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DERIVATIONS INTO ITERATED DUALS OF IDEALS OF BANACH ALGEBRAS

BY

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Abstract. The notions of approximate $n-I$-weak amenability and approximate $n-$ideal amenability in Banach algebras are introduced. General theory is developed for these notions, and in particular, we show that every symmetric Segal algebra $S(G)$ on an amenable group G is approximately permanently ideally amenable.

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1. Introduction. In [8], GORGI and YAZDANPANAH, introduced two notions of amenability for a Banach algebra A. The two notions are the concepts of I-weak amenability and ideal amenability for Banach algebras, where I is a closed two-sided ideal in A. They related these concepts to weak amenability of Banach algebras, and showed that ideal amenability is different from amenability and weak amenability. Also in [4], DALES, GHAHRAMANI and GRONBAEK introduced the concept of n-weak amenability for Banach algebras for $n \in \mathbb{N}$. They determined some relations between m- and n-weak amenability for general Banach algebras and for Banach algebras in various classes, and proved that, for each $n \in \mathbb{N}$, $(n+2)$- weak amenability always implies n-weak amenability. Let A be a weakly amenable Banach algebra. Then it is also proved in [4] that in the case where A is an ideal in its second dual $(A'', \Box)$, A is necessarily $(2m-1)$-weakly amenable for each $m \in \mathbb{N}$.

Another variation of the notion of amenability for Banach algebras was also introduced by GHAHRAMANI and LOY in [6]. Let A be a Banach algebra and let X be a Banach A-bimodule. A derivation $D : A \to X$ is *approximately inner* if there is a net (x_α) in X such that

$$D(a) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|.\|)$. The Banach algebra A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is approximately inner.

The basic properties of approximately amenable Banach algebras were established in [6], see also [1]. Certainly every amenable Banach algebra is approximately amenable; a commutative, approximately amenable Banach algebra is weakly amenable; examples of commutative, approximately amenable Banach algebras which are not amenable are given in [6, Example 6.1]. Characterizations of approximately amenable Banach algebras were also established in [6], they are analogous to the characterization of amenable Banach algebras as those with a bounded approximate diagonal.

In this paper, we shall extend the notions of approximate amenability and n-weak amenability in Banach algebras to that of ideal amenability and I-weak amenability. We shall study the concepts of approximately $n-I-$weak amenability and approximately $n-$ideal amenability. As in the [4], we determine the relations between approximate $n-I$ and $m-I-$ weak amenability for general Banach algebras. Many of the proofs to follow are variants on the classical arguments, with due care given to possible unboundedness.

2. Preliminaries.

First, we recall some standard notions; for further details, see [3], [12], [14] and [17].

Let A be an algebra and let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = Da \cdot b + a \cdot Db, \quad (a, b \in A)\,.$$

For example, $\delta_x : a \to a \cdot x - x \cdot a$ is a derivation; derivations of this form are the *inner derivations*.

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \le k \|a\| \|x\|, \quad \|x \cdot a\| \le k \|a\| \|x\|, \quad (a \in A,\, x \in X)\,.$$

By renorming X, we can suppose that $k = 1$. For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations specified for by

$$\langle x,\, a \cdot \lambda\rangle = \langle x \cdot a,\, \lambda\rangle, \quad \langle x,\, \lambda \cdot a\rangle = \langle a \cdot x,\, \lambda\rangle, \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X. Successively, the duals $X^{(n)}$ are Banach A-bimodules; in particular $A^{(n)}$ is a Banach A-bimodule for each $n \in \mathbb{N}$. We take $X^{(0)} = X$. Also, every closed two-sided ideal I of A is Banach A-bimodule and I' the dual space of I is a dual A-bimodule.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach A-bimodule X, *weakly amenable* if $\mathcal{H}^1(A, A') = \{0\}$, *$n-$weakly amenable* if $\mathcal{H}^1(A, A^{(n)}) = \{0\}$, and*permanently weakly amenable* if it is $n-$ weakly amenable for every $n \in \mathbb{N}$. For instance, the group algebra, $L^1(G)$ of a locally compact group G is always weakly amenable ([11]), and is amenable if and only if G is amenable in the classical sense ([10]). Also, a C^*-algebra is always weakly amenable ([9]) and is amenable if and only if it is nuclear ([2]). Also, each C^*-algebra is permanently weakly amenable ([4]) and each group algebra is n-weakly amenable whenever n is odd ([4]).

Recently, the authors in [8], defined A as *I-weakly amenable* if $\mathcal{H}^1(A, I') = \{0\}$ for a closed two-sided ideal I of A, and ideally amenable if it is I-weakly amenable for every closed two-sided ideal I of A. Clearly, an amenable Banach algebra is ideally amenable and an ideally amenable Banach algebra is weakly amenable.

Our definition which shall describe the main new property that we shall study in this work is as follows:

Definition 2.1. *Let A be a Banach algebra, let I be a closed two-sided ideal in A and let $n \in \mathbb{N}$.*

1. *A is approximately $n-I$-weakly amenable if every derivation $D : A \to I^{(n)}$ is approximately inner.*

2. *A is approximately $n-$ ideally amenable if it is approximately $n-I$-weakly amenable for every closed two-sided I of A.*

3. *A is approximately permanently $I-$ weakly amenable if it is approximately $n-I-$ weakly amenable for each $n \in \mathbb{N}$.*

4. *A is approximately permanently ideally amenable if it is approximately $n-I-$ weakly amenable for every closed two-sided ideal I in A and for each $n \in \mathbb{N}$.*

Remark 2.2. We have the following trivial observations:

1. An approximately $n-$ ideally amenable Banach algebra is approximately $n-$ weakly amenable.

2. An approximately permanently ideally amenable Banach algebra is approximately permanently weakly amenable.

3. General results. Let A be a Banach algebra and let I be a closed two-sided ideal in A, then I is a Banach $A-$ bimodule, let $n \in \mathbb{N}$. The adjoint of the injective map $i : I^{(n-1)} \to I^{(n+1)}$ is the projection $P : I^{(n+2)} \to I^{(n)}$ defined by $P(\Lambda) = \Lambda \backslash i(I^{(n-1)})$. (See [4] for further details on P.)

Proposition 3.1. *Let A be a Banach algebra and let I be a closed two-sided ideal in A with an approximate identity, then every ideal in I is an ideal in A.*

Proof. This is proposition 3.14(1) of [13].

Proposition 3.2. *Let A be a Banach algebra and I be a closed two-sided ideal in A, and let $n \in \mathbb{N}$. Suppose A is approximately $(n+2)-I-$ weakly amenable. Then A is approximately $n-I-$ weakly amenable.*

Proof. Let $D : A \to I^{(n)}$ be a continuous derivation. Clearly $\varphi : I^{(n)} \to I^{(n+2)}$ is a $A-$ module homomorphism, so $\varphi \circ D : A \to I^{(n+2)}$ is a derivation, and so $\varphi \circ D$ is approximately inner, that is, there exists a net $(i_\alpha) \subset I^{(n+2)}$ such that for every $a \in A$ $\varphi \circ D(a) = \lim_\alpha (a \cdot i_\alpha - i_\alpha \cdot a)$. Let $P : I^{(n+2)} \to I^{(n)}$ be the projection described above, then for every $a \in A$, we have $D(a) = P \circ \varphi \circ D(a) = \lim_\alpha (a \cdot P(i_\alpha) - P(i_\alpha) \cdot a)$ and so D is approximately inner. Thus A is approximately $n-I-$ weakly amenable.$\square$

Corollary 3.3. *Let A be a Banach algebra, and let $n \in \mathbb{N}$. Suppose A is approximately $(n+2)-$ideally amenable. Then A is approximately $n-$ideally amenable.*

Proof. Follows from Proposition 3.2.

Theorem 3.4. *Let A be a Banach algebra and I be a closed two-sided ideal in A with a bounded approximate identity. Suppose A is approximately $n-$ideally amenable. Then I is approximately n-ideally amenable.*

Proof. Since I has a bounded approximate identity, then by Proposition 3.1, every ideal of I is also an ideal of A. Let J be a closed two-sided ideal in I, and $D : I \to J^{(n)}$ be a derivation, then by [17, Proposition 2.1.6], D can be extended to a derivation $\tilde{D} : A \to J^{(n)}$ and since A is approximately $n-$ideally amenable, there is a net $(j_\alpha) \subset J^{(n)}$ such that $\tilde{D}(a) = \lim_\alpha (a \cdot j_\alpha - j_\alpha \cdot a) \quad (a \in A)$. Then $D(i) = \tilde{D}(i) = \lim_\alpha (i \cdot j_\alpha - j_\alpha \cdot i)$ for each $i \in I$, and so D is approximately inner. Thus I is approximately $n-$ideally amenable. □

Corollary 3.5 *Let A be a Banach algebra and I be a closed two-sided ideal in A with a bounded approximate identity. Suppose A is approximately permanently ideally amenable. Then I is approximately permanently ideally amenable.*

Proof. Follows from Theorem 3.4

Let A be a Banach algebra, we recall from [3] that a left [right] multiplier on A is an element L [R] in $L(A)$ such that $L(ab) = L(a)b$ [$R(ab) = aR(b)$] $(a, b \in A)$. A multiplier is a pair (L, R) where L and R are left and right multipliers on A respectively and $aL(b) = R(a)b \quad (a, b \in A)$. The set of all multipliers on A is denoted by $\mathcal{M}(A)$. It is called multiplier algebra of A.

Let A be a Banach algebra, let $B(A)$ be the Banach algebra of bounded linear operators on A and let $\mathcal{M}(A)$ be multiplier algebra of A. That is, $\mathcal{M}(A) = \{(L, R) : L, R \in L(A), L(ab) = L(a)b, R(ab) = aR(b), aL(b) = R(a)b, a, b \in A\}$.

As norm closed subalgebra of $B(A) \times B(A)^{op}$ (where $B(A)^{op}$ is the opposite algebra of $B(A)$), $\mathcal{M}(A)$ is a Banach algebra.

For $a \in A, L_a, R_a$ will denote the linear maps $b \to ab$ and $b \to ba$ on A, respectively. Then $(L_a, R_a) \in \mathcal{M}(A)$ with $\|L_a\| \leq \|a\|$ and $\|R_a\| \leq \|a\|$. It is easy to see that $a \to La$ (Resp. $a \to R_a$) is injective if and only if A is left (Resp. right) faithful. In particular, if A has a bounded approximate identity $\{e_\alpha\}$ of bound m, then $\|L_a\| \geq m^{-1}\|a\|$ and $\|R_a\| \geq m^{-1}\|a\|$ for all $a \in A$. In this case, A is identified with a norm closed ideal in $\mathcal{M}(A)$.

For each Banach A-bimodule X, $\mathcal{M}(A)$ acts on X through

$$T \cdot x = \lim_\alpha L(e_\alpha) \cdot x, \quad x \cdot T = \lim_\alpha x \cdot R(e_\alpha), \quad x \in X, T = (L, R) \in \mathcal{M}(A),$$

thus X is a Banach $\mathcal{M}(A)$-bimodule.

Also, A embeds in $\mathcal{M}(A)$ through $a \to (L_a, R_a)$, in the case where A is faithful, A is an ideal in $\mathcal{M}(A)$. Given a continuous derivation $D : \mathcal{M}(A) \to X'$, the restriction $\tilde{D}$ of D to A is also a continuous derivation. □

The following result is very useful and important.

Proposition 3.6. *Let A be a Banach algebra. Suppose A is faithful and has a bounded approximate identity, then every ideal in A is an ideal in $\mathcal{M}(A)$.*

Proof. This is Proposition 3.4(2) of [13].

Proposition 3.7. *Let A be a faithful Banach algebra with a bounded approximate identity and $\mathcal{M}(A)$ be the multiplier algebra of A. Suppose $\mathcal{M}(A)$ is approximately permanently ideally amenable, then A is approximately permanently ideally amenable.*

Proof. Follows from the fact that A is a closed two-sided ideal in $\mathcal{M}(A)$ and Theorem 3.4. □

Example 3.8. For a locally compact group G, $L^1(G)$ is approximately permanently ideally amenable if $\mathcal{M}(L^1(G)$ is approximately permanently ideally amenable.

Proof. Since $L^1(G)$ is left and right faithful and also has a bounded approximate identity, then the result follows from Proposition 3.7. □

Proposition 3.9. *Let A be a Banach algebra and let J be a closed two-sided ideal in A with a bounded approximate identity. Then for every closed two-sided ideal I in J, J is approximately $n-I$-weakly amenable if and only if A is approximately $n-I$-weakly amenable for $n \in \mathbb{N}$.*

Proof. Since J has a bounded approximate identity, then by Proposition 3.1, every ideal I of J is also an ideal of A. Let (e_α) be a bounded approximate identity for J, for every continuous derivation $D : J \to I^{(n)}$, the map $\tilde{D} : A \to I^{(n)}$ defined by $\tilde{D}(a) = w^* - (D(a \cdot e_\alpha) - a \cdot D(e_\alpha))$ is a continuous derivation by [17, Proposition 2.1.6]. By using the fact that if $D = 0$, then $\tilde{D} = 0$, since $JI = IJ = I$, then it follows easily that D is approximately inner if and only if $\tilde{D}$ is approximately inner. □

Corollary 3.10. *Let G be a locally compact group. For every closed two-sided ideal I of $L^1(G), L^1(G)$ is approximately $n - I$-weakly amenable if and only if $\mathcal{M}(L^1(G))$ is approximately $n - I$-weakly amenable for $n \in \mathbb{N}$.*

Proposition 3.11. *Let A be a Banach algebra and let $n \in \mathbb{N}$. Suppose $A^\sharp$ is approximately $n-$ ideally amenable. Then A is approximately $n-$ ideally amenable.*

Proof. Let I be a closed two-sided ideal in A, clearly I is an ideal in $A^\sharp$. Let $D : A \to I^{(n)}$ be a continuous derivation and define $\tilde{D} : A^\sharp \to I^{(n)}$ by $\tilde{D}(a + \alpha) = D(a), \quad (a \in A, \alpha \in \mathbb{C})$. It is easy to see that $\tilde{D}$ is a derivation. Since $A^\sharp$ is approximately $n-$ ideally amenable, $\tilde{D}$ is approximately inner, and so D is approximately inner. Thus A is approximately $n-$ ideally amenable. □

4. Segal algebras on locally compact groups. Segal algebras were first defined by Reifer for group algebras; see [16] for example. Let A be a Banach algebra. A Segal algebra is a subspace B of A such that

1. B is dense in A.

2. B is a left ideal of A.

3. B admits a norm $\|.\|_B$ under which it is complete and a contractive A-module.

4. B is essential A-module; $A \cdot B$ is $\|.\|_B$-dense in B.

   We further say that B is symmetric if it is also a contractive essential right A-module.

   In the case $A = L^1(G)$, where G is a locally compact group, we write $S(G)$ instead of B and further insist that

5. $S(G)$ is closed under left translations; $l_x f \in S(G)$ for all $x \in G$ and $f \in S(G)$ where $l_x f(y) = f(x^{-1}y)$ for $y \in G$. Condition 3. on $B = S(G)$ is equivalent to
   3.′ the map $(x, f) \to l_x f : G \times S(G) \to S(G)$ is continuous with $\|l_x f\|_S = \|f\|_S$ for all $x \in G$ and $f \in S(G)$.

Moreover, symmetry for $S(G)$ is equivalent to having that $S(G)$ is closed under right actions; $r_x f \in S(G)$ for $x \in G$ and $f \in S(G)$, where $r_x f(y) = f(y^{-1}x) \quad (y \in G)$, with the actions being continuous and isometric. For further information on Segal algebras see [18] and [15].

The symmetric Segal algebras include all Segal algebras on locally compact abelian groups. Also, every symmetric Segal algebra on a locally compact group G is a two-sided ideal in $L^1(G)$ and has an approximate identity see ([7]).

As in [7], we have the following theorem

Theorem 4.1. *Let G be an amenable group, and let $S(G)$ be a symmetric Segal algebra on G. Suppose X is a Banach $L^1(G)$-bimodule. Then every continuous derivation from $S(G)$ into X is approximately inner.*

Using Theorem 4.1, we have the following result

Theorem 4.2. *Let G be an amenable group. Then every symmetric Segal algebra $S(G)$ on G is approximately permanently ideally amenable.*

Proof. For every closed two-sided ideal I of $S(G), I$ is an ideal of $L^1(G)$ since $S(G)$ is an ideal in $L^1(G)$ and has an approximate identity by Proposition 3.1. Thus both I and $I^{(n)}$ are Banach $L^1(G)$-bimodules. And so for each $n \in \mathbb{N}$, every continuous derivation $D : S(G) \to I^{(n)}$ is approximately inner by Theorem 4.1 with $X = I^{(n)}$.

We recall from [6] that a Banach algebra A is essentially amenable if every derivation $D : A \to X'$ is inner for any neo-unital $A-$bimodule X. We also recall that an $A-$bimodule X is neo-unital if

$$X = A \cdot X \cdot A = \{a \cdot y \cdot b : a, b \in A, y \in X\}.$$

Certainly every essentially amenable Banach algebra is approximately essentially amenable. □

The following result is also from [6].

Theorem 4.3. *The following are essentially amenable:*

1. *any symmetric Segal algebra on G, for any amenable locally compact group G.*
2. *$l^p(S)$ for any set $S, 1 \le p < \infty$.*

In [5], Feichtinger defined, for any abelian group G, a Segal algebra $S_\circ(G)$ of $L^1(G)$. This Segal algebra is the minimal Segal algebra on $L^1(G)$ which is closed under pointwise multiplication by characters and on which multiplication by any character is isometry.

Theorem 4.4. *The Feichtinger's Segal algebra on an infinite compact abelian group is essentally amenable.*

Proof. For a compact and abelian group G, the Feichtinger's Segal algebra on G is

$$S_\circ(G) = \{f = \sum_{\gamma \in \hat{G}} C_\gamma \chi_\gamma : \|f\| = \sum |c_\gamma| < \infty\}$$

where χ_γ is the character of G associated with $\gamma \in \hat{G}$ and $\hat{G}$ is the dual group.

Hence, $S_\circ(G) \cong l^1(\hat{G})$, where $l^1(\hat{G})$ is equipped with the pointwise product. By Theorem 4.3 (2), $l^1(S)$ is essentially amenable for any set S, and so $S_\circ(G)$ is essentially amenable. □

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