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HARMONIC AND BIHARMONIC MAPS AT IAŞI

BY

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Abstract. We report on the achievements of the geometers from Iaşi in the field of harmonic and biharmonic maps.

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Nowadays, the theory of harmonic maps between Riemannian manifolds is a very important field of Riemannian geometry. The harmonic maps $\phi : (M,g) \to (N,h)$ are critical points of the *energy functional* E which is defined on the infinit dimensional manifold of the smooth maps from M to N,

$$E : C^\infty(M,N) \to \mathbb{R}, \quad E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 \, v_g.$$

Here $d\phi$ is the differential of the map ϕ and $\|\cdot\|$ denotes the Hilbert-Schmidt norm. The Euler-Lagrange equation associated to the energy E is given by the vanishing of the *tension field* $\tau(\phi) = \operatorname{trace} \nabla d\phi$ and it is a second order elliptic equation, as one can see from its local expression

$$-\Delta\phi^\alpha + {}^N\Gamma^\alpha_{\beta\sigma} \frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\sigma}{\partial x^j} g^{ij} = 0,$$

where Γ denotes the Christoffel symbols and Δ is the Laplacian acting on functions. From here raises the deep link established by the harmonic maps

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between the geometry of the domain and target manifolds, on one hand, and various problems from the theory of partial differential equations and calculus of variations, on the other.

Although the notion of harmonic map was known before 1964, the paper "Harmonic mappings of Riemannian manifolds", by Eells and Sampson, imposed this new research field (see [25]). In this famous work the authors proved, amongst other very interesting results, that if (M, g) and (N, h) are two compact Riemannian manifolds such that (N, h) has non-positive sectional curvature, then the homotopy class of any map $\psi : M \to N$ has a harmonic representative $\phi : (M, g) \to (N, h)$, which is a minimum of the energy functional E. This existence result was proved by using analytic techniques.

Since then many great mathematicians have contributed to the development of the harmonic maps theory from both the analytic and the geometric point of view: T. Aubin, P. Baird, F. Burstall, R. Caddeo, J. Jost, L. Lemaire, A. Lichnerowicz, Y. Ohnita, A. Sanini, R. Schoen, K. Uhlenbeck, H. Urakawa, C.M. Wood, J.C. Wood, Y.L. Xin, S.T. Yau, etc.. Among the Romanian mathematicians we recall V. Balan, S. Dragomir, C. Gherghe, S. Ianuş, R. Pantilie, and at Iaşi results in this field were obtained by A. Balmuş, C.L. Bejan, D. Fetcu, C. Oniciuc, V. Oproiu and N. Papaghiuc.

In the first paper on harmonic maps published by a geometer from Iaşi, which appeared in 1988, Bejan established the first and second variation formulas for harmonic maps between semi-Riemannian manifolds (see [13]).

Then followed the study of the harmonicity in the context of the tangent and cotangent bundles. Given a smooth map $\phi : (M, g) \to (N, h)$, the harmonicity of its differential $d\phi : TM \to TN$ was investigated by Oproiu in [55], where TM and TN were endowed with certain semi-Riemannian metrics. Further, a different direction was followed by the same author in [56], where the harmonicity of 1-forms, thought of as maps from (M, g) to the cotangent bundle T^*M, was studied. There was considered on T^*M a semi-Riemannian metric G defined by

$$G(\theta^V, \omega^V) = 0, \quad G(\theta^V, X^H) = \theta(X), \quad G(X^H, Y^H) = c(X, Y),$$

where c is a symmetric $(0, 2)$-tensor field on M, θ^V is the vertical lift of the 1-form θ and X^H is the horizontal lift of the vector field X. The horizontal distribution on T^*M was the usual one. The author obtained

Theorem ([56]). *(i) If η is a geodesic 1-form and c is a harmonic symmetric $(0,2)$-tensor field on M, then $\eta : (M, g) \to (T^*M, G)$ is a harmonic map.*

(ii) If c is the Ricci tensor field of (M, g), then η is harmonic if and only if η is a geodesic 1-form.

(iii) If θ is a 1-form on M and c is the Lie derivative of g with respect to $\theta^\sharp$, then η is harmonic if $\eta + \theta$ is geodesic; $-\theta$ is harmonic.

(iv) If (M, J, g) is a Kähler manifold and η is a 1-form on M such that $\eta^\sharp$ is a holomorphic vector field on M, then η is harmonic if and only if c is a harmonic tensor field.

An extended chapter in the theory of harmonic maps is represented by the study of the harmonicity of vector fields $\xi \in C(TM)$ thought of as maps from the base manifold (M, g) to its tangent bundle (TM, G), where G is a (semi-)Riemannian metric. The first papers on this subject were those written by Nouhaud, Ishihara and Piu (see [46, 37, 60]), in which the complete lift of the metric g, or the Sasaki metric, were considered on TM. M.P. Piu also studied the harmonicity of unit vector fields thought of as maps in the unit tangent bundle $T_1M = \{v \in TM : g(v, v) = 1\}$ equipped with the Sasaki metric. Following the same direction, one studied the harmonicity of vector fields and unit vector fields when the tangent bundle was endowed with the Cheeger-Gromoll metric ([47]), with metrics of natural lift type (see [12, 50]), and with metrics of Sasaki type or of complete lift type (see [48, 49]). In the last two cases the metrics were constructed by considering on TM a new horizontal distribution, different from the usual one, i.e. that induced by the Levi-Civita connection of g.

The results above are, in general, quite rigid, since the vector fields are harmonic maps when they are parallel or Killing. For example,

Theorem ([47]). *Let $\xi \in C(TM)$ be a unit vector field on the compact manifold (M, g). Consider G to be the Cheeger-Gromoll metric on TM. Then $\xi : (M, g) \to (TM, G)$ is a harmonic map if and only if ξ is parallel. Moreover, if ξ is a Killing unit vector field and (M, g) has constant sectional curvature, then $\xi : (M, g) \to (T_1M, G)$ is a harmonic map.*

Somehow contrary to these rigid results, the following was obtained.

Theorem ([48]). *For any vector field $\xi \in C(TM)$, a Riemannian metric G of Sasaki type on TM can be constructed such that ξ becomes a harmonic map.*

The proof is based on the fact that if a vector field $X : M \to TM$ is a Riemannian immersion, where TM is endowed with a Riemannian metric such that the canonical projection $\pi : TM \to M$ is a Riemannian submersion, then X is a totally geodesic map, and thus harmonic. Then, for a fixed vector field ξ, the required metric G was obtained by considering on TM a new horizontal distribution H' depending on ξ: $H' = \mathrm{span}\{\frac{\delta'}{\delta x^i} = \frac{\partial}{\partial x^i} - (\Gamma^j_{ik} y^k + B^j_i(x))\frac{\partial}{\partial y^j}\}$, where $B = -\nabla\xi$ and (x^i, y^j) denote the canonical coordinates on TM. We recall here that the usual horizontal distribution H is generated by $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^j_{ik} y^k \frac{\partial}{\partial y^j}$. The metric G has the expression

$$G(X^V, Y^V) = g(X, Y), \quad G(X^H, Y^V) = g(B(X), Y),$$
$$G(X^H, Y^H) = g(X, Y) + g(B(X), B(Y)),$$

or, equivalently,

$$G(X^V, Y^V) = g(X, Y), \quad G(X^{H'}, Y^V) = 0, \quad G(X^{H'}, Y^{H'}) = g(X, Y),$$

where X^V, X^H and $X^{H'}$ denote the vertical lift, the horizontal lift with respect to H and the horizontal lift with respect to H' of the vector field X.

A particular interest was paid to harmonic maps between manifolds equipped with certain structures. A well known result of Lichnerowicz states that any holomorphic map with compact domain between two almost Kähler manifolds is harmonic with the minimum energy in its homotopy class (see [40]). Further results concerning the theory of harmonic maps were obtained in the context of contact geometry (see, for example, [32, 34]). A notion which generalizes both complex and contact manifolds is that of framed φ-manifolds, and the harmonic maps in this general setup were studied (see, for example, [26]). The case of complex Sasakian structures was also analyzed. In this context, three classes of harmonic maps between complex Sasakian manifolds were found.

Theorem ([27]). *Consider M and N to be two complex Sasakian manifolds, (J, G, H, u, v, U, V, g) and $(J', G', H', u', v', U', V', h)$ being the corresponding structures. If $\phi : M \to N$ is a smooth map satisfying one of the following conditions:*

(*i*) $d\phi = G'\, d\phi\, G + H'\, d\phi\, H$,

(*ii*) $d\phi\, G = G'\, d\phi$,

(*iii*) $d\phi\, H = H'\, d\phi$,

then ϕ is a harmonic map.

As harmonic maps arise from a variational principle, an important problem is the study of the second variation for the energy functional. In this regard, the stability of the identity map of a complex Sasakian manifold was investigated. Let $(M, J, G, H, u, v, U, V, g)$ be a compact $(4m+2)$-dimensional complex Sasakian manifold with constant GH-sectional curvature c and $\mathbf{1} : M \to M$ the identity map. By computing the Hessian of the identity map on gradient vector fields corresponding to the first non-zero eigenvalue λ_1 of the Laplacian on M, the following was obtained

Theorem ([27]). *Let M be a compact $(4m+2)$-dimensional complex Sasakian manifold with constant GH-sectional curvature $c \leq \frac{m-2}{m+2}$. If the first eigenvalue λ_1 of the Laplacian Δ, acting on $C^\infty(M)$, satisfies $\lambda_1 < 2(mc + 3m + 2c + 2)$, then the identity map $\mathbf{1} : M \to M$ is unstable.*

Harmonic maps and morphisms between almost symplectic manifolds, almost para-Hermitian manifolds or tangent bundles were studied in [1, 14, 16]. The authors also introduced and studied in [15] the notion of f-pluriharmonicity, and related it to $\pm f$-holomorphicity. In particular, they generalized a known result of Sampson, obtaining

Theorem ([15]). *If $\phi : M \to N$ is a pluriharmonic map from a Kähler manifold into a Riemannian manifold with strictly negative* (*resp. strictly positive*) *complex sectional curvature at every point, then* $\operatorname{rank}(d\phi) \leq 2$.

The research in the field of Riemannian geometry in the last 10-15 years was partly characterized by the study of some fourth order partial differential equations. Whether the origin of these equations is analytic, as in the case of the Paneitz operator, or geometric, as in the case of Willmore surfaces, they represent a generalization of the concept of harmonic map. A natural extension of the harmonic maps was suggested by Eells and Sampson themselves in [25].

The first results in this field were obtained by Jiang in 1986 (see [38]). He derived the Euler-Lagrange equation of the *bienergy functional*

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \quad E_2(\phi) = \frac{1}{2} \int_M \|\tau(\phi)\|^2 \, v_g.$$

This equation characterizes *biharmonic maps* and is given by the vanishing of the *bitension field*

$$\tau_2(\phi) = -\Delta\tau(\phi) - \operatorname{trace} R^N(d\phi, \tau(\phi))d\phi,$$

where Δ is the Laplacian in the induced bundle $\phi^{-1}(TN)$ and R^N is the curvature tensor field of (N, h). The *biharmonic equation* $\tau_2(\phi) = 0$ is a fourth order elliptic equation. Any harmonic map is a minimum for the bi-energy functional and thus biharmonic. Therefore only proper-biharmonic, i.e. non-harmonic biharmonic, maps are studied.

The theory of biharmonic maps has two main directions: one concerns the analytic properties (see [20, 39, 45, 64]) and the other deals with geometric aspects. In the following we shall only refer to the latter.

The harmonicity of the identity map $\mathbf{1} : (M, g) \to (M, g)$ under a conformal deformation of the domain or target manifold is trivial: for $m = 2$ it is harmonic for any conformal factor, and for $m > 2$ it is harmonic if and only if homothetic. On the contrary, the biharmonicity of the identity map $\mathbf{1} : (M, e^{2\rho}g) \to (M, g)$, or $\mathbf{1} : (M, g) \to (M, e^{2\rho}g)$, turned out to be very interesting (see [2, 5]). There exists a surprising link between the biharmonicity of the identity map and the property of the function ρ of being isoparametric, when (M, g) is an Einstein manifold.

Theorem ([5]). *Let (M^m, g), $m > 2$, $m \neq 4$, be an Einstein space and $\widetilde{g} = e^{2\rho}g$ a metric conformally equivalent to g such that $\mathbf{1} : (M, g) \to (M, \widetilde{g})$ is biharmonic. Then $\rho : M \to \mathbb{R}$ is isoparametric. Conversely, if $f : M \to \mathbb{R}$ is isoparametric, then away from critical points of f, there exists a reparametrization $\rho = \rho \circ f$ such that $\mathbf{1} : (M, g) \to (M, \widetilde{g})$ is biharmonic.*

This problem was taken further: one can ask whether a conformal change of the domain or the target metric of a harmonic map $\psi : (M, g) \to (N, h)$ or a (harmonic) conformal submersion could render it proper-biharmonic. Some interesting answers can be found in [5, 59].

Then biharmonic conformal diffeomorphisms were studied and it was proved that in dimension 6 any biharmonic conformal diffeomorphism has biharmonic inverse (see [3]). Biharmonic conformal immersions were also studied (see [57]) and a series of examples of such immersions of the cylinder into $\mathbb{R}^3$ and of the four-dimensional Euclidean space into the sphere and the hyperbolic space were constructed.

In [52] one studied the biharmonicity of Riemannian submersions between Riemannian manifolds and some non-existence results were obtained in the case of target manifolds with non-positive Ricci curvature. On the other hand, for a Riemannian submersion with basic tension field, i.e. $\tau(\phi) = \xi \circ \phi$ with ξ a vector field on the target manifold, if ξ is a unit

Killing vector field, then the map ϕ is proper-biharmonic. Some examples of such proper-biharmonic Riemannian submersions were constructed.

New examples of proper-biharmonic maps were also obtained by using warped product manifolds (see [7]).

An important problem for the theory of biharmonic maps consists in the study of proper-biharmonic submanifolds, i.e. submanifolds with non-harmonic (non-minimal) biharmonic inclusion map. The space forms were the first ambient spaces investigated regarding to the existence and classification of proper-biharmonic submanifolds. All the known results obtained for proper-biharmonic submanifolds in Euclidean and hyperbolic spaces were non-existence results (see, for example, [8, 22, 24, 33]). Therefore, the following was conjectured.

Generalized Chen conjecture. *Biharmonic submanifolds of a non-positive sectional curvature manifold are minimal.*

The situation proved to be quite different in the case of the unit Euclidean sphere $\mathbb{S}^n$. The hypersphere of radius $1/\sqrt{2}$, i.e. $\mathbb{S}^{n-1}(1/\sqrt{2})$, and the generalized Clifford torus $\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2})$, $n_1 + n_2 = n - 1$ and $n_1 \neq n_2$, are the main examples of proper-biharmonic submanifolds in $\mathbb{S}^n$ (see [17, 38]).

The biharmonic equation for submanifolds in spheres assumes a simple form $\Delta H = mH$, where H denotes the mean curvature vector field of the submanifold. Nevertheless, the key result which allowed the development of further studies was the seemingly more complicated characterization of proper-biharmonic submanifolds in spheres in terms of the second fundamental form and of the mean curvature vector field, obtained by splitting the biharmonic equation in its tangent and normal components (see [21, 52]),

$$\begin{cases} \Delta^{\perp} H + \operatorname{trace} B(\cdot, A_H(\cdot)) - mH = 0 \\ 4 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + m \operatorname{grad}(\|H\|^2) = 0. \end{cases}$$

Then, the efforts were concentrated on the construction of examples, on one hand, and on the classification of proper-biharmonic submanifolds in spheres, on the other.

The main examples presented above suggested two methods of construction for proper-biharmonic submanifolds in spheres.

Theorem ([18])**.** *Let M be a minimal submanifold of $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$. Then M is proper-biharmonic in $\mathbb{S}^n$ if and only if $a = 1/\sqrt{2}$.*

Theorem ([18]). *Let $M_1^{m_1}$ and $M_2^{m_2}$ be two minimal submanifolds of $\mathbb{S}^{n_1}(a_1)$ and $\mathbb{S}^{n_2}(a_2)$, respectively, where $n_1+n_2=n-1$, $a_1^2+a_2^2=1$. Then $M_1 \times M_2$ is proper-biharmonic in $\mathbb{S}^n$ if and only if $a_1=a_2=1/\sqrt{2}$ and $m_1 \neq m_2$.*

The classification of proper-biharmonic submanifolds in spheres was initiated with the case of $\mathbb{S}^3$.

Theorem ([17]). *An arc length parameterized curve in $\mathbb{S}^3$ is proper-biharmonic if and only if it is a geodesic of the minimal Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2}) \subset \mathbb{S}^3$ with slope different from ± 1. A surface is proper-biharmonic in $\mathbb{S}^3$ if and only if it is an open part of $\mathbb{S}^2(1/\sqrt{2}) \subset \mathbb{S}^3$.*

Further, all proper-biharmonic curves in $\mathbb{S}^n$ were determined (see [18]). It was proved that, up to a totally geodesic embedding, these are the proper-biharmonic curves of $\mathbb{S}^3$.

The biharmonic equation simplifies in codimension one, and results on proper-biharmonic hypersurfaces in spheres were obtained by dividing the study according to the number of their principal curvatures. If a hypersurface in $\mathbb{S}^{m+1}$ is proper-biharmonic and umbilical, i.e. all its principal curvatures are equal, then it is an open part of $\mathbb{S}^m(1/\sqrt{2})$. For what concerns proper-biharmonic hypersurfaces with at most two distinct principal curvatures, the following was obtained.

Theorem ([8]). *A hypersurface with at most two distinct principal curvatures is proper-biharmonic in $\mathbb{S}^{m+1}$ if and only if it is an open part of $\mathbb{S}^m(1/\sqrt{2})$ or of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1+m_2=m$, $m_1 \neq m_2$.*

The case of biharmonic hypersurfaces with at most three distinct principal curvatures was approached by first proving that proper-biharmonic hypersurfaces with constant mean curvature in $\mathbb{S}^{m+1}$, $m \geq 2$, have positive constant scalar curvature (see [8]). Then, using certain results on isoparametric hypersurfaces, it was proved that there exist no compact proper-biharmonic hypersurfaces of constant mean curvature and with three distinct principal curvatures everywhere in the unit Euclidean sphere (see [9]).

The fact that a proper-biharmonic hypersurface in $\mathbb{S}^4$ must have constant mean curvature (detailed proofs can be found in [6]) and thus constant scalar curvature, and a result in [19], concerning isoparametric hypersurfaces, led to the full classification of compact proper-biharmonic hypersurfaces in $\mathbb{S}^4$.

Theorem ([9]). *The only compact proper-biharmonic hypersurfaces in* $\mathbb{S}^4$ *are the hypersphere* $\mathbb{S}^3(1/\sqrt{2})$ *and the torus* $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$.

In higher codimension, in order to obtain classification results, conditions on the mean curvature vector field were imposed. In this direction, there were obtained some results which, in particular, allowed the classification of proper-biharmonic constant mean curvature surfaces in $\mathbb{S}^4$.

Theorem ([10]). *The only proper-biharmonic constant mean curvature surfaces in* $\mathbb{S}^4$ *are the minimal surfaces in* $\mathbb{S}^3(1/\sqrt{2})$.

Among other properties, it was proved in [53] that proper-biharmonic submanifolds of constant mean curvature $\|H\|$ in $\mathbb{S}^n$ satisfy $\|H\| \in (0, 1]$, with $\|H\| = 1$ if and only if they are minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$. Taking this further, the type (in the sense of B.-Y. Chen) of compact proper-biharmonic submanifolds of constant mean curvature in $\mathbb{S}^n$ was studied and it turned out that, depending on the value of their mean curvature, these are of 1-type or of 2-type as submanifolds of $\mathbb{R}^{n+1}$ (see [8, 11]). We mention here that spherical submanifolds of low finite type were extensively studied, M. Barros and B.-Y. Chen saying: *mass-symmetric, 2-type submanifolds of* $\mathbb{S}^n$ *are the "simplest" submanifolds of* $\mathbb{R}^{n+1}$*, next to minimal submanifolds of* $\mathbb{S}^n$.

Conclusively, although important progress has been made in recent years, the classification of proper-biharmonic submanifolds in spheres is still an open problem. All the properties and the classification results listed above suggested the following two conjectures.

Conjecture ([8]). *The only proper-biharmonic hypersurfaces in* $\mathbb{S}^{m+1}$ *are the open parts of hyperspheres* $\mathbb{S}^m(1/\sqrt{2})$ *or of generalized Clifford tori* $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

Conjecture ([8]). *Any proper-biharmonic submanifold in* $\mathbb{S}^n$ *has constant mean curvature.*

After space forms, a series of non-constant sectional curvature spaces were considered as ambient spaces for the study of proper-biharmonic submanifolds (see, for example, [4, 23, 30, 31, 35, 36, 54, 58, 61]). In this respect, one of the research directions was the study of proper-biharmonic submanifolds in Sasakian space forms.

The parametric equations of all proper-biharmonic Legendre curves and Hopf cylinders in the 3-dimensional unit sphere endowed with a deformed

Sasakian structure were obtained (see [29]). Then all proper-biharmonic Legendre curves in arbitrary dimensional Sasakian spaces were classified. It was proved that they are helices (see [30]), that certain angles involving their Frenet frame field are constant and, by using the $(2n+1)$-dimensional unit sphere endowed with its canonical and deformed Sasakian structures (see [63]), the explicit parametric equations of such curves were given.

Considering the flow of the characteristic vector field of a Sasakian space form, proper-biharmonic anti-invariant submanifolds were obtained from proper-biharmonic integral submanifolds. Then all proper-biharmonic surfaces invariant under the flow-action of the characteristic vector field in a Sasakian space form were found.

For the Boothby-Wang fibration $\pi : N \to \bar{N}$ of a strictly regular Sasakian space form, the characterization of proper-biharmonic Hopf cylinders over submanifolds of $\bar{N}$ was obtained. The Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ is a well-known example of Boothby-Wang fibration and, using the Takagi list (see [62]), all proper-biharmonic Hopf cylinders over homogeneous real hypersurfaces in the complex projective space $\mathbb{C}P^n$ were given (see [31]).

Results on proper-biharmonic submanifolds of a complex space form and, in particular, of the complex projective space were also proved (see [28]). There was obtained the relation between the bitension field of the inclusion $\bar{\jmath} : \overline{M} \to \mathbb{C}P^n$ of a submanifold in $\mathbb{C}P^n$ and the bitension field of the inclusion $\jmath : M \to \mathbb{S}^{2n+1}$ of the corresponding Hopf cylinder in $\mathbb{S}^{2n+1}$,

$$(\tau_2(\bar{\jmath}))^H = \tau_2(\jmath) - 4\hat{J}(\hat{J}\tau(\jmath))^\top + 2\,\mathrm{div}((\hat{J}\tau(\jmath))^\top)\xi,$$

where $\hat{J}$ denotes the canonical complex structure of $\mathbb{R}^{2n+2}$ and ξ is the Hopf vector field on $\mathbb{S}^{2n+1}$. Then, using the following theorem, new families of proper-biharmonic submanifolds of $\mathbb{C}P^n$ were found.

Theorem ([28]). *Let $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ be the Hopf map. Let $M = M_1^{m_1} \times M_2^{m_2}$ be the product of two minimal submanifolds of $\mathbb{S}^{2p+1}(a)$ and $\mathbb{S}^{2q+1}(b)$, respectively, $p+q = n-1$, $a^2+b^2 = 1$. Assume that M is invariant under the action of the one-parameter group of isometries generated by the Hopf vector field ξ on $\mathbb{S}^{2n+1}$. Then $\pi(M)$ is a proper-biharmonic submanifold of $\mathbb{C}P^n$ if and only if M is (-4)-biharmonic, that is*

$$\frac{a}{b}m_2 - \frac{b}{a}m_1 \neq 0 \quad \textit{and} \quad \frac{b^2}{a^2}m_1 + \frac{a^2}{b^2}m_2 = 4 + m_1 + m_2.$$

Proper-biharmonic curves in $\mathbb{C}P^n$ were characterized in terms of their curvatures and complex torsions and, using the classification of holomorphic helices of $\mathbb{C}P^2$, all proper-biharmonic curves in $\mathbb{C}P^2$ were determined.

The second variation for the bienergy functional was first studied in a general setting in [38] and then concrete results on the stability of biharmonic maps to spheres were obtained (see [42, 43, 51]). Since biharmonic Riemannian immersions in spheres are stable, i.e. of index 0, if and only if they are minimal, the aim was to evaluate the (biharmonic) index of the proper-biharmonic Riemannian immersions in spheres. A first result was the following.

Theorem ([42]). *The index of the proper-biharmonic hypersurface $\mathbb{S}^m(1/\sqrt{2})$ in $\mathbb{S}^{m+1}$ is exactly one.*

Theorem ([41]). *Let $\psi : \mathbb{S}^m(r) \to \mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$ be a minimal immersion, $r \geq 1/\sqrt{2}$, and $\phi : \mathbb{S}^m(r) \to \mathbb{S}^n$ the corresponding biharmonic map. Then*

(i) $\mathrm{index}(\phi) \geq m+2$ *if either* $r^2 > \frac{1+\sqrt{m^2+1}}{2m}$, *or* $m \geq 5$ *and* $r^2 > \frac{(m-2)^2}{2m(m-4)}$,

(ii) $\mathrm{index}(\phi) \geq 2m+3$ *if* $m \geq 5$ *and* $r^2 > \frac{(m-2)(1+\sqrt{m^2-4m+1})}{2m(m-4)}$.

When ψ is the minimal generalized Veronese map one gets

Corollary ([42]). *The biharmonic map derived from the generalized Veronese map $\psi : \mathbb{S}^m(\sqrt{\frac{m+1}{m}}) \to \mathbb{S}^{m+p}(\frac{1}{\sqrt{2}})$, $p = \frac{(m-1)(m+2)}{2}$, has* index *at least $m+2$, when $m \leq 4$, and at least $2m+3$, when $m > 4$.*

Instability results were obtained not only for Riemannian immersions. Accordingly, for the usual harmonic Hopf map $\psi : \mathbb{S}^3(\sqrt{2}) \to \mathbb{S}^2(1/\sqrt{2})$ with harmonic index 4 the following was obtained.

Theorem ([43]). *The index of the proper-biharmonic subimmersion $\phi : \mathbb{S}^3(\sqrt{2}) \to \mathbb{S}^3$, which is derived from the usual harmonic Hopf map $\psi : \mathbb{S}^3(\sqrt{2}) \to \mathbb{S}^2(1/\sqrt{2})$, is at least* 11.

For a fixed map $\phi : M \to (N, h)$ one can consider the bienergy functional defined on the set of all metrics on the domain manifold, $F(g) = E_2(\phi)$. It was proved in [44] that the critical points of this new functional are given by the vanishing of the stress-bienergy tensor

$$\begin{aligned} S_2(X,Y) &= \frac{1}{2}\|\tau(\phi)\|^2\langle X,Y\rangle + \langle d\phi, \nabla\tau(\phi)\rangle\langle X,Y\rangle \\ &\quad -\langle d\phi(X), \nabla_Y\tau(\phi)\rangle - \langle d\phi(Y), \nabla_X\tau(\phi)\rangle, \end{aligned}$$

which was first introduced by G.Y. Jiang. The study of the stress-bienergy tensor allowed the construction of new examples of proper-biharmonic maps. For instance,

Proposition ([44]). *Let* $\mathbf{1} : (M^m, g) \to (M^m, \widetilde{g} = e^{2\rho} g)$ *be the identity map,* $m \neq 2$, *M non-compact and* $\rho \in C^\infty(M)$.

(*i*) *If* $\rho = \ln \sqrt{f}$, *f a non-constant affine positive function on* $(M, \widetilde{g})$, *then* $\mathbf{1}$ *is proper-biharmonic.*

(*ii*) *If* ρ *is a non-constant affine function on* $(M, \widetilde{g})$, *then* $\mathbf{1}$ *is proper-biharmonic if and only if* $m = 4$.

Important links between the stress-bienergy tensor of submanifolds and their property of being parallel or pseudo-umbilical were established.

Theorem ([44]). *A hypersurface* $\mathbf{i} : M^m \to N^{m+1}$, $m \neq 4$, *has* $\nabla S_2 = 0$ *if and only if it is parallel.*

Theorem ([44]). *Let* $\phi : (M^m, g) \to (N, h)$ *be a non-minimal Riemannian immersion. Then* $S_2 = c\, g$ *if and only if* ϕ *is pseudo-umbilical and, if* $m \neq 4$, *the norm of* $\tau(\phi)$ *is constant.*

Furthermore, interesting relations between the stress-bienergy tensor of a submanifold in the Euclidean space and its associated Gauss map were proved.

Harmonic and biharmonic maps have represented two of the main research directions for geometers from Iaşi. Recent research projects foresee new results on the harmonicity of vector fields and on the classification and properties of biharmonic submanifolds in complex projective spaces and Sasakian space forms.

REFERENCES

1. Baird, P.; Bejan, C.L. – *Quasi-harmonic maps between almost symplectic manifolds*, Harmonic morphisms, harmonic maps, and related topics, Brest 1997, 75–95.
2. Baird, P.; Kamissoko, D. – *On constructing biharmonic maps and metrics*, Ann. Global Anal. Geom., 23 (2003), 65–75.

3. Baird, P.; Fardoun, A.; Ouakkas, S. – *Conformal and semi-conformal biharmonic maps*, Ann. Global Anal. Geom., 34 (2008), 403–414.

4. Balmuş, A. – *On the biharmonic curves of the Euclidean and Berger* 3*-dimensional spheres*, Sci. Ann. Univ. Agric. Sci. Vet. Med., 47 (2004), 87–96.

5. Balmuş, A. – *Biharmonic properties and conformal changes*, An. Ştiinţ. Univ. Al.I. Cuza Iaşi. Mat. (N.S.), 50 (2004), 361–372.

6. Balmuş, A. – *Biharmonic Maps and Submanifolds*, Ph.D. Thesis, DGDS Monographs 10, Geometry Balkan Press, Bucureşti, 2009, `http://www.mathem.pub.ro/dgds/mono/dgdsmono.htm`.

7. Balmuş, A.; Montaldo, S.; Oniciuc, C. – *Biharmonic maps between warped product manifolds*, J. Geom. Phys., 57 (2007), 449–466.

8. Balmuş, A.; Montaldo, S.; Oniciuc, C. – *Classification results for biharmonic submanifolds in spheres*, Israel J. Math., 168 (2008), 201–220.

9. Balmuş, A.; Montaldo, S.; Oniciuc, C. – *Biharmonic hypersurfaces in 4-dimensional space forms*, Math. Nachr., to appear.

10. Balmuş, A.; Oniciuc, C. – *Biharmonic surfaces of* $\mathbb{S}^4$, Kyushu J. Math., 63 (2009), 339–345.

11. Balmuş, A.; Oniciuc, C. – *Properties of biharmonic submanifolds in spheres*, Proceedings of the 11-th International Conference on Geometry, Integrability and Quantization, Varna 2009, Bulgaria, to appear.

12. Balmuş, A.; Oniciuc, C.; Papaghiuc, N. – *Harmonic properties on the tangent bundle*, An. Univ. Vest Timisoara, Ser. Mat.-Inform., 42 (2004), 17–28.

13. Bejan, C.L. – *On harmonic maps between semi-Riemannian manifolds*, The XVIII-th National Conference on Geometry and Topology (Oradea, 1987), Univ. "Babeş-Bolyai", Cluj-Napoca (1988), 31–34.

14. Bejan, C.L.; Benyounes, M. – *Harmonic maps between almost para-Hermitian manifolds*, New developments in differential geometry, Budapest, 1996, 67–76.

15. Bejan, C.L.; Benyounes, M. – *f-pluriharmonic maps on manifolds with f-structures*, SUT J. Math., 39 (2003), 161–170.

16. Bejan, C.L.; Binh, T.Q. – *Harmonic maps and morphisms from the tangent bundle*, Acta Sci. Math. (Szeged), 66 (2000), 385–401.

17. Caddeo, R.; Montaldo, S.; Oniciuc, C. – *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math., 12 (2001), 867–876.

18. Caddeo, R.; Montaldo, S.; Oniciuc, C. – *Biharmonic submanifolds in spheres*, Israel J. Math., 130 (2002), 109–123.

19. Chang, S. – *A closed hypersurface with constant scalar curvature and mean curvature in* $\mathbb{S}^4$ *is isoparametric*, Comm. Anal. Geom., 1 (1993), 71–100.

20. Chang, S.-Y.A.; Wang, L.; Yang, P. – *A regularity theory of biharmonic maps*, Comm. Pure Appl. Math., 52 (1999), 1113–1137.

21. Chen, B.-Y. – *Total Mean Curvature and Submanifolds of Finite Type*. Series in Pure Mathematics, 1. World Scientific Publishing Co., Singapore, 1984.

22. Chen, B.-Y.; Ishikawa, S. – *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math., 52 (1998), 167–185.

23. Chiang, Y.-J.; Sun, H. – 2-*harmonic totally real submanifolds in a complex projective space*, Bull. Inst. Math. Acad. Sinica, 27 (1999), 99–107.

24. Dimitric, I. – *Submanifolds of* $\mathbb{E}^m$ *with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinica, 20 (1992), 53–65.

25. Eells, J.; Sampson, J.H. – *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., 86 (1964), 109–160.

26. Fetcu, D. – *Properties of Harmonic Mappings and Morphisms*, Ph.D. Thesis, DGDS Monographs 5, Geometry Balkan Press, Bucureşti, 2005, `http://www.mathem.pub.ro/dgds/mono/dgdsmono.htm`.

27. Fetcu, D. – *Harmonic maps between complex Sasakian manifolds*, Rend. Semin. Mat. Univ. Politec. Torino, 64 (2006), 319–329.

28. Fetcu, D.; Loubeau, E.; Montaldo, S.; Oniciuc,C. – *Biharmonic submanifolds of* $\mathbb{C}P^n$, Math. Z., to appear.

29. Fetcu, D.; Oniciuc, C. – *Explicit formulas for biharmonic submanifolds in non-Euclidean 3-spheres*, Abh. Math. Sem. Univ. Hamburg, 77 (2007), 179–190.

30. Fetcu, D.; Oniciuc, C. – *Explicit formulas for biharmonic submanifolds in Sasakian space forms*, Pacific J. Math., 240 (2009), 85–107.

31. Fetcu, D.; Oniciuc, C. – *Biharmonic hypersurfaces in Sasakian space forms*, Differential Geom. Appl., 27 (2009), 713-722.

32. Gherghe, C.; Ianuş, S.; Pastore, A.M. – *CR-manifolds, harmonic maps and stability*, J. Geom., 71 (2001), 42–53.

33. Hasanis, Th.; Vlachos, Th. – *Hypersurfaces in* $\mathbb{E}^4$ *with harmonic mean curvature vector field*, Math. Nachr., 172 (1995), 145–169.

34. Ianuş, S.; Pastore, A.M. – *Harmonic maps on contact metric manifolds*, Ann. Math. Blaise Pascal, 2 (1995), 43–53.

35. Ichiyama, T.; Inoguchi, J.; Urakawa, H. – *Bi-harmonic maps and bi-Yang-Mills fields*, Note Mat., 28 (suppl.n.1) (2008), 233–275.

36. Inoguchi, J. – *Submanifolds with harmonic mean curvature vector field in contact 3-manifolds*, Colloq. Math., 100 (2004), 163–179.

37. Ishihara, S. – *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ., 13 (1979), 23–27.

38. Jiang, G.Y. – *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A, 7 (1986), 389–402.

39. Lamm, T. – *Heat flow for extrinsic biharmonic maps with small initial energy*, Ann. Global Anal. Geom., 26 (2004), 369–384.

40. Lichnerowicz, A. – *Applications harmonique et variétés kählteriennes*, Symp. Math. III (INDAM, Rome 1969), 341–402.

41. Loubeau, E.; Oniciuc, C. – *The stability of biharmonic maps*, preprint.

42. Loubeau, E.; Oniciuc, C. – *The index of biharmonic maps in spheres*, Compositio Math., 141 (2005), 729–745.

43. Loubeau, E.; Oniciuc, C. – *On the biharmonic and harmonic indices of the Hopf map*, Trans. Amer. Math. Soc., 359 (2007), 5239–5256.

44. Loubeau, E.; Montaldo, S.; Oniciuc, C. – *The stress-energy tensor for biharmonic maps*, Math. Z., 259 (2008), 503–524.

45. Moser, R. – *A variational problem pertaining to biharmonic maps*, Comm. in Partial Differential Equations, 33 (2008), 1654–1689.

46. Nouhaud, O. – *Applications harmoniques d'une variété riemanniene dans son fibré tangent. Généralization*, Comp. Rend. Acad. Sci. Paris, 284 (1977), 815–818.

47. Oniciuc, C. – *The tangent bundles and harmonicity*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), 43 (1997), 151–172.

48. Oniciuc, C. – *Nonlinear connections on tangent bundle and harmonicity*, Ital. J. Pure Appl. Math., 6 (1999), 109–122.

49. Oniciuc, C. – *Pseudo-Riemannian metrics on tangent bundle and harmonic problems*, Bull. Belg. Math. Soc. Simon Stevin, 7 (2000), 443–454.

50. Oniciuc, C. – *Harmonic sections in the unitary tangent bundle*, Demonstratio Math., 34 (2001), 681–692.

51. Oniciuc, C. – *On the second variation formula for biharmonic maps to a sphere*, Publ. Math. Debrecen, 61 (2002), 613–622.

52. Oniciuc, C. – *Biharmonic maps between Riemannian manifolds*, An. Ştiinţ. Univ. Al.I. Cuza Iaşi Mat (N.S.), 48 (2002), 237–248.

53. Oniciuc, C. – *Tangency and Harmonicity Properties*, Ph.D. Thesis, DGDS Monographs 3, Geometry Balkan Press, Bucureşti, 2003, `http://www.mathem.pub.ro/dgds/mono/dgdsmono.htm`.

54. Oniciuc, C. – *New examples of biharmonic maps in spheres*, Colloq. Math., 97 (2003), 131–139.

55. Oproiu, V. – *Harmonic maps between tangent bundles*, Rend. Sem. Mat. Univ. Politec. Torino, 47 (1989), 47–55.

56. Oproiu, V. – *On the harmonic sections of cotangent bundles*, Rend. Sem. Fac. Sci. Univ. Cagliari, 59 (1989), 177–184.

57. Ou, Y.-L. – *On conformal biharmonic immersions*, Ann. Global Anal. Geom., 36 (2009), 133-142.

58. Ou, Y.-L. – *Biharmonic hypersurfaces in Riemannian manifolds*, preprint 2009, arXiv: math.DG/0901.1507v1.

59. OUAKKAS, S. – *Biharmonic maps, conformal deformations and the Hopf maps*, Differential Geom. Appl., 26 (2008), 495–502.

60. PIU, M.P. – *Campi di vettori ed applicazioni armoniche*, Rend. Sem. Fac. Sci. Univ. Cagliari, 52 (1982), 85–94.

61. SASAHARA, T. – *Biharmonic Lagrangian surfaces of constant mean curvature in complex space forms*, Glasg. Math. J., 49 (2007), 497–507.

62. TAKAGI, R. – *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math., 10 (1973), 495–506.

63. TANNO, S. – *The topology of contact Riemannian manifolds*, Ill. J. Math., 12 (1968), 700–717.

64. WANG, C. – *Remarks on biharmonic maps into spheres*, Calc. Var. Partial Differential Equations, 21 (2004), 221–242.

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