

ON APPROXIMATE IDEAL AMENABILITY IN BANACH ALGEBRAS

BY

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Abstract. The notions of approximate I -weak amenability and approximate ideal amenability in Banach algebras are introduced. General theory is developed for these notions, and in particular, we show that the approximate ideal amenability of the Segal algebra $S(G)$ in $L^1(G)$ implies that G , and hence $L^1(G)$ is amenable.

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1. Introduction. In [6], GORGI and YAZDANPANAHI, introduced two notions of amenability for a Banach algebra A . The two notions are the concepts of I -weak amenability and ideal amenability for Banach algebras, where I is a closed two-sided ideal in A . They related these concepts to weak amenability of Banach algebras, and showed that ideal amenability is different from amenability and weak amenability.

Another variation of the notion of amenability for Banach algebras was also introduced by GHAHRAMANI and LOY in [4]. Let A be a Banach algebra and let X be a Banach A -bimodule. A derivation $D : A \rightarrow X$ is *approximately inner* if there is a net (x_α) in X such that

$$D(a) = \lim_{\alpha} (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|\cdot\|)$. The Banach algebra A is *approximately amenable* if, for each Banach A -bimodule X , every continuous derivation $D : A \rightarrow X'$ is approximately inner.

The basic properties of approximately amenable Banach algebras were established in [4]. Certainly every amenable Banach algebra is approximately amenable; a commutative, approximately amenable Banach algebra is weakly amenable; examples of commutative, approximately amenable Banach algebras which are not amenable are given in [4, Example 6.1]. Characterizations of approximately amenable Banach algebras were also established in [4], they are analogous to the characterization of amenable Banach algebras as those with a bounded approximate diagonal.

In this paper, we shall extend the notion of approximate amenability in Banach algebras to that of ideal amenability and I -weak amenability. The question is which of the standard constructions for ideally amenable Banach algebras work for the concept of approximate ideally amenable Banach algebras. Many of the proofs to follow are variants on the classical arguments, with due care given to possible unboundedness.

2. Preliminaries. First, we recall some standard notions; for further details, see [2], [3], [11], [14] and [16].

Let A be an algebra and let X be an A -bimodule. A *derivation* from A to X is a linear map $D : A \rightarrow X$ such that

$$D(ab) = Da \cdot b + a \cdot Db, \quad (a, b \in A).$$

For example, $\delta_x : a \rightarrow a \cdot x - x \cdot a$ is a derivation; derivations of this form are the *inner derivations*.

Let A be a Banach algebra, and let X be an A -bimodule. Then X is a Banach A -bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k \|a\| \|x\|, \quad \|x \cdot a\| \leq k \|a\| \|x\|, \quad (a \in A, x \in X).$$

By renorming X , we can suppose that $k = 1$. For example, A itself is Banach A -bimodule, and X' , the dual space of a Banach A -bimodule X , is a Banach A -bimodule with respect to the module operations specified for by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle, \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle, \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X . In particular every closed two-sided ideal I of A is Banach A -bimodule and I' the dual space of I is a dual A -bimodule.

Let A be a Banach algebra, and let X be a Banach A -bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X , $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X , and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X) / \mathcal{N}^1(A, X).$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach A -bimodule X and *weakly amenable* if $\mathcal{H}^1(A, A') = \{0\}$. For instance, the group algebra, $L^1(G)$ of a locally compact group G is always weakly amenable ([9]), and is amenable if and only if G is amenable in the classical sense ([8]). Also, a C^* -algebra is always weakly amenable ([7]) and is amenable if and only if it is nuclear ([1]). Recently, the authors in [6], defined A as *I -weakly amenable* if $\mathcal{H}^1(A, I') = \{0\}$ for a closed two-sided ideal I of A , and *ideally amenable* if it is I -weakly amenable for every closed two-sided ideal I of A . Clearly, an amenable Banach algebra is ideally amenable and an ideally amenable Banach algebra is weakly amenable.

Our definition which shall describe the main new property that we shall study in this work is as follows:

Definition 2.1. *Let A be a Banach algebra and let I be a closed two-sided ideal in A .*

1. *A is approximately I -weakly amenable if every derivation $D : A \rightarrow I'$ is approximately inner*
2. *A is approximately ideally amenable if it is approximately I -weakly amenable for every closed two-sided I of A .*

Remark 2.2. We have the following trivial observations:

1. An approximate amenable Banach algebra is approximately ideally amenable.
2. An approximate ideally amenable Banach algebra is approximately weakly amenable.
3. Every weakly amenable Banach algebra is approximately weakly amenable.

3. General results. We recall that a *character* on A is a non-zero homomorphism from A into the scalar field. The set of all characters on A

is the character space of A , denoted by Φ_A . The kernel of $\varphi \in \Phi_A \cup \{0\}$ is denoted by M_φ . Let $\varphi \in \Phi_A \cup \{0\}$. A linear functional d on A is a *point derivation* at φ if

$$d(ab) = d(a)\varphi(b) + \varphi(a)d(b), \quad (a, b \in A).$$

Proposition 3.1. *Let A be an approximately ideally amenable Banach algebra. Then there are no non-zero, continuous point derivations on A .*

Proof. Let I be an arbitrary closed two-sided ideal of A and let d be a continuous point derivation of A at $\varphi \in \Phi_A$. Then the map $D : A \rightarrow I'$ defined by $D(a) = d(a)\varphi$, ($a \in A$) is a derivation since

$$D(ab) = d(a)\varphi(b)\varphi + \varphi(a)d(b)\varphi = d(a)\varphi \cdot b + a \cdot \varphi d(b), \quad (a, b \in A).$$

Since A is approximately ideally amenable, then D is approximately inner, that is, there exists a net $(i_\alpha) \subset I'$ such that for every $a \in A$,

$$D(a) = \lim_{\alpha} (a \cdot i_\alpha - i_\alpha \cdot a).$$

Clearly we have, for $a \in A$,

$$d(a)\varphi(a) = \lim_{\alpha} \delta_{i_\alpha}(a)(a) = \lim_{\alpha} \langle a \cdot i_\alpha - i_\alpha \cdot a, a \rangle = \lim_{\alpha} i_\alpha(a^2 - a^2) = 0,$$

and so $d \mid (A/M_\varphi) = 0$. Thus $d = 0$.

The proof of our next result follow the same line as that of [4, Proposition 2.4]. The major points are that

1. Every derivation from A can be extended to a derivation from $A^\#$, such that the extended derivation is inner if and only if the original derivation was

2. If D is a derivation from $A^\#$ to an A -bimodule I' , and e is the identity of $A^\#$, then there is an inner derivation $\tilde{D} : A^\# \rightarrow I'$ such that $(D - \tilde{D})(e) = 0$. \square

Proposition 3.2. *Let A be a Banach algebra. Then A is approximately ideally amenable if and only if $A^\#$ is approximately ideally amenable.*

Proposition 3.3. *Let A be a Banach algebra with an approximate identity. Suppose I is a closed two-sided ideal in A such that the left (or right) action of A on I is trivial. Then A is approximately I -weakly amenable.*

Proof. Suppose the right action of A on I is trivial, that is, $I.A = \{0\}$, then it is clear that $A.I' = \{0\}$. Let (e_α) be an approximate identity in A and let $D : A \rightarrow I'$ be a continuous derivation. Then

$$\begin{aligned} D(a) &= \lim_{\alpha} D(e_\alpha \cdot a) = \lim_{\alpha} [D(e_\alpha) \cdot a + e_\alpha \cdot D(a)] \\ &= \lim_{\alpha} D(e_\alpha) \cdot a = \lim_{\alpha} [D(e_\alpha) \cdot a - a \cdot D(e_\alpha)] \end{aligned}$$

and so D is approximately inner. Thus A is approximately I -weakly amenable.

Let A be a Banach algebra, we recall from [2] that a left [right] multiplier on A is an element L [R] in $L(A)$ such that $L(ab) = L(a)b$ [$R(ab) = aR(b)$] ($a, b \in A$). A multiplier is a pair (L, R) where L and R are left and right multipliers on A respectively and $aL(b) = R(a)b$ ($a, b \in A$). The set of all multipliers on A is denoted by $\mathcal{M}(A)$. It is called multiplier algebra of A .

Let A be a Banach algebra, let $B(A)$ be the Banach algebra of bounded linear operators on A and let $\mathcal{M}(A)$ be multiplier algebra of A . That is, $\mathcal{M}(A) = \{(L, R) : L, R \in L(A), L(ab) = L(a)b, R(ab) = aR(b), aL(b) = R(a)b, a, b \in A\}$.

As norm closed subalgebra of $B(A) \times B(A)^{op}$ (where $B(A)^{op}$ is the opposite algebra of $B(A)$), $\mathcal{M}(A)$ is a Banach algebra.

For $a \in A$, L_a, R_a will denote the linear maps $b \rightarrow ab$ and $b \rightarrow ba$ on A , respectively. Then $(L_a, R_a) \in \mathcal{M}(A)$ with $\|L_a\| \leq \|a\|$ and $\|R_a\| \leq \|a\|$. It is easy to see that $a \rightarrow L_a$ (Resp. $a \rightarrow R_a$) is injective if and only if A is left (Resp. right) faithful. In particular, if A has a bounded approximate identity $\{e_\alpha\}$ of bound m , then $\|L_a\| \geq m^{-1}\|a\|$ and $\|R_a\| \geq m^{-1}\|a\|$ for all $a \in A$. In this case, A is identified with a norm closed ideal in $\mathcal{M}(A)$.

For each Banach A -bimodule X , $\mathcal{M}(A)$ acts on X through

$$T \cdot x = \lim_{\alpha} L(e_\alpha) \cdot x, \quad x \cdot T = \lim_{\alpha} x \cdot R(e_\alpha), \quad x \in X, T = (L, R) \in \mathcal{M}(A),$$

thus X is a Banach $\mathcal{M}(A)$ -bimodule.

Also, A embeds in $\mathcal{M}(A)$ through $a \rightarrow (L_a, R_a)$, in the case where A is faithful, A is an ideal in $\mathcal{M}(A)$. Given a continuous derivation $D : \mathcal{M}(A) \rightarrow X'$, the restriction \tilde{D} of D to A is also a continuous derivation. \square

The following result is very useful and important.

Proposition 3.4. *Let A be a Banach algebra.*

1. If I is an ideal in A with an approximate identity, then every ideal in I is an ideal in A .
2. If A is faithful and has a bounded approximate identity, then every ideal in A is an ideal in $\mathcal{M}(A)$.

Proof. 1. This is Proposition 2.9.4 of [2].

2. Since A is faithful, then A is an ideal in $\mathcal{M}(A)$. Also since A has a bounded approximate identity, then the result follows from (1). \square

Theorem 3.5. *Let A be a Banach algebra and I be a closed two-sided ideal of A with a bounded approximate identity. Suppose A is approximately ideally amenable, then I is approximately ideally amenable.*

Proof. Since I has a bounded approximate identity, then by Proposition 3.4(1), every ideal of I is also an ideal of A . Let J be a closed two-sided ideal in I , and $D : I \rightarrow J'$ be a derivation, then by [13, Proposition 2.1.6], D can be extended to a derivation $\tilde{D} : A \rightarrow J'$ and since A is approximately ideally amenable, there is a net $(j_\alpha) \subset J'$ such that $\tilde{D}(a) = \lim_\alpha (a \cdot j_\alpha - j_\alpha \cdot a)$ ($a \in A$). Then

$$D(i) = \tilde{D}(i) = \lim_\alpha (i \cdot j_\alpha - j_\alpha \cdot i)$$

for each $i \in I$, and so D is approximately inner. Thus I is approximately ideally amenable. \square

Corollary 3.6. *Let A be a faithful Banach algebra with a bounded approximate identity and $\mathcal{M}(A)$ be the multiplier algebra of A . If $\mathcal{M}(A)$ is approximately ideally amenable, then A is approximately ideally amenable.*

Proof. Follows from the fact that A is a closed two-sided ideal in $\mathcal{M}(A)$ and Theorem 3.5. \square

Example 3.7. For a locally compact group G , $L^1(G)$ is approximately ideally amenable if $\mathcal{M}(L^1(G))$ is approximately ideally amenable.

Proof. Since $L^1(G)$ is left and right faithful and also has a bounded approximate identity, then the result follows from Corollary 3.6. \square

Theorem 3.8. *Let A be a Banach algebra and let J be a closed two-sided ideal in A with a bounded approximate identity. Then for every closed*

two-sided ideal I in J , J is approximately I -weakly amenable if and only if A is approximately I -weakly amenable.

Proof. Since J has a bounded approximate identity, then Proposition 3.4(1), every ideal I of J is also an ideal of A . Let (e_α) be a bounded approximate identity for J , for every continuous derivation $D : J \rightarrow I'$, the map $\tilde{D} : A \rightarrow I'$ defined by $\tilde{D}(a) = w^* - (D(a \cdot e_\alpha) - a \cdot D(e_\alpha))$ is a continuous derivation by [13, Proposition 2.1.6]. By using the fact that if $D = 0$, then $\tilde{D} = 0$, since $JI = IJ = I$, then it follows easily that D is approximately inner if and only if \tilde{D} is approximately inner. \square

Corollary 3.9. *Let G be a locally compact group. For every closed two-sided ideal I of $L^1(G)$, $L^1(G)$ is approximately I -weakly amenable if and only if $\mathcal{M}(L^1(G))$ is approximately I -weakly amenable.*

4. Segal algebras on locally compact groups. Segal algebras were first defined by Reifer for group algebras; see [13] for example. Let A be a Banach algebra. A segal algebra is a subspace B of A such that

1. B is dense in A
2. B is a left ideal of A
3. B admits a norm $\|\cdot\|_B$ under which it is complete and a contractive A -module
4. B is essential A -module; $A \cdot B$ is $\|\cdot\|_B$ -dense in B .

We further say that B is symmetric if it is also a contractive essential right A -module.

In the case $A = L^1(G)$, where G is a locally compact group, we write $S(G)$ instead of B and further insist that

5. $S(G)$ is closed under left translations; $l_x f \in S(G)$ for all $x \in G$ and $f \in S(G)$ where $l_x f(y) = f(x^{-1}y)$ for $y \in G$. Condition 3. on $B = S(G)$ is equivalent to
 - 3.' the map $(x, f) \rightarrow l_x f : G \times S(G) \rightarrow S(G)$ is continuous with $\|l_x f\|_S = \|f\|_S$ for all $x \in G$ and $f \in S(G)$.

Moreover, symmetry for $S(G)$ is equivalent to having that $S(G)$ is closed under right actions; $r_x f \in S(G)$ for $x \in G$ and $f \in S(G)$, where $r_x f(y) = f(y^{-1}x)$ ($y \in G$), with the actions being continuous and isometric. For further information on Segal algebras see [12] and [15].

The symmetric Segal algebras include all Segal algebras on locally compact abelian groups. Also, every symmetric Segal algebra on a locally compact group G is a two-sided ideal in $L^1(G)$ and has an approximate identity see ([5]).

As in [5], we have the following theorem

Theorem 4.1. *Let G be an amenable group, and let $S(G)$ be a symmetric Segal algebra on G . Suppose X is a Banach $L^1(G)$ -bimodule. Then every continuous derivation from $S(G)$ into X is approximately inner.*

Using Theorem 4.1, we have the following result

Theorem 4.2. *Let G be an amenable group. Then every symmetric Segal algebra $S(G)$ on G is approximately ideally amenable.*

Proof. For every closed two-sided ideal I of $S(G)$, I is an ideal of $L^1(G)$ since $S(G)$ is an ideal in $L^1(G)$ and has an approximate identity by Proposition 3.4(1). Thus both I and I' are Banach $L^1(G)$ -bimodules. And so the result follows from the above theorem with $X = I'$.

It was shown in [10] that if B is a Segal algebra in a Banach algebra A , then the mapping $J \rightarrow \bar{J}^A$ is a bijection from the set of all closed ideals in B onto the set of all closed ideals in A , where for a set $J \subset B$, \bar{J}^A denotes the closure of J in A . \square

We have the following result from [10, Proposition 2.7]

Proposition 4.3. *Let B be an abstract Segal algebra in a Banach algebra A , let J be a closed ideal of B and let $I = \bar{J}^A$. Suppose that A and J both have right approximate identities. Then I has a right approximate identity*

Our next result shows that the approximate ideal amenability of the Segal algebra $S(G)$ in $L^1(G)$ implies that G , and hence $L^1(G)$ is amenable.

Theorem 4.4. *If $S(G)$ is approximately ideally amenable, then G is an amenable group.*

Proof. Let $L_0^1(G) = \{f \in L^1(G) : \int_G f(x)dx = 0\}$ be the augmentation ideal in $L^1(G)$. It is easily checked that $L_0^1(G)$ is a closed ideal of $L^1(G)$ with codimension one. Let $J = L_0^1(G) \cap S(G)$, then J is a closed ideal of $S(G)$ with codimension one also. Since ideals with finite dimension or codimension are trivially complemented and every complemented subspace is weakly complemented, then if $S(G)$ is approximately ideally amenable, J has a right approximate identity, see ([4, Corollary 2.4]. Therefore, by Proposition 4.3, $L_0^1(G)$ has a right approximate identity. Thus G is amenable by [16, Theorem 5.2]. \square

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