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ON A GENERAL MIXED VOLTERRA-FREDHOLM INTEGRAL EQUATION

BY

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Abstract. In this paper we study some basic properties of solutions of a general mixed Volterra Fredholm integral equation. A variant of a certain integral inequality with explicit estimate is obtained and used to establish the results.

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Key words: Volterra-Fredholm integral equation, integral inequality, explicit estimate, estimate on the solution, uniqueness of solutions, continuous dependence of solution.

1. Introduction. Let R be the set of real numbers, $R_+ = [0,\infty)$ and $B \subset R^n$, $B = \prod_{i=1}^{n}[a_i, b_i]$, $(a_i < b_i)$. Let $\Delta = B \times R_+$ and denote by $C(S_1, S_2)$, the class of continuous functions from the set S_1 to the set S_2. Consider the general mixed Volterra-Fredholm integral equation

$$u(x,t) = h(x,t) + \int_0^t \int_B F(x,t,y,s,u(y,s),(Tu)(y,s))\,dyds, \tag{1.1}$$

for $(x,t) \in \Delta$, where

$$(Tu)(x,t) = \int_0^t \int_B K(x,t,z,\tau,u(z,\tau))\,dzd\tau, \tag{1.2}$$

and $h \in C(\Delta, R)$, $K \in C\left(\Delta^2 \times R, R\right)$, $F \in C\left(\Delta^2 \times R^2, R\right)$. The equations of the form (1.1) arise in a natural way while studing some initial boundary value problems for partial differential equations of the parabolic type, the integrodifferential equations occuring in reactor dynamics and mathematical epidemiology, see [1, p. 18], [6, Chapter VI] and [2-5,10]. The equation (1.1) appears to be Volterra type in t, and of Fredholm type with respect to x and hence we view it as a mixed Volterra-Fredholm integral equation. Here, we note that one can formulate the existence result for the solution of equation (1.1) by modifying the idea employed in [8], see also [9]. The main purpose of this paper is to study some basic properties of solutions of equation (1.1) under some suitable conditions on the functions involved therein. The analysis used in the proof is based on variant of a certain integral inequality given in [7].

2. A basic integral inequality. We need the following variant of the integral inequality given in [7, Theorem 1.7.2, part (iv), p.37].

Theorem 1. *Let $u(x,t)$, $p(x,t)$, $q(x,t)$, $f(x,t)$, $g(x,t) \in C(\Delta, R_+)$ and suppose that*

$$\begin{aligned} u(x,t) &\leq p(x,t) + q(x,t) \int_0^t \int_B f(y,s) \Big[u(y,s) \\ &\quad + q(y,s) \int_0^s \int_B g(z,\tau)\, u(z,\tau)\, dz d\tau \Big] dy ds, \end{aligned} \tag{2.1}$$

for $(x,t) \in \Delta$. Then

$$\begin{aligned} u(x,t) &\leq p(x,t) + q(x,t) \int_0^t \int_B p(y,s) \left[f(y,s) + g(y,s) \right] \\ &\quad \times \exp\left(\int_s^t \int_B q(z,\tau) \left[f(z,\tau) + g(z,\tau) \right] dz d\tau \right) dy ds, \end{aligned} \tag{2.2}$$

for $(x,t) \in \Delta$.

Proof. Introduce the notation

$$e(s) = \int_B f(y,s) \left[u(y,s) + q(y,s) \int_0^s \int_B g(z,\tau)\, u(z,\tau)\, dz d\tau \right] dy. \tag{2.3}$$

Then the inequality (2.1) can be restated as

$$u(x,t) \leq p(x,t) + q(x,t) \int_0^t e(s)\, ds, \tag{2.4}$$

for $(x,t) \in \Delta$. Define

$$m(t) = \int_0^t e(s)\,ds, \tag{2.5}$$

for $t \in R_+$, then $m(0) = 0$ and from (2.4) we get

$$u(x,t) \le p(x,t) + q(x,t)\,m(t), \tag{2.6}$$

for $(x,t) \in \Delta$. From (2.5), (2.3) and (2.6) we observe that

$$\begin{aligned} m'(t) &= e(t) = \int_B f(y,t)\left[u(y,t) + q(y,t)\int_0^t \int_B g(z,\tau)u(z,\tau)dzd\tau\right] dy \\ &\le \int_B f(y,t)\,[\,p(y,t) + q(y,t)\,m(t) \\ &\quad + q(y,t)\int_0^t \int_B g(z,\tau)\left[p(z,\tau) + q(z,\tau)\,m(\tau)\right] dzd\tau\Big]\, dy, \end{aligned} \tag{2.7}$$

for $t \in R_+$. Introducing the notation

$$r(\tau) = \int_B g(z,\tau)\left[p(z,\tau) + q(z,\tau)\,m(\tau)\right] dz, \tag{2.8}$$

the inequality (2.7) can be written as

$$m'(t) \le \int_B f(y,t)\left[p(y,t) + q(y,t)\left\{m(t) + \int_0^t r(\tau)\,d\tau\right\}\right] dy, \tag{2.9}$$

for $t \in R_+$. Define

$$v(t) = m(t) + \int_0^t r(\tau)\,d\tau, \tag{2.10}$$

then $m(t) \le v(t), t \in R_+, v(0) = m(0) = 0$ and

$$m'(t) \le \int_B f(y,t)\left[p(y,t) + q(y,t)\,v(t)\right] dy. \tag{2.11}$$

From (2.10), (2.11), (2.8) and the fact that $m(t) \leq v(t)$, $t \in R_+$, we observe that

$$\begin{aligned} v'(t) &= m'(t) + r(t) \leq \int_B f(y,t)\left[p(y,t) + q(y,t)v(t)\right]dy \\ &\quad + \int_B g(z,t)\left[p(z,t) + q(z,t)m(t)\right]dz \\ &\leq v(t)\int_B q(y,t)\left[f(y,t) + g(y,t)\right]dy \\ &\quad + \int_B p(y,t)\left[f(y,t) + g(y,t)\right]dy. \end{aligned} \tag{2.12}$$

Multiplying both sides of (2.12) by the integrating factor

$$\exp\left(-\int_0^t \left\{\int_B q(z,\tau)\left[f(z,\tau) + g(z,\tau)\right]dz\right\}d\tau\right),$$

we have

$$\begin{aligned} &\left[v(t)\exp\left(-\int_0^t \left\{\int_B q(z,\tau)\left[f(z,\tau) + g(z,\tau)\right]dz\right\}d\tau\right)\right]' \\ &\leq \left\{\int_B p(y,t)\left[f(y,t) + g(y,t)\right]dy\right\} \\ &\times \exp\left(-\int_0^t \left\{\int_B q(z,\tau)\left[f(z,\tau) + g(z,\tau)\right]dz\right\}d\tau\right). \end{aligned} \tag{2.13}$$

By setting $t = s$ in (2.13) and integrating it with respect to s from 0 to t, $t \in R_+$ we get

$$\begin{aligned} v(t) &\leq \int_0^t \int_B p(y,s)\left[f(y,s) + g(y,s)\right] \\ &\quad \times \exp\left(\int_s^t \int_B q(z,\tau)\left[f(z,\tau) + g(z,\tau)\right]dzd\tau\right)dyds. \end{aligned} \tag{2.14}$$

Using the fact that $m(t) \leq v(t)$, $t \in R_+$ in (2.14) and then using the bound on $m(t)$ in (2.6) we get the required inequality in (2.2). The proof is complete. □

3. Properties of solutions. In this section we apply the inequality established in Theorem 1 to study some basic properties of the solutions of equation (1.1).

First, we shall give the following theorem concerning the estimate on the solution of equation (1.1).

Theorem 2. *Suppose that the functions F, K in equations* (1.1), (1.2) *satisfy the conditions*

$$|F(x,t,y,s,u,v) - F(x,t,y,s,\bar{u},\bar{v})| \leq q(x,t)f(y,s)[|u-\bar{u}| + |v-\bar{v}|], \tag{3.1}$$

$$|K(y,s,z,\tau,u) - K(y,s,z,\tau,\bar{u})| \leq q(y,s)\,g(z,\tau)\,|u-\bar{u}|, \tag{3.2}$$

where $q, f, g \in C(\Delta, R_+)$. *Let*

$$c = \sup_{(x,t)\in\Delta} \left| h(x,t) + \int_0^t \int_B F(x,t,y,s,o,(To)(y,s))\,dyds \right|. \tag{3.3}$$

If $u(x,t)$ *is any solution of equation* (1.1) *on* Δ, *then*

$$\begin{aligned} |u(x,t)| \leq\; & c\Big[1 + q(x,t)\int_0^t \int_B [f(y,s) + g(y,s)] \\ & \times \exp\left(\int_s^t \int_B q(z,\tau)[f(z,\tau) + g(z,\tau)]\,dzd\tau\right) dyds\Big], \end{aligned} \tag{3.4}$$

for $(x,t) \in \Delta$.

Proof. Using the fact that $u(x,t)$ is a solution of equation (1.1) and the hypotheses, we have

$$\begin{aligned} |u(x,t)| \leq\; & \left| h(x,t) + \int_0^t \int_B F(x,t,y,s,o,(To)(y,s))\,dyds \right| \\ & + \int_0^t \int_B |F(x,t,y,s,u(y,s),(Tu)(y,s)) \\ & - F(x,t,y,s,o,(To)(y,s))|\,dyds \\ & \leq c + q(x,t)\int_0^t \int_B f(y,s)\Big[|u(y,s)| \\ & + q(y,s)\int_0^s \int_B g(z,\tau)|u(z,\tau)|\,dzd\tau\Big] dyds. \end{aligned} \tag{3.5}$$

Now applying the Theorem 1 to (3.5) it results (3.4).

Next theorem deals with the uniqueness of solutions of equation (1.1).

Theorem 3. *Suppose that the functions F, K in equation* (1.1) *satisfy the conditions* (3.1), (3.2) *respectively. Then the equation* (1.1) *has at most one solution on* Δ.

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of equation (1.1) on Δ. Using our hypotheses, we have

$$\begin{aligned} |u_1(x,t)-u_2(x,t)| &\leq \int_0^t \int_B |F(x,t,y,s,u_1(y,s),(Tu_1)(y,s)) \\ &\quad -F(x,t,y,s,u_2(y,s),(Tu_2)(y,s))|dyds \\ &\leq q(x,t)\int_0^t \int_B f(y,s)[|u_1(y,s)-u_2(y,s)| \\ &\quad +q(y,s)\int_0^s \int_B g(z,\tau)|u_1(z,\tau)-u_2(z,\tau)|dzd\tau]dyds. \end{aligned} \tag{3.6}$$

Using the Theorem 1, the relation (3.6) yields $|u_1(x,t)-u_2(x,t)| \leq 0$, which implies $u_1(x,t) = u_2(x,t)$. Thus there is at most one solution of equation (1.1) on Δ.

Finally, we present a result on the continuous dependence of solution of equation (1.1) on the functions involved therein. Consider the equation (1.1) and the corresponding equation

$$v(x,t) = \bar{h}(x,t) + \int_0^t \bar{F}\left(x,t,y,s,v(y,s),\left(\bar{T}v\right)(y,s)\right) dyds, \tag{3.7}$$

for $(x,t) \in \Delta$, where

$$\left(\bar{T}v\right)(x,t) = \int_0^t \int_B \bar{K}(x,t,z,\tau,v(z,\tau))\, dzd\tau, \tag{3.8}$$

and $\bar{h} \in C(\Delta, R)$, $\bar{K} \in C\left(\Delta^2 \times R, R\right)$, $\bar{F} \in C\left(\Delta^2 \times R^2, R\right)$. □

Theorem 4. *Suppose that the functions F, K in equation* (1.1) *satisfy the conditions* (3.1), (3.2)*, respectively. Furthermore, suppose that*

$$\begin{aligned} &\left|h(x,t)-\bar{h}(x,t)\right| + \int_0^t \int_B |F(x,t,y,s,v(y,s),(Tv)(y,s)) \\ &-\bar{F}\left(x,t,y,s,v(y,s),\left(\bar{T}v\right)(y,s)\right)\Big| dyds \leq \varepsilon, \end{aligned} \tag{3.9}$$

where h, F, Tu *and* $\bar{h}, \bar{F}, \bar{T}v$ *are as in equations* $(1.1), (1.2)$ *and* $(3.7), (3.8)$, *respectively,* $\varepsilon > 0$ *is an arbitrary small constant and* $v(x,t)$ *is a solution of equation* (3.7). *Then the solution* $u(x,t)$, $(x,t) \in \Delta$ *of equation* (1.1) *depends continuously on the functions involved in equation* (1.1).

Proof. Let $w(x,t) = |u(x,t) - v(x,t)|$, $(x,t) \in \Delta$. Using the facts that $u(x,t)$ and $v(x,t)$ are the solutions of equations (1.1) and (3.7) and our hypotheses, we have

$$\begin{aligned} w(x,t) &\leq |h(x,t) - \bar{h}(x,t)| \\ &\quad + \int_0^t \int_B |F(x,t,y,s,u(y,s),(Tu)(y,s)) \\ &\quad - F(x,t,y,s,v(y,s),(Tv)(y,s))| dyds \\ &\quad + \int_0^t \int_B |F(x,t,y,s,v(y,s),(Tv)(y,s)) \\ &\quad - \bar{F}(x,t,y,s,v(y,s),(\bar{T}v)(y,s))| dyds \\ &\leq \varepsilon + q(x,t) \int_0^t \int_B f(y,s)[w(y,s) \\ &\quad + q(y,s) \int_0^s \int_B g(z,\tau) w(z,\tau) dz d\tau] dyds. \end{aligned} \tag{3.10}$$

Now using the Theorem 1, the relation (3.10) yields

$$\begin{aligned} |u(x,t) - v(x,t)| &\leq \varepsilon \Bigg[1 + q(x,t) \int_0^t \int_B [f(y,s) + g(y,s)] \\ &\times \exp\left(\int_s^t \int_B q(z,\tau) [f(z,\tau) + g(z,\tau)] dz d\tau \right) dyds \Bigg], \end{aligned} \tag{3.11}$$

for $(x,t) \in \Delta$. From (3.11) it follows that the solution of equation (1.1) depends continuously on the functions involved therein. □

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REFERENCES

1. Corduneanu, C. – *Integral Equations and Applications,* Cambridge University Press, 1991.

2. Diekmann, O. – *Run for your life. A note on the asymptotic speed of propagation of an epidemic,* J. Differential Equations, 33 (1979), 58–73.

3. Friedman, A. – *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

4. Ladyženskaja, O.A.; Solonnikov, V.A.; Uralceva, N.N. – *Linear and Quasi-linear Equations of Parabolic type,* Amer.Math.Soc., Providance, R.I., 1968.

5. Levin, J.J.; Nohel, J.A. – *The integrodifferential equations of a class of nuclear reactors with delayed neutrons,* Arch. Rational Mech. Anal., 31(1968), 151–172.

6. Mikhaĭlov, V.P. – *Partial differential equations,* Translated from the Russian by P.C. Sinha, "Mir", Moscow, distributed by Imported Publications, Inc., Chicago, Ill., 1978.

7. Pachpatte, B.G. – *Inequalities for Differential and Integral Equations,* Mathematics in Science and Engineering, 197, Academic Press, Inc., San Diego, CA, 1998.

8. Pachpatte, B.G. – *On mixed Volterra-Fredholm type integral equations,* Indian J. Pure Appl. Math., 17 (1986), ,488–496

9. Pachpatte, B.G. – *On Volterra-Fredholm integral equation in two variables,* Demonstratio Math., 40 (2007), 839–852.

10. Pao, C.V. – *Solution of a nonlinear integrodifferential system arising in nuclear reactor dynamics,* J. Math. Anal. Appl., 48 (1974), 470–492.

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