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CHARACTERIZATIONS OF QUIET QUASI-UNIFORMITY AND ALMOST QUIET QUASI-UNIFORMITY IN TERMS OF COVERS

BY

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Abstract. In this paper quiet quasi-uniformity and almost quiet quasi-uniformity are characterized in terms of strong quasi-uniform covers. Also some applications of this type of covers are given in the last section.

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Key words: Quiet quasi-uniform space, almost quiet quasi-uniform space, strong quasi-uniform cover.

1. Introduction.

In [5] Pervin proved that every topological space is quasi-uniformizable in the sense that τ is a topology on a set X if and only if there exists a quasi-uniformity $\mathcal{V}$ on X, which induces τ. Later in [4] Mukherjee and Debray introduced strong quasi-uniform covers and characterized quasi-uniformity in terms of strong quasi-uniform covers.

In this paper, our aim is to characterize quiet quasi-uniformity, introduced by Doitchinov in [1], and almost quiet quasi-uniformity introduced by Ganguly, Dutta and Chattopadhyay in [3] in terms of quasi-uniform covers. To characterize both quiet quasi-uniformity and almost quiet quasi-uniformity we use strong quasi-uniform covers suitably. We shall also derive characterizations of some topological concepts in terms of covers for quasi-uniform spaces.

Throughout the paper, for $int(cl(A))$ where $A \subseteq X$, we shall use the notation $\dot{\overline{A}}$.

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Definition 1.1 ([1]). *A quasi-uniform space* $(X,\mathcal{U})$ *is called a quiet quasi-uniform space if the following property hold : for any* $U \in \mathcal{U}$, *there exists a* $V \in \mathcal{U}$ *such that, if* $x', x'' \in X$ *and* $\{x'_\alpha : \alpha \in A\}$ *and* $\{x''_\beta : \beta \in B\}$ *are two nets in* X, *then from* $(x', x'_\alpha) \in V$ *for* $\alpha \in A$, $(x''_\beta, x'') \in V$ *for* $\beta \in B$ *and* $(x''_\beta, x'_\alpha) \to 0$ *[i.e., for any* $W \in \mathcal{U}$, *there exist* α_W *and* β_W *belonging to* A *and* B *respectively such that* $(x''_\beta, x'_\alpha) \in W$ *for* $\alpha \geq \alpha_W$, $\beta \geq \beta_W$*] it follows that* $(x', x'') \in U$. *When a* $V \in \mathcal{U}$ *is connected with some* $U \in \mathcal{U}$ *by the above property, we then say that* V *is* Q *-subordinated to* U.

Definition 1.2 ([3]). *A topological space* (X, τ) *is said to be almost quiet quasi-uniformizable if and only if there exists a compatible quasi-uniformity* $\mathcal{U}$ *with the following properties : for* $U \in \mathcal{U}$ *and* $x \in X$, *there exists* $V_x \in \mathcal{U}$ *for which the following conditions hold : if* $\{x_\alpha : \alpha \in A\}$ & $\{y_\beta : \beta \in B\}$ *be two nets such that* $(x, x_\alpha) \in V_x$ *for* $\alpha \in A$, $(y_\beta, y) \in V_x$ *(for some* $y \in X$*), for* $\beta \in B$, *and* $(y_\beta, x_\alpha) \to 0$, *then* $y \in \overline{\dot{U}(x)}$, *where the closure and the interior of* $U(x)$ *and* $\overline{U(x)}$ *respectively are taken under the topology* τ; *we call* V_x *subordinated to* U *with respect to* x.

Definition 1.3 ([4]). *A cover* $\mathcal{U}$ *of a quasi-uniform space* $(X, \mathcal{V})$ *is said to be a strong quasi-uniform cover, if there exists some* $V \in \mathcal{V}$ *such that for each* $x \in X$, $V(x) \subseteq \cap\{H \in \mathcal{U} : x \in H\}$, *where as usual* $V(x) = \{y \in X : (x, y) \in V\}$.

2. Characterization of quiet quasi-uniformity in terms of strong quasi-uniform covers.

In this article we shall use strong quasi-uniform covers to characterize quiet quasi-uniformity.

Theorem 2.1. *Let* $(X, \mathcal{V})$ *be a quiet quasi-uniform space and* $\mathcal{C}$ *be the collection of all strong quasi-uniform covers of* X. *Then the followings hold:*

(i) If $\mathcal{U} \in \mathcal{C}$ *and* $\mathcal{U}_1$ *is a cover of* X *such that for each* $x \in X$, $\cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : x \in K\}$, *then* $\mathcal{U}_1 \in \mathcal{C}$.

(ii) If $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}$, *then there is a* $\mathcal{U} \in \mathcal{C}$ *such that* $\cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : x \in K\}$ *and* $\cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_2 : x \in K\}$, *for each* $x \in X$.

(iii) For $\mathcal{U} \in \mathcal{C}$, *if* $\{x_\alpha : \alpha \in A\}$ *and* $\{y_\beta : \beta \in B\}$ *be two nets in* $(X, \mathcal{V})$ *such that* $x_\alpha \in \cap\{H \in \mathcal{U} : x \in H\}$, $y_\beta \in \cap\{K \in \mathcal{U} : y \in K\}$ *for* $\alpha \in A$, $\beta \in B$ *and* $(y_\beta, x_\alpha) \to 0$, *then* $y \in \cap\{H \in \mathcal{U} : x \in H\}$.

Conversely, let X *be a set and* $\mathcal{C}$ *be the collection of all covers of* X *satisfying conditions* $(i), (ii)$ *and* (iii). *Then there exists some quiet quasi-*

uniformity $\mathcal{V}$ on X w.r.t. which $\mathcal{C}$ is precisely the collection of all strong quasi-uniform covers of X.

Proof. Let $\mathcal{C}$ be the collection of all strong quasi-uniform covers of a quiet quasi-uniform space $(X, \mathcal{V})$.

(i) If $\mathcal{U} \in \mathcal{C}$, then there exists $V \in \mathcal{V}$ such that

$$V(x) \subseteq \cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : x \in K\}, \text{ for each } x \in X.$$

Hence $\mathcal{U}_1 \in \mathcal{C}$.

(ii) Let $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}$. Then there exist $V, W \in \mathcal{V}$ such that $V(x) \subseteq \cap\{H \in \mathcal{U}_1 : x \in H\}$ and $W(x) \subseteq \cap\{K \in \mathcal{U}_2 : x \in K\}$ for each $x \in X$. Consider $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$. Then we get, $(V \cap W)(x) \subseteq \cap\{H \in \mathcal{U} : x \in H\}$, for each $x \in X$.

For, $y \in (V \cap W)(x) \Rightarrow y \in V(x)$ and $y \in W(x) \Rightarrow y \in \cap\{H \in \mathcal{U}_1 : x \in H\}$ and $y \in \cap\{K \in \mathcal{U}_2 : x \in K\} \Rightarrow y \in \cap\{H \in \mathcal{U}_1 \cup \mathcal{U}_2 : x \in H\}$. Now, $\mathcal{V}$ being a quasi-uniformity on X, $V \cap W = U$ (say) $\in \mathcal{V}$. Thus $\mathcal{U}$ is a strong quasi-uniform cover. Now, $\cap\{H \in \mathcal{U}_1 \cup \mathcal{U}_2 : x \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : x \in K\}$ and $\cap\{H \in \mathcal{U}_1 \cup \mathcal{U}_2 : x \in H\} \subseteq \cap\{K \in \mathcal{U}_2 : x \in K\}$ for each $x \in X$.

(iii) Given that $\mathcal{U} \in \mathcal{C}$ and $\{x_\alpha : \alpha \in A\}$, $\{y_\beta : \beta \in B\}$ are two nets in $(X, \mathcal{V})$ such that $x_\alpha \in \cap\{H \in \mathcal{U} : x \in H\}$, $y_\beta \in \cap\{K \in \mathcal{U} : y \in K\}$ for $\alpha \in A$, $\beta \in B$ with $(y_\beta, x_\alpha) \to 0$. Since $\mathcal{U}$ is a strong quasi-uniform cover, there exists $U \in \mathcal{V}$ such that $U(x) \subseteq \cap\{H \in \mathcal{U} : x \in H\}$ for each $x \in X$. Choose $V \in \mathcal{V}$ subordinated to U. Then $(x, x_\alpha) \in V$, $(y_\beta, y) \in V$ with $(y_\beta, x_\alpha) \to 0$ will give $(x, y) \in U \Rightarrow y \in U(x) \subseteq \cap\{H \in \mathcal{U} : x \in H\}$, i.e., $y \in \cap\{H \in \mathcal{U} : x \in H\}$.

For the converse part, let $\mathcal{C}$ be the collection of all covers of X satisfying conditions (i), (ii) and (iii). For each $\mathcal{U} \in \mathcal{C}$, define $V(\mathcal{U}) = \cup\{(\{x\} \times [\cap\{H \in \mathcal{U} : x \in H\}]) : x \neq x_\alpha, y_\beta \text{ for } \alpha \in A, \beta \in B\}$ and $W(\mathcal{U}) = \cup\{([\cap\{K \in \mathcal{U} : y \in K\}] \times \{y\}) : y \neq x_\alpha, y_\beta \text{ for } \alpha \in A, \beta \in B\}$.

We first show that the collection $\mathcal{B} = \{V(\mathcal{U}) \cup W(\mathcal{U}) : \mathcal{U} \in \mathcal{C}\}$ forms a base for some quiet quasi-uniformity on X. Obviously $\Delta \subseteq V(\mathcal{U}) \cup W(\mathcal{U})$ for each $\mathcal{U} \in \mathcal{C}$, where Δ is the diagonal in $X \times X$. Next let, $V(\mathcal{U}_1) \cup W(\mathcal{U}_1)$, $V(\mathcal{U}_2) \cup W(\mathcal{U}_2) \in \mathcal{B}$. Then $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}$. By condition (ii), there exists $\mathcal{U} \in \mathcal{C}$ such that $\cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : x \in K\}$ and $\cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_2 : x \in K\}$ for each $x \in X$. Let $(x, y) \in V(\mathcal{U}) \cup W(\mathcal{U})$. Then $(x, y) \in V(\mathcal{U})$ or $(x, y) \in W(\mathcal{U})$. If $(x, y) \in V(\mathcal{U})$, then $(x, y) \in \{x\} \times [\cap\{H \in \mathcal{U} : x \in H\}] \Rightarrow y \in \cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : x \in K\}$ and $y \in \cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_2 : x \in K\}$. Thus $(x, y) \in V(\mathcal{U}_1) \cap V(\mathcal{U}_2)$.

If $(x,y) \in W(\mathcal{U})$ then $(x,y) \in [\cap\{H \in \mathcal{U} : y \in H\}] \times \{y\} \Rightarrow x \in \cap\{H \in \mathcal{U} : y \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : y \in K\}$ and $x \in \cap\{H \in \mathcal{U} : y \in H\} \subseteq \cap\{K \in \mathcal{U}_2 : y \in K\}$. Thus $(x,y) \in W(\mathcal{U}_1) \cap W(\mathcal{U}_2)$. Thus $(x,y) \in (V(\mathcal{U}_1) \cap V(\mathcal{U}_2)) \cup (W(\mathcal{U}_1) \cap W(\mathcal{U}_2)) \subseteq (V(\mathcal{U}_1) \cup W(\mathcal{U}_1)) \cap (V(\mathcal{U}_2) \cup W(\mathcal{U}_2))$. Hence $V(\mathcal{U}) \cup W(\mathcal{U}) \subseteq (V(\mathcal{U}_1) \cup W(\mathcal{U}_1)) \cap (V(\mathcal{U}_2) \cup W(\mathcal{U}_2))$.

Next we prove that $\mathcal{B}$ forms a base for some quiet quasi-uniformity on X. Let $\{x_\alpha : \alpha \in A\}$, $\{y_\beta : \beta \in B\}$ be two nets in X with $(x, x_\alpha) \in V(\mathcal{U}) \cup W(\mathcal{U})$, $(y_\beta, y) \in V(\mathcal{U}) \cup W(\mathcal{U})$ and $(y_\beta, x_\alpha) \to 0$. Then $x_\alpha \in \cap\{H \in \mathcal{U} : x \in H\}$, $y_\beta \in \cap\{K \in \mathcal{U} : y \in K\}$ with $(y_\beta, x_\alpha) \to 0$. Using condition (iii), we get, $y \in \cap\{H \in \mathcal{U} : x \in H\}$, i.e., $(x,y) \in V(\mathcal{U}) \cup W(\mathcal{U})$. Thus our assertion is proved.

Now, for any $\mathcal{U} \in \mathcal{C}$ and any $x \in X$ we have, $V(\mathcal{U})(x) = \{H \in \mathcal{U} : x \in H\}$ so that $\mathcal{U}$ is a strong quasi-uniform cover of X. For the converse, let $\mathcal{U}$ be any strong quasi-uniform cover. Then there is $V(\mathcal{U}_1) \cup W(\mathcal{U}_1) \in \mathcal{B}$ for some $\mathcal{U}_1 \in \mathcal{C}$ such that $(V(\mathcal{U}_1) \cup W(\mathcal{U}_1))(x) \subseteq \cap\{H \in \mathcal{U} : x \in H\}$, for each $x \in X$, i.e., $V(\mathcal{U}_1)(x) \subseteq \cap\{H \in \mathcal{U} : x \in H\}$ for each $x \in X$. Now as $\mathcal{U}_1 \in \mathcal{C}$, we have by condition (i), $\mathcal{U} \in \mathcal{C}$. Same is the case for $W(\mathcal{U})(x)$. This completes the proof. □

3. Characterization of almost quiet quasi-uniformity in terms of strong quasi-uniform covers. Here we characterize almost quiet quasi-uniformity in terms of strong quasi-uniform covers.

Theorem 3.1. *Let $(X, \mathcal{V})$ be an almost quiet quasi-uniform space and $\mathcal{C}$ be the collection of all strong quasi-uniform covers of X. Then the followings hold:*

(i) If $\mathcal{U} \in \mathcal{C}$ and $\mathcal{U}_1$ is a cover of X such that for each $x \in X$, $\cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : x \in K\}$, then $\mathcal{U}_1 \in \mathcal{C}$.

(ii) If $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{C}$, then there exists $\mathcal{U} \in \mathcal{C}$ such that $\cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_1 : x \in K\}$ and $\cap\{H \in \mathcal{U} : x \in H\} \subseteq \cap\{K \in \mathcal{U}_2 : x \in K\}$, for each $x \in X$.

(iii) For $\mathcal{U} \in \mathcal{C}$, if $\{x_\alpha : \alpha \in A\}$ and $\{y_\beta : \beta \in B\}$ be two nets in $(X, \mathcal{V})$ such that $x_\alpha \in \cap\{H \in \mathcal{U} : x \in H\}$, $y_\beta \in \cap\{K \in \mathcal{U} : y \in K\}$ for $\alpha \in A$, $\beta \in B$ and $(y_\beta, x_\alpha) \to 0$, then $y \in \cap\{H \in \mathcal{U} : x \in H\}$.

Conversely, let X be a set and $\mathcal{C}$ be the collection of all covers of X satisfying conditions $(i), (ii)$ and (iii). Then there exists some almost quiet quasi-uniformity $\mathcal{V}$ on X w.r.t. which $\mathcal{C}$ is precisely the collection of all strong quasi-uniform covers of X.

Proof. The proof of the necessity part can be done similarly as for Theorem 2.1.

For the sufficient part, we define for each $\mathcal{U} \in \mathcal{C}$, $V(\mathcal{U}) = \cup\{(\{x\} \times [\cap\{H \in \mathcal{U} : x \in H\}]) : x \neq x_\alpha, y_\beta \text{ for } \alpha \in A, \beta \in B\}$ and $W(\mathcal{U}) = \cup\{([\cap\{K \in \mathcal{U} : x \in K\}] \times \{y\}) : y \neq x_\alpha, y_\beta \text{ for } \alpha \in A, \beta \in B\}$.

Define $\mathcal{B} = \{V(\mathcal{U}) \cup W(\mathcal{U}) : \mathcal{U} \in \mathcal{C}\}$. Then as in Theorem 2.1, it can be proved that $\mathcal{B}$ is a base for some quasi-uniformity $\mathcal{V}$ on X. We only prove that $\mathcal{V}$ is an almost quiet quasi-uniformity on X. For this, let $\{x_\alpha : \alpha \in A\}$, $\{y_\beta : \beta \in B\}$ be two nets in X with $(x, x_\alpha) \in V(\mathcal{U}) \cup W(\mathcal{U})$, $(y_\beta, y) \in V(\mathcal{U}) \cup W(\mathcal{U})$ and $(y_\beta, x_\alpha) \to 0$. Then $(x, x_\alpha) \in \{x\} \times [\cap\{H \in \mathcal{U} : x \in H\}]$ and $(y_\beta, y) \in [\cap\{K \in \mathcal{U} : y \in K\}] \times \{y\}$, i.e., $x_\alpha \in \cap\{H \in \mathcal{U} : x \in H\}$, $y_\beta \in \cap\{K \in \mathcal{U} : y \in K\}$ with $(y_\beta, x_\alpha) \to 0$. Using condition (iii) we get, $y \in \cap\{H \in \mathcal{U} : x \in H\}$, i.e., $(x, y) \in \dot{V}(\mathcal{U}) \cup W(\mathcal{U}) \Rightarrow y \in (V(\mathcal{U}) \cup W(\mathcal{U}))(x) = V(\mathcal{U})(x) \cup W(\mathcal{U})(x) \subseteq \overline{V(\mathcal{U})(x) \cup W(\mathcal{U})(x)}$. Thus our assertion is proved.□

4. Applications. In this section we use the concept of strong quasi-uniform covers to characterize some topological concepts.

Definition 4.1 ([2])**.** *A collection $\mathcal{C}$ of subsets of a topological space is interior preserving if $\mathcal{C}' \in \mathcal{C}$, then $int \cap \{C : C \in \mathcal{C}'\} = \cap\{intC : C \in \mathcal{C}'\}$.*

Remark 4.2 ([2])**.** If $\mathcal{C}$ is a collection of subsets of a set X and $x \in X$, then $\mathcal{C}_x = \{C \in \mathcal{C} : x \in C\}$, so that $\cap\mathcal{C}_x = \cap\{C \in \mathcal{C} : x \in C\}$. In terms of this notation, a collection $\mathcal{C}$ of open subsets of a topological space is interior preserving iff for each $x \in X$, $\cap\mathcal{C}_x \in \tau$.

Definition 4.3 ([2])**.** *A topological space is orthocompact provided that every open cover has an interior preserving open refinement.*

Definition 4.4. *A covering $\{A_\alpha : \alpha \in \Lambda\}$ of a topological space (X, τ) is called point-finite if for each $x \in X$, there exists at most finitely many indices $\alpha \in \Lambda$ such that $x \in A_\alpha$.*

Definition 4.5 ([2])**.** *A topological space (X, τ) is metacompact if each open covering has a point-finite open refinement.*

Definition 4.6 ([2])**.** *Let $\mathcal{A}$ be a nonempty family of covers of a set X and for each $\mathcal{C} \in \mathcal{A}$, let $U_{\mathcal{C}} = \{(x, y) : x \in X$ and $y \in \cap\{C \in \mathcal{C} : x \in C\}\}$. It can be then checked that each such $U_{\mathcal{C}}$ is reflexive and transitive relation and that the family $\{U_{\mathcal{C}} : \mathcal{C} \in \mathcal{A}\}$ is a subbase for a quasi-uniformity on X, to be denoted by $\mathcal{U}_{\mathcal{A}}$.*

Theorem 4.7. *A topological space (X, τ) is orthocompact iff for every open cover $\mathcal{C}$ of (X, τ), there exists a strong quasi-uniform cover $\mathcal{C}^*$ of $(X, \mathcal{U}_\mathcal{A})$ such that $\mathcal{C}^*$ refines $\mathcal{C}$, where $\mathcal{A}$ is the collection of all interior preserving open covers of X.*

Proof. First let (X, τ) be orthocompact and let $\mathcal{C}$ be an open cover of (X, τ). Then there is an interior preserving open refinement $\mathcal{C}^*$ of $\mathcal{C}$. Now $\mathcal{C}^*$ being an interior preserving open cover of X, $U_{\mathcal{C}^*} \in \mathcal{U}_\mathcal{A}$ where $U_{\mathcal{C}^*} = \{(x, y) \in X \times X : x \in X \text{ and } y \in \cap\{C \in \mathcal{C}^* : x \in C\}\}$. Thus for each $x \in X$, $U_{\mathcal{C}^*}(x) = \cap\{C \in \mathcal{C}^* : x \in C\}$. Hence $\mathcal{C}^*$ is a strong quasi-uniform cover of $(X, \mathcal{U}_\mathcal{A})$.

Conversely, let $\mathcal{C}$ be an open cover of (X, τ). By the given condition, there is a strong quasi-uniform cover $\mathcal{C}^*$ of $(X, \mathcal{U}_\mathcal{A})$ such that $\mathcal{C}^*$ refines $\mathcal{C}$. Since $\mathcal{C}^*$ is a strong quasi-uniform cover of $(X, \mathcal{U}_\mathcal{A})$, there exists finitely many $U_{\mathcal{C}_1}$, $U_{\mathcal{C}_2}, ..., U_{\mathcal{C}_n} \in \mathcal{U}_\mathcal{A}$ such that $U(x) \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$, for each $x \in X$, where $U = U_{\mathcal{C}_1} \cap U_{\mathcal{C}_2} \cap ... \cap U_{\mathcal{C}_n}$, i.e., $\bigcap_{i=1}^n [\cap\{C \in \mathcal{C}_i : x \in C\}] \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$... (i), i.e., $\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \cup \mathcal{C}_n : x \in C\} \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$.

Let $\mathcal{C}' = \{\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \mathcal{C}_n : x \in C\} : x \in X\}$. As $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n \in \mathcal{A}$, $U_{\mathcal{C}_i}(x) \in \tau$ for $i = 1, 2, ..., n$ and for all $x \in X$. Hence $\bigcap_{i=1}^n U_{\mathcal{C}_i}(x) \in \tau$, for all $x \in X$. Thus each set of $\mathcal{C}'$ is open. Also $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n$ being interior preserving, $\mathcal{C}'$ is also so. Again by (i) $\mathcal{C}'$ refines $\mathcal{C}^*$ and $\mathcal{C}^*$ also refines $\mathcal{C}$. Hence $\mathcal{C}'$ refines $\mathcal{C}$. Thus there is an interior preserving open refinement $\mathcal{C}'$ of $\mathcal{C}$ of (X, τ). Hence (X, τ) is orthocompact. □

Theorem 4.8. *A topological space (X, τ) is metacompact iff for every open cover $\mathcal{C}$ of (X, τ), there exists a strong quasi-uniform cover $\mathcal{C}^*$ of $(X, \mathcal{U}_\mathcal{A})$ such that $\mathcal{C}^*$ refines $\mathcal{C}$, where $\mathcal{A}$ is the collection of all point-finite open covers of X.*

Proof. Let (X, τ) be metacompact and let $\mathcal{C}$ be an open cover of (X, τ). Then there exists a point-finite open refinement $\mathcal{C}^*$ of $\mathcal{C}$. Now $\mathcal{C}^*$ being a point-finite open cover of X, $U_{\mathcal{C}^*} \in \mathcal{U}_\mathcal{A}$ where $U_{\mathcal{C}^*} = \{(x, y) \in X \times X : x \in X \text{ and } y \in \cap\{C \in \mathcal{C}^* : x \in C\}\}$. Then for each $x \in X$, $U_{\mathcal{C}^*}(x) = \cap\{C \in \mathcal{C}^* : x \in C\}$. Hence $\mathcal{C}^*$ is a strong quasi-uniform cover of $(X, \mathcal{U}_\mathcal{A})$.

Conversely, let $\mathcal{C}$ be an open cover of (X, τ). By the given condition, there is a strong quasi-uniform cover $\mathcal{C}^*$ of $(X, \mathcal{U}_\mathcal{A})$ such that $\mathcal{C}^*$ refines $\mathcal{C}$. Since $\mathcal{C}^*$ is a strong quasi-uniform cover of $(X, \mathcal{U}_\mathcal{A})$, there exists finitely many $U_{\mathcal{C}_1}$, $U_{\mathcal{C}_2}, ..., U_{\mathcal{C}_n} \in \mathcal{U}_\mathcal{A}$ such that $U(x) \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$, for each

$x \in X$, where $U = U_{\mathcal{C}_1} \cap U_{\mathcal{C}_2} \cap ... \cap U_{\mathcal{C}_n}$, i.e., $\bigcap_{i=1}^{n}[\cap\{C \in \mathcal{C}_i : x \in C\}] \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$... (i), i.e., $\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \cup \mathcal{C}_n : x \in C\} \subseteq \cap\{C \in \mathcal{C}^* : x \in C\}$.

Let $\mathcal{C}' = \{\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ...\mathcal{C}_n : x \in C\} : x \in X\}$. As $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n$ are point-finite, $\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \cup \mathcal{C}_n : x \in C\}$ is a finite intersection for each $x \in X$. Hence each set $\cap\{C \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \cup \mathcal{C}_n : x \in C\}$ is an open set in (X, τ). Now $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n$ being point-finite, $\mathcal{C}'$ is also so. By (i) $\mathcal{C}'$ is a refinement of $\mathcal{C}^*$ which in turn is a refinement of $\mathcal{C}$. Thus $\mathcal{C}'$ is an open point-finite refinement of $\mathcal{C}$. Hence (X, τ) becomes metacompact. □

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