpha} \right\} \end{aligned} \tag{31}$$

where $D_t^\sigma[\bullet]$ is operator of the σ^{th} derivative with respect to time $t$in the form (19).

By eliminating of the rheological reaction forces of the standard light hereditary elements from basic equations obtained by principle of dynamical equilibrium or principle of work (27), and corresponding constitutive equations (relations) of the stress-strain state in the standard light hereditary elements (22) and (28), is possible to obtain extended (modification of the) Lagrange's equations second kind. In the tensor covariant form for discrete hereditary and creep fractional order system these integro-fractional order differential equations, expressed by generalized coordinates, are in the following forms:

$$\begin{aligned} & a_{\alpha\beta} \frac{D\dot{q}^\beta}{dt} + \sum_{\nu=1}^{\nu=N} \left\{ \sum_{j=1}^{j=N} \sum_{k=1}^{k=K_\nu} b_{(\nu,\nu+j)k,\alpha} (c_{(\nu,\nu+j)k} \right. \\ & \left. + c_{0(\nu,\nu+j)k}) \rho_{(\nu,\nu+j)k}(t) \right\} = Q_\alpha + Q_\alpha^* \\ & + \sum_{\nu=1}^{\nu=N} \left\{ \sum_{j=1}^{j=N} \sum_{k=1}^{k=K_\nu} b_{(\nu,\nu+j)k,\alpha} c_{(\nu,\nu+j)k} \cdot \right. \\ & \left. \cdot \int_{-\infty}^{t} R_{(\nu,\nu+j)k}(t-\tau) \rho_{(\nu,\nu+j)k}(\tau) d\tau \right\} \\ & - \sum_{\nu=1}^{\nu=N} \left\{ \sum_{j=1}^{j=N} \sum_{k=1}^{k=K_\nu} b_{(\nu,\nu+j)k,\alpha} c_{\sigma(\nu,\nu+j)k} D_t^\sigma [\rho_{(\nu,\nu+j)k}(t)] \right\}. \end{aligned} \tag{32}$$

It is easy visible that for arbitrary number $K$of the standard light hereditary elements sample coupled between material particles, is very easy to obtain Lagrange's equations second kind in covariant, integro-differential and expressed in the previous form (32) if rheological coordinates are expressed by generalized coordinates. In this case we obtained system of the n integrodifferential equations containing only n generalized coordinates q^{α}, $\alpha = 1, 2, 3, \dots, n (= 3N - s)$ and this system is now independent of the other additional equations as well as of the rheological reactive forces. In this case is possible to fined solutions along generalized coordinates, only from this system of the integro-fractional order differential equations. In this case we have *dynamically determined (or defined) discrete hereditary and creep fractional order system.*

If these integro-fractional order differential equations contain rheological coordinate which is not possible expressed by generalized coordinates of the hereditary and creep fractional order discrete system, then we have dynamically *undefined (or undetermined) discrete hereditary and creep fractional order system.* In this case it is *necessary introduce a number of internal coordinates as fictive generalized coordinates* depending of the number of *the system internal degrees of the freedom* then these n integro-fractional order differential equations are not enough for solving problem and must be add corresponding number same type integro-fractional order differential equations with respect to the fictive generalized coordinates in accordance with system internal degree of freedom of the discrete hereditary and creep-fractional order system.

Previous system of the integro-fractional order differential equations (31) are *universal* and it is possible to obtain these equations for arbitrary numbers K of the standard light hereditary elements and corresponding number of the standard creep fractional order elements simple coupled with material particles in the considered system.

6. Consideration of the dynamically defined and undefined discrete hereditary systems. Number of rheological coordinates depend of the number of standard light hereditary elements. And, it is possible that number of these rheological coordinates is same, less or more then number of the generalized coordinates. Also,the number of the these standard light hereditary elements as well as creep elements can be less, or more them numbers of the degrees of the freedom of the discrete hereditary system.

Standard light hereditary elements as well as standard light elastic or fractional order elements are not reason for increasing number of the degrees of freedom of the discrete hereditary system, but is a main reason for possible increase number of the internal degrees of the freedom in the discrete hereditary system depending of the internal coupling between material particles and between standard light hereditary element with massless nodes between them.

For obtaining Lagrange's equations for discrete hereditary defined system dynamics in differential form, it is necessary to eliminate rheological reaction forces from basic equations (27) obtained by principle of dynamical equilibrium or principle of the work, and corresponding constitutive equations (relations) (16)–(19) of the stress-strain state in the standard light hereditary elements in the differential form. It is possible to obtain extended (modification of the) Lagrange's equations second kind only in a few cases. In the tensor covariant form for discrete hereditary system expressed by generalized coordinates, these differential equations are not possible to obtain in every case.

Let us to consider three possible cases.

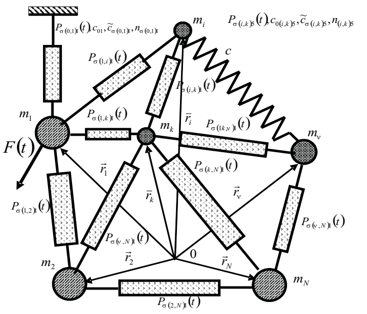

Figure 1: Model of the dynamically defined discrete hereditary systems (in plane) with finite number of the degree of freedom and number of the standard light hereditary elements equal or less them degrees of freedom $K = \sum_{v=1}^{N} K_v \leq n = 3N - s$

First case. Dynamically defined (determined) discrete hereditary system (see figure 1.) If there are number of the standard light hereditary elements less them number of the degree of freedom or equal $K = \sum_{\nu=1}^{N} K_\nu \leq n = 3N - s$, and every standard light hereditary element is connected between two material particle with corresponding rheological coordinate between two material particle, then it is possible to solve previous system of the equations of the system motion. It is possible to solve with respect to the all $K = \sum_{\nu=1}^{N} K_\nu \leq n = 3N - s$ rheological reactive forces of the hereditary standard light elements and then in these equations introduce constitutive relations of the every standard light element and eliminate all rheological reactive forces induced in the system by standard light elements and obtain system of the differential equations third order with respect to the generalized coordinates, followed by $3n$ initial conditions, correspond to the n generalized coordinates at the initial moment. Then we have discrete hereditary system *dynamically defined.* (determined) From $3n$ initial conditions, correspond the generalized coordinate, generalized velocity and generalized acceleration is possible to obtain $3n$ integral unsown constants from solutions for the generalized coordinates of the hereditary system dynamics.

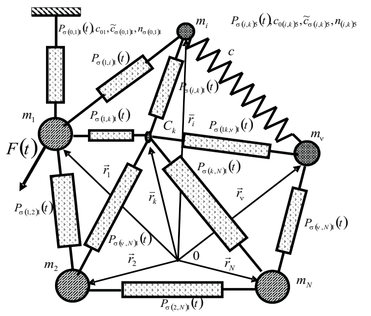

Figure 2: Model of the dynamically undefined discrete hereditary systems (in plane) with finite number of the degree of freedom and number of the standard light hereditary elements equal or less them degrees of freedom $K = \sum_{v=1}^{N} K_v \leq n = 3N - s$

Second case. Dynamically no (un) defined (no determined) discrete hereditary system (see figure 2). If there are number $K = \sum_{\nu=1}^{N} K_\nu \leq n = 3N - s$, of the standard light hereditary elements, less them number of the degrees of freedom or equal, and that every standard light hereditary element is not connected between two material particle with corresponding rheological coordinate between two material particle, but with mass less node, one or more, then it is not possible to solve previous system of the equations of the system motion with respect to the all $K = \sum_{\nu=1}^{N} K_\nu \leq n = 3N - s$ rhelogical reactive forces of the hereditary standard light elements and then in these equations introduce constitutive relations of the every standard light element and eliminate all rheological reactive forces induced in the system by standard light elements. In this case it not possible to obtain system of the differential equations third orders only with respect to the generalized coordinates, because these equations contain additional coordinates of the dynamically undefined (undetermined) system followed by initial conditions $3n$ correspond to the n generalized coordinates at the initial moment and also by additional initial conditions of the additionally introduced internal degree of freedom coordinates. Necessary for slowing this task.

For expressing equations of motion of the dynamically undetermined (undefined) discrete hereditary system only by generalized coordinates, and eliminating rheonomic reactive forces it is necessary to introduce additional number of corresponding internal generalized coordinates depending of internal system degrees of freedom in accordance with number of the dynamic undefiniteness of the discrete hereditary system. For that reason it is necessary to introduce the corresponding numbers of the fictive material points with masses equal zero. These fictive material points must to put in the nodes of the standard light elements intercoupling. Number of introduced fictive material particles, and corresponding additional number of the additional generalized coordinates depend of the properties of these interconnection (internal coupling) between standard light hereditary elements, as well as of the internal degrees of hereditary discrete system freedom as a properties of the dynamical undefinitness.

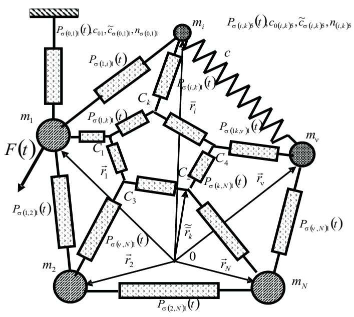

Figure 3: Model of the dynamically undefined discrete hereditary systems (in plane) with finite number of the degree of freedom and number of the standard light hereditary elements large them degrees of freedom $K = \sum_{v=1}^{N} K_v > n = 3N - s$

Third case. Dynamically no (un) defined (no determined) discrete hereditary system (see figure 3). For the case that the number K of the standard light hereditary elements more them number $n = 3N - s$ of the degree of freedom $K = \sum_{\nu=1}^{N} K_\nu > n = 3N - s$ then it is not possible to solve system of the equations of the system motion with respect to the to the all $K = \sum_{\nu=1}^{N} K_\nu > n = 3N - s$ rheological reactive forces of the hereditary standard light elements, but only with respect to the $n = 3N - s$ rheological reactive forces of the hereditary standard light elements in arbitrary chooses (variant) in the functions of the $n = 3N - s$ generalized coordinates and other $K - n = \sum_{\nu=1}^{N} K_\nu - n = \sum_{\nu=1}^{N} K_\nu - (3N - s)$ rheological reactive forces of the hereditary standard light elements an to obtain two subsystems of the motion of the undefined (undetermined) discrete hereditary system. For expressing equations of motion of the dynamically undetermined discrete hereditary system only by generalized coordinates, and eliminating rheonomic reactive forces it is necessary to introduce additional number of generalized coordinates depending of internal system degrees of freedom in accordance with number of the dynamic undefiniteness of the discrete hereditary system. By introducing fictive material points with masses equal zero and also corresponding additional initial conditions.

7. Concluding remarks. In this paper we derive the covariant integro-differential equations fractional order derivatives of the discrete creep-hereditary systems with creep and hereditary standard light elements between mass particles. In this paper we introduce standard creep light element as a new possibility of the realization of real creping coupling between bodies in the multi body system.

Tensor equations of dynamics of a discrete hereditary dynamically defined (determined) and undefined (undetermined) systems with finite numbers of the constraints and standard light hereditary and creep elements are composed. Covariant integro-differential equations of fractional order of the motion of the discrete creep-hereditary system are composed and discussed in the light of the dynamically determined and undetermined discrete heresitary system.

On the basis of the analysis of the discrete creep-hereditary oscillatory systems the Goroshko's definition on dynamically determinate or indeterminate discrete hereditary systems was confirmed.

For next step in consideration and for solving dynamics of the discrete hereditary system by using previous system (27) of the covariant equations of the system dynamics, with corresponding additional unnecessary constitutive relations and equations, three possibilities and different cases are identified. These three possibilities for obtaining these equations are: in two integro-differential forms and also in differential form. Derivations of these form are not possible in every case. Reason is that there are possibilities depend of the discrete hereditary system definiteness or no-definiteness. This is connected with relations between number of the degrees of freedom and number of the discrete standard light hereditary elements in the discrete hereditary system, and also of the types of the coupling with material particles and massless nodes un the discrete hereditary system.

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