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A DECOMPOSITION OF THE BUNDLE OF SECOND ORDER JETS ON A COMPLEX MANIFOLD

BY

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Abstract. We define the $(2,0)$, $(1,1)$ and $(0,2)$-jets on a complex manifold M. We prove that the fiber bundle of 2-jets on M is a direct sum of the bundles of the above types of jets. We describe the 1-dimensional Dolbeault cohomology group in terms of fields of $(0,2)$-jets.

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1. Introduction. The study of jets on a manifold is an important task for the geometers and the mathematical physicists. An introduction to the language of jet bundles could be found in [1], [8]. The domain of complex manifolds is also an important subject of research, [3], [7]. The jets on a complex manifold are useful tools in study of its submanifolds, for example, [2], [6]. They could be related with the cohomology of that manifold and with its Chern classes, [9]. The holomorphic jets are mainly considered but there are also another types of jets which could be interesting.

Let M be a n-dimensional complex manifold. We consider that the indices k, l, k', l'... take the values $1, ..., n$. Such a manifold carries a real $2n$-dimensional differentiable structures and it is well-known the decomposition of the complexified tangent space

$$T_{\mathbf{C}}M = T^{1,0}M \oplus T^{0,1}M, \tag{1.1}$$

where $T^{1,0}M$ is an holomorphic vector bundle over M, namely the *holomorphic tangent bundle* of M, while the second term is a vector bundle over M, but not a holomorphic one.

If M is considered as a real manifold, its second order jet manifold $J^2(M)$ is a fiber bundle over M, [8]. The above decomposition of $T_{\mathbf{C}}M$ suggests us to looking for a decomposition of the complexified space of $J^2(M)$. The main result of this paper is

$$J^2_{\mathbf{C}}(M) = J^{2,0}(M) \oplus J^{1,1}(M) \oplus J^{0,2}(M), \tag{1.2}$$

where the terms are fiber bundles over the complex manifold M, the first one being an holomorphic bundle which contains the holomorphic second order jets on M (holomorphic 2-jets on M) in sense of [9].

2. Second order jets on a complex manifold. Let M be a n-dimensional complex manifold with local complex coordinate system $(z^1, ..., z^n)$. The corresponding real coordinate system is $(x^1, y^1, x^2, y^2, ..., x^n, y^n)$, where $z^k = x^k + i \cdot y^k$, $i^2 = -1$. There are well-known relations

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), \quad \frac{\partial}{\partial \overline{z^k}} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right), \tag{2.1}$$

where $\overline{z}$ is the complex conjugate of z.

Let $\Omega^0(M)$ be the set of complex functions on M whose real and imaginary parts are differentiable real functions. Then, for every $f \in \Omega^0(M)$, $f = u + i \cdot v$, we have

$$\begin{aligned} \frac{\partial f}{\partial z^k} &= \frac{1}{2}\left[\frac{\partial u}{\partial x^k} + \frac{\partial v}{\partial y^k} + i\left(\frac{\partial v}{\partial x^k} - \frac{\partial u}{\partial y^k}\right)\right], \\ \frac{\partial f}{\partial \overline{z^k}} &= \frac{1}{2}\left[\frac{\partial u}{\partial x^k} - \frac{\partial v}{\partial y^k} + i\left(\frac{\partial v}{\partial x^k} + \frac{\partial u}{\partial y^k}\right)\right]. \end{aligned} \tag{2.2}$$

A function $f \in \Omega^0(M)$ is called *holomorphic* if $\frac{\partial f}{\partial \overline{z^k}} = 0$.

If $(U, (z^k))$, $(\tilde{U}, (\tilde{z}^k))$ are two local charts whose domains overlap, the transition functions

$$\tilde{z}^k = \tilde{z}^k(z^1, ..., z^n), \tag{2.3}$$

are holomorphic ones. As a consequence, the conjugates of the coordinates functions $\tilde{z}^k$ depend only on $\overline{z^1}, ..., \overline{z^n}$, so

$$\frac{\partial \overline{\tilde{z}^k}}{\partial z^l} = 0. \tag{2.4}$$

From [8], two real maps $u, u' : M \to \mathbf{R}$ determine the same second order (real) jet (or 2-jet) at $x \in M$ if $u(x) = u'(x) = 0$ and in a local chart at x, they have the same derivative of first and second order at x. This relation is an equivalence one and the equivalence class of u with respect with this relation is called the *2-jet of u at x*. We remark that the notion of 2-jet of curves on a manifold is the start point for the geometry of higher order, [5].

In a local chart $(U, (x^1, y^1, ..., x^n, y^n))$ around x, the 2-jet of u at x has the expression:

$$\begin{aligned} j^2_x u \;=\; & \frac{\partial u}{\partial x^k}(x) j^2_x x^k + \frac{\partial u}{\partial y^k}(x) j^2_x y^k + \frac{1}{2}\frac{\partial^2 u}{\partial x^k \partial x^l}(x) j^2_x x^k \cdot j^2_x x^l \\ & + \frac{\partial^2 u}{\partial x^k \partial y^l}(x) j^2_x x^k \cdot j^2_x y^l + \frac{1}{2}\frac{\partial^2 u}{\partial y^k \partial y^l}(x) j^2_x y^k \cdot j^2_x y^l. \end{aligned} \tag{2.5}$$

The real 2-jets $\{j^2_x x^k, j^2_x y^k, j^2_x x^k \cdot j^2_x x^l, j^2_x x^k \cdot j^2_x y^l, j^2_x y^k \cdot j^2_x y^l\}$ form a basis in the vectorial space of real 2-jets on M at x. Generally, by a real 2-jet at x we shall mean every expression locally given by a linear combination of the elements from the above base, where the coefficients change at local chart changing like the coefficients from (2.5). The set of all real 2-jets at x is denoted by $J^2_x(M)$.

The set $J^2(M) = \cup_{x \in M} J^2_x(M)$ is a fiber bundle over M, called the *real 2-jet bundle of M*. The sections of this bundle are called *real fields of 2-jets on M* and their set is denoted by J^2M. The field of 2-jet of u, denoted j^2u, has the locally expression (2.5) where we delete x.

The local expression of a real field of 2-jets on M is

$$\begin{aligned} \alpha^2_{\mathbf{R}} \;=\; & a_k \cdot j^2 x^k + b_k j^2 y^k + \frac{1}{2} a_{kl} \cdot j^2 x^k \cdot j^2 x^l \\ & + c_{kl} \cdot j^2 x^k \cdot j^2 y^l + \frac{1}{2} b_{kl} \cdot j^2 y^k \cdot j^2 y^l, \end{aligned} \tag{2.6}$$

where $a_k, b_k, a_{kl}, b_{kl}, c_{kl}$ are differentiable functions on U which satisfy $a_{kl} = a_{lk}, b_{kl} = b_{lk}$, (not necessarily $c_{kl} = c_{lk}$) and if $(U, (z^k))$, $(\tilde{U}, (\tilde{z}^k))$ are two local charts which domains overlap, in $U \cap \tilde{U}$ this functions verify the same equations as the functions $\frac{\partial u}{\partial x^k}, \frac{\partial u}{\partial y^k}, \frac{\partial^2 u}{\partial x^k \partial x^l}, \frac{\partial^2 u}{\partial x^k \partial y^l}, \frac{\partial^2 u}{\partial y^k \partial y^l}$.

Definition 2.1. *We say that two complex functions $f, g \in \Omega^0(M)$ determine the same 2-jet at a fixed point x if their real and imaginary parts determine the same real 2-jets at x, respectively. For $f = u + i \cdot v$, we define*

$$j^2_x f = j^2_x u + i \cdot j^2_x v. \tag{2.7}$$

Proposition 2.1. *For $f, g \in \Omega^0(M)$ the following assertions are equivalent:*

a) $j^2_x f = j^2_x g$*;*

b) $f(x) = g(x) = 0$ *and in a local complex chart*

$$\frac{\partial f}{\partial z^k}(x) = \frac{\partial g}{\partial z^k}(x), \frac{\partial f}{\partial \overline{z^k}}(x) = \frac{\partial g}{\partial \overline{z^k}}(x), \frac{\partial^2 f}{\partial z^k \partial z^l}(x) = \frac{\partial^2 g}{\partial z^k \partial z^l}(x),$$

$$\frac{\partial^2 f}{\partial z^k \partial \overline{z^l}}(x) = \frac{\partial^2 g}{\partial z^k \partial \overline{z^l}}(x), \frac{\partial^2 f}{\partial \overline{z^k} \partial \overline{z^l}}(x) = \frac{\partial^2 g}{\partial \overline{z^k} \partial \overline{z^l}}(x),$$

for all $k, l = 1, ..., n$.

Proof. By a straightforward calculation, using relations (2.1) and (2.2), we obtain the result. Moreover, the conditions from b) have geometrical meaning. Indeed, if $(U, (z^k))$, $(\tilde{U}, (\tilde{z}^k))$ are two local charts which domains overlap, taking into account that the coordinate functions $z^k = z^k(\tilde{z}^1, ..., \tilde{z}^n)$ are holomorphic and relations (2.4), in $U \cap \tilde{U}$ we have

$$\begin{aligned}
&\frac{\partial f}{\partial \tilde{z}^{k'}} = \frac{\partial f}{\partial z^k} \cdot \frac{\partial z^k}{\partial \tilde{z}^{k'}}, \quad \frac{\partial f}{\partial \overline{\tilde{z}^{k'}}} = \frac{\partial f}{\partial \overline{z^k}} \frac{\partial \overline{z^k}}{\partial \overline{\tilde{z}^{k'}}},\\
&\frac{\partial^2 f}{\partial \tilde{z}^{k'} \partial \tilde{z}^{l'}} = \frac{\partial f}{\partial z^k} \cdot \frac{\partial^2 z^k}{\partial \tilde{z}^{k'} \partial \tilde{z}^{l'}} + \frac{\partial^2 f}{\partial z^k \partial z^l} \cdot \frac{\partial z^k}{\partial \tilde{z}^{k'}} \frac{\partial z^l}{\partial \tilde{z}^{l'}}\\
&\frac{\partial^2 f}{\partial \overline{\tilde{z}^{k'}} \partial \overline{\tilde{z}^{l'}}} = \frac{\partial f}{\partial \overline{z^k}} \cdot \frac{\partial^2 \overline{z^k}}{\partial \overline{\tilde{z}^{k'}} \partial \overline{\tilde{z}^{l'}}} + \frac{\partial^2 f}{\partial \overline{z^k} \partial \overline{z^l}} \cdot \frac{\partial \overline{z^k}}{\partial \overline{\tilde{z}^{k'}}} \frac{\partial \overline{z^l}}{\partial \overline{\tilde{z}^{l'}}},\\
&\frac{\partial^2 f}{\partial \tilde{z}^{k'} \partial \overline{\tilde{z}^{l'}}} = \frac{\partial^2 f}{\partial z^k \partial \overline{z^l}} \cdot \frac{\partial z^k}{\partial \tilde{z}^{k'}} \frac{\partial \overline{z^l}}{\partial \overline{\tilde{z}^{l'}}}.
\end{aligned} \tag{2.8}$$

□

As a consequence of the definition 2.1, the field of 2-jet of f is $j^2 f = j^2 u + i \cdot j^2 v$ and the fields of 2-jets of the coordinate functions in a local chart are

$$j^2 z^k = j^2 x^k + i \cdot j^2 y^k.$$

Moreover, for their conjugates we have

$$j^2 \overline{z^k} = j^2 x^k - i \cdot j^2 y^k,$$

so we can write

$$j^2x^k = \frac{1}{2}\left(j^2z^k + j^2\overline{z^k}\right), \quad j^2y^k = \frac{1}{2i}\left(j^2z^k - j^2\overline{z^k}\right). \tag{2.9}$$

Taking into account (2.5), (2.7) and (2.9), we have the local expression of $j^2 f$:

$$\begin{aligned} j^2 f = {} & \frac{\partial f}{\partial z^k} j^2 z^k + \frac{\partial f}{\partial \overline{z^k}} j^2\overline{z^k} + \frac{1}{2}\frac{\partial^2 f}{\partial z^k \partial z^l} j^2z^k \cdot j^2 z^l \\ & + \frac{\partial^2 f}{\partial z^k \partial \overline{z^l}} j^2 z^k \cdot j^2\overline{z^l} + \frac{1}{2}\frac{\partial^2 f}{\partial \overline{z^k}\partial\overline{z^l}} j^2\overline{z^k}\cdot j^2\overline{z^l}. \end{aligned} \tag{2.10}$$

Let $(U, (z^k))$, $(\tilde{U}, (\tilde{z}^k))$ be two local charts which domains overlap. Taking into account that the coordinate functions $z^k = z^k(\tilde{z}^1, ..., \tilde{z}^n)$ are holomorphic and relations (2.4), in $U \cap \tilde{U}$ we have

$$\begin{aligned} j^2\tilde{z}^{k'} &= \frac{\partial \tilde{z}^{k'}}{\partial z^k} j^2 z^k + \frac{1}{2}\frac{\partial^2 \tilde{z}^{k'}}{\partial z^k \partial z^l} j^2 z^k \cdot j^2 z^l, \\ j^2\overline{\tilde{z}^{k'}} &= \frac{\partial \overline{\tilde{z}^{k'}}}{\partial \overline{z^k}} j^2\overline{z^k} + \frac{1}{2}\frac{\partial^2 \overline{\tilde{z}^{k'}}}{\partial\overline{z^k}\partial\overline{z^l}} j^2\overline{z^k}\cdot j^2\overline{z^l}. \end{aligned} \tag{2.11}$$

By (2.9) and the changing rules for the 2-jets of real coordinates we also have:

$$\begin{aligned} j^2\tilde{z}^{k'} \cdot j^2\tilde{z}^{l'} &= \frac{\partial \tilde{z}^{k'}}{\partial z^k}\frac{\partial \tilde{z}^{l'}}{\partial z^l} j^2 z^k \cdot j^2 z^l, \\ j^2\tilde{z}^{k'} \cdot j^2\overline{\tilde{z}^{l'}} &= \frac{\partial \tilde{z}^{k'}}{\partial z^k}\frac{\partial \overline{\tilde{z}^{l'}}}{\partial \overline{z^l}} j^2 z^k \cdot j^2\overline{z^l}, \\ j^2\overline{\tilde{z}^{k'}} \cdot j^2\overline{\tilde{z}^{l'}} &= \frac{\partial \overline{\tilde{z}^{k'}}}{\partial \overline{z^k}}\frac{\partial \overline{\tilde{z}^{l'}}}{\partial \overline{z^l}} j^2\overline{z^k} \cdot j^2\overline{z^l}. \end{aligned} \tag{2.12}$$

Let $J^2_{\mathbf{C}}(M)$ be the complexification of $J^2(M)$ and we call its elements the *2-jets on* M(meaning the complex 2-jets). Obviously, $j^2_x f \in J^2_{\mathbf{C}}(M)$. The elements of $J^2_{\mathbf{C}}(M)$ are $\omega^2 = \alpha^2_{\mathbf{R}} + i \cdot \beta^2_{\mathbf{R}}$, for every $\alpha^2_{\mathbf{R}}, \beta^2_{\mathbf{R}} \in J^2(M)$. If $\alpha^2_{\mathbf{R}}$ is locally given by (2.6) and the locally expression of $\beta^2_{\mathbf{R}}$ is

$$\begin{aligned} \beta^2_{\mathbf{R}} = {} & \alpha_k \cdot j^2x^k + \beta_k j^2 y^k + \frac{1}{2}\alpha_{kl}\cdot j^2x^k \cdot j^2 x^l \\ & + \gamma_{kl} \cdot j^2x^k \cdot j^2y^l + \frac{1}{2}\beta_{kl}\cdot j^2y^k \cdot j^2y^l, \end{aligned}$$

then the locally expression of ω^2 is

$$\begin{aligned}\omega^2 &= \omega_k \cdot j^2 z^k + \theta_k j^2\overline{z^k} + \frac{1}{2}\omega_{kl} \cdot j^2 z^k \cdot j^2 z^l \\ &+\lambda_{kl} \cdot j^2 z^k \cdot j^2\overline{z^l} + \frac{1}{2}\theta_{kl} \cdot j^2\overline{z^k} \cdot j^2\overline{z^l},\end{aligned} \tag{2.13}$$

where

$$\omega_k = \frac{1}{2}[a_k + \beta_k + i \cdot (\alpha_k - b_k)], \quad \theta_k = \frac{1}{2}[a_k - \beta_k + i \cdot (\alpha_k + b_k)],$$

$$\omega_{kl} = \frac{1}{4}[a_{kl} + \gamma_{kl} + \gamma_{lk} - b_{kl} + i \cdot (\alpha_{kl} - c_{kl} - c_{lk} - \beta_{kl})],$$

$$\theta_{kl} = \frac{1}{4}[a_{kl} - \gamma_{kl} - \gamma_{lk} - b_{kl} + i \cdot (\alpha_{kl} + c_{kl} + c_{lk} - \beta_{kl})],$$

$$\lambda_{kl} = \frac{1}{4}[a_{kl} - \gamma_{kl} + \gamma_{lk} + b_{kl} + i \cdot (\alpha_{kl} + c_{kl} - c_{lk} + \beta_{kl})].$$

The above relations were obtained by a straightforward calculation using (2.7) and (2.9).

Returning now to the 2-jet (2.10) of a complex function f, relations (2.8), (2.11) and (2.12) show that the following expressions have geometrical meaning:

$$\begin{aligned} j^{2,0}f &= \frac{\partial f}{\partial z^k} j^2 z^k + \frac{1}{2}\frac{\partial^2 f}{\partial z^k \partial z^l} j^2 z^k \cdot j^2 z^l, \\ j^{1,1}f &= \frac{\partial^2 f}{\partial z^k \partial \overline{z^l}} j^2 z^k \cdot j^2 \overline{z^l}, \\ j^{0,2}f &= \frac{\partial f}{\partial \overline{z^k}} j^2 \overline{z^k} + \frac{1}{2}\frac{\partial^2 f}{\partial \overline{z^k} \partial \overline{z^l}} j^2 \overline{z^k} \cdot j^2 \overline{z^l}. \end{aligned} \tag{2.14}$$

Remark 2.1. From the above relations ones can see that

$$j^{0,2}f = \overline{j^{2,0}\overline{f}},$$

All the above considerations were a reason for introduction of some new types of 2-jets on a complex manifold.

3. New types of 2-jets on a complex manifold. Let $f, g \in \Omega^0(M)$, where M is a complex manifold, like in the previous section.

Definition 3.1. *We say that f, g determine the same*

a) (2,0)-jet at $x \in M$ if $f(x) = g(x) = 0$ and in a local complex chart

$$\frac{\partial f}{\partial z^k} = \frac{\partial g}{\partial z^k}, \quad \frac{\partial^2 f}{\partial z^k \partial z^l} = \frac{\partial^2 g}{\partial z^k \partial z^l};$$

b) (1,1)-jet at $x \in M$ if $f(x) = g(x) = 0$ and in a local complex chart

$$\frac{\partial^2 f}{\partial z^k \partial \overline{z^l}} = \frac{\partial^2 g}{\partial z^k \partial \overline{z^l}};$$

c) (0,2)-jet at $x \in M$ if $f(x) = g(x) = 0$ and in a local complex chart

$$\frac{\partial f}{\partial \overline{z^k}} = \frac{\partial g}{\partial \overline{z^k}}, \quad \frac{\partial^2 f}{\partial \overline{z^k} \partial \overline{z^l}} = \frac{\partial^2 g}{\partial \overline{z^k} \partial \overline{z^l}};$$

for all $k, l = 1, ..., n$.

From the relations (2.8), the above definition is correct. The relations to determine the same $(2,0)$-, $(1,1)$-, $(0,2)$-jet at x are equivalence relations. We shall denote by $j_x^{2,0} f$, $j_x^{1,1} f$, $j_x^{0,2} f$, the equivalence classes of f with respect to these relations, respectively. Obviously, their local expressions are given by (2.14) considered at x.

Generally, we say that

a) a *(2,0)-jet at $x \in M$* is locally given by

$$\omega_x^{2,0} = \omega_k(x) \cdot j^2 z^k + \frac{1}{2} \omega_{kl}(x) \cdot j^2 z^k \cdot j^2 z^l; \tag{3.1}$$

b) a *(1,1)-jet at $x \in M$* is locally given by

$$\omega_x^{1,1} = \lambda_{kl}(x) \cdot j^2 z^k \cdot j^2 \overline{z^l}; \tag{3.2}$$

c) a *(0,2)-jet at $x \in M$* is locally given by

$$\omega_x^{0,2} = \theta_k(x) \cdot j^2 \overline{z^k} + \frac{1}{2} \theta_{kl}(x) \cdot j^2 \overline{z^k} \cdot j^2 \overline{z^l}, \tag{3.3}$$

where $\omega_k, \theta_k, \omega_{kl}, \lambda_{kl}, \theta_{kl} \in \Omega^0(U)$ satisfy $\omega_{kl} = \omega_{lk}$, $\theta_{kl} = \theta_{lk}$ and if $(\tilde{U}, (\tilde{z}^k))$ is another local chart at x, in $U \cap \tilde{U}$ we have

$$\tilde{\omega}_{k'} = \omega_k \frac{\partial z^k}{\partial \tilde{z}^{k'}}, \quad \tilde{\omega}_{k'l'} = \omega_k \frac{\partial^2 z^k}{\partial \tilde{z}^{k'} \partial \tilde{z}^{l'}} + \omega_{kl} \frac{\partial z^k}{\partial \tilde{z}^{k'}} \cdot \frac{\partial z^l}{\partial \tilde{z}^{l'}}, \tag{3.4}$$

$$\tilde{\theta}_{k'} = \theta_k \frac{\partial \overline{z^k}}{\partial \overline{\tilde{z}^{k'}}}, \quad \tilde{\theta}_{k'l'} = \theta_k \frac{\partial^2 \overline{z^k}}{\partial \overline{\tilde{z}^{k'}} \partial \overline{\tilde{z}^{l'}}} + \theta_{kl} \frac{\partial \overline{z^k}}{\partial \overline{\tilde{z}^{k'}}} \cdot \frac{\partial \overline{z^l}}{\partial \overline{\tilde{z}^{l'}}},$$

$$\tilde{\lambda}_{k'l'} = \lambda_{kl} \frac{\partial z^k}{\partial \tilde{z}^{k'}} \cdot \frac{\partial \overline{z^l}}{\partial \overline{\tilde{z}^{l'}}}.$$

We denote by $J_x^{2,0}(M)$, $J_x^{1,1}(M)$, $J_x^{0,2}(M)$ the set of all (2,0)-, (1,1)-, (0,2)-jets at $x \in M$, respectively.

Remark 3.1. A (2,0)-jet on M locally given by (3.1) with coefficient functions ω_k, ω_{kl} holomorphic functions is called *an holomorphic 2-jet on* M. This is exactly the notion of the 2-jet forms from [9].

The spaces $J^{2,0}(M) = \cup_{x\in M} J_x^{2,0}(M)$, $J^{1,1}(M) = \cup_{x\in M} J_x^{1,1}(M)$, $J^{0,2}(M) = \cup_{x\in M} J_x^{0,2}(M)$ are fiber bundles over the complex manifold M. Their sections are called *fields of (2,0)-, (1,1)-, (0,2)-jets*, respectively, and the spaces of these sections are denoted by $J^{2,0}M$, $J^{1,1}M$, $J^{0,2}M$.

The bundle $J^{2,0}(M)$ is just an holomorphic bundle over M. Indeed, let $\pi^{2,0} : J^{2,0}(M) \to M$ be the surjection $\pi^{2,0}(\omega^{2,0}) = x$ iff $\omega^{2,0} \in J_x^{2,0}(M)$, and $\big((\pi^{2,0})^{-1}(U), \varphi\big)$ a bundle chart, where U is a domain from M and

$$\varphi : (\pi^{2,0})^{-1}(U) \to U \times \mathbf{C}^{\frac{n(n+3)}{2}}, \quad \varphi(\omega_x^{2,0}) = (x, \omega_k(x), \omega_{kl}(x)),$$

for every (2,0)-jet locally given in U by (3.1). The transition functions of this bundle are given by the matrix

$$\begin{pmatrix} \dfrac{\partial z^k}{\partial \tilde{z}^{k'}} & 0_n \\ \dfrac{\partial^2 z^k}{\partial \tilde{z}^{k'} \partial \tilde{z}^{l'}} & \dfrac{\partial z^k}{\partial \tilde{z}^{k'}} \cdot \dfrac{\partial z^l}{\partial \tilde{z}^{l'}} \end{pmatrix},$$

whose elements are holomorphic functions.

The bundles $J^{1,1}(M)$, $J^{0,2}(M)$ are not holomorphic ones.

From the local expression (2.13) of a 2-jet on M ones can see that the expressions (3.1), (3.2), (3.3) are also 2-jets on M, so $J^{2,0}(M)$ $J^{1,1}(M)$,

$J^{0,2}(M)$ are subsets of $J^2(M)$. Moreover, they are also subbundles of $J^2(M)$ and every $\omega^2 \in J^2(M)$ locally given by (2.13) is a sum

$$\omega^2 = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}. \tag{3.5}$$

It result that we have

Theorem 3.1. *The bundle $J^2(M)$ admits the decomposition*

$$J^2(M) = J^{2,0}(M) \oplus J^{1,1}(M) \oplus J^{0,2}(M). \tag{3.6}$$

A consequence of the relation (3.6) is the decomposition of the module of fields of 2-jets on M:

$$\mathcal{J}^2 M = \mathcal{J}^{2,0} M \oplus \mathcal{J}^{1,1} M \oplus \mathcal{J}^{0,2} M. \tag{3.7}$$

We can define now the following maps

(3.8)
$$j^{2,0} : \Omega^0(M) \to \mathcal{J}^{2,0}M,\ j^{1,1} : \Omega^0(M) \to \mathcal{J}^{1,1}M,\ j^{0,2} : \Omega^0(M) \to \mathcal{J}^{0,2}M,$$

which assign to a complex function $f \in \Omega^0(M)$ its field of (2,0)-, (1,1)-, (0,2)-jets, respectively, locally given by (2.14). Taking into account (2.10), we have

Proposition 3.1. *For two complex functions $f, g \in \Omega^0(M)$, the following assertions are equivalent*

a) $j^2 f = j^2 g$;

b) $j^{2,0} f = j^{2,0} g$; $j^{1,1} f = j^{1,1} g$; $j^{0,2} f = j^{0,2} g$.

A immediately consequence of the above proposition is the decomposition

$$j^2 = j^{2,0} + j^{1,1} + j^{0,2}. \tag{3.9}$$

4. An isomorphism for the 1-dimensional Dolbeault cohomology group. Following some ideas from [4] we shall prove that the 1-dimensional Dolbeault cohomology group $H^{0,1}(M)$ of the complex manifold M could be described in terms of (0,2)-jets on M.

Let d be the exterior derivative on M. It is well-known that

$$d = d_{10} + d_{01}, \tag{4.1}$$

where d_{01} send (p,q)-forms into $(p,q+1)$-forms on M. The module of (0,1)-forms on M is denoted by $\Omega^{0,1}(M)$.

Let $Z^{0,1}(M)$ be the group of $(0,1)$-forms θ which satisfies $d_{01}\theta = 0$, namely the *closed* (0,1)-forms, and $B^{0,1}(M)$ the group of (0,1)-forms θ which could be write $\theta = d_{01}f$ for some $f \in \Omega^0(M)$, namely the *exact* (0,1)-forms. The 1-dimensional Dolbeault cohomology group is the quotient group

$$H^{0,1}(M) = Z^{0,1}(M)/B^{0,1}(M). \tag{4.2}$$

Let D be the map

$$D : \Omega^{0,1}(M) \to J^{0,2}M$$

who assigns to a (0,1)-form θ locally given in $(U,(z^1,...,z^n))$ by $\theta = \theta_k \cdot d\overline{z^k}$ the field of (0,2)-jets locally given in U by

$$D(\theta) = \theta_k \cdot j^2\overline{z^k} + \frac{1}{4}\left(\frac{\partial\theta_k}{\partial\overline{z^l}} + \frac{\partial\theta_l}{\partial\overline{z^k}}\right) \cdot j^2\overline{z^k} \cdot j^2\overline{z^l}. \tag{4.3}$$

We can easily see that D has geometrical meaning.

Proposition 4.1. *We have* $D \circ d_{01} = j^{0,2}$.

Proof. For every $f \in \Omega^0(M)$, $d_{01}f$ is a (0,1)-form locally given by

$$d_{01}f = \frac{\partial f}{\partial\overline{z^k}} \cdot d\overline{z^k}.$$

Then,

$$D(d_{01}f) = \frac{\partial f}{\partial\overline{z^k}} j^2\overline{z^k} + \frac{1}{2}\frac{\partial^2 f}{\partial\overline{z^k}\partial\overline{z^l}} j^2\overline{z^k} \cdot j^2\overline{z^l},$$

which is just the local expression of $j^{0,2}f$. □

Definition 4.1. *A field* $\omega^{0,2}$ *of (0,2)-jets is called exact if* $\omega^{0,2} = j^{0,2}f$ *for some* $f \in \Omega^0(M)$ *and it is called closed if it is locally exact.*

Let us denote by $E^2(M)$, $C^2(M)$ the sets of all fields of (0,2)-jets which are exact, respectively closed.

Proposition 4.2. *We have the equalities*

$$D(Z^{0,1}(M)) = C^2(M), \quad D(B^{0,1}(M)) = E^2(M). \tag{4.4}$$

Proof. Let $\theta \in Z^{0,1}(M)$, so it is a (0,1)-form which $d_{01}\theta = 0$. From the Dolbeault lemma it results that in every domain U from M there is $f_U \in \Omega^0(U)$ such that in U, $\theta = d_{01}f_U$. Taking into account proposition 4.1 we obtain that in U, $D(\theta) = j^{0,2}f_U$, so $D(\theta)$ is locally exact. Hence, $D(Z^{0,1}(M)) \subseteq C^2(M)$.

Let be now $\omega^{0,2} \in C^2(M)$, so it is a field of (0,2)-jets locally exact. That means that for every domain U there is a function $f_U \in \Omega^0(U)$ such that in U $\omega^{0,2} = j^{0,2}f_U$. We define a (0,1)-form θ putting $\theta = d_{01}f_U$ in U. Since $\omega^{0,2}$ is globally defined, so it is θ. Hence, $\theta \in Z^{0,1}(M)$. Moreover, in U we have $D(\theta) = D(d_{01}f_U) = j^{0,2}f_U$, using proposition 4.1. We obtain $C^2(M) \subseteq D(Z^{0,1}(M)$.

An analogous argument justify the second equality. □

Now we can give the main result of this section:

Theorem 4.1. *There is an isomorphism between the 1-dimensional Dolbeault cohomology group of M and the quotient group $C^2(M)/E^2(M)$.*

Proof. Let ς be the map

$$\varsigma : H^{0,1}(M) \to C^2(M)/E^2(M), \quad \varsigma([\theta]) = [D(\theta)].$$

for $\theta \in Z^{0,1}(M)$. Proposition 4.2 assures either the well-definition of ς, either its bijectivity. Obviously, this map is a morphism of groups. □

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