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A LIMIT HARVESTING PROBLEM OF POPULATION DYNAMICS WITH LOGISTIC TERM

BY

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Abstract. In this paper, a nonlinear age-dependent population model with logistic term is considered. In the first part is reminded a large time behavior result, and next an optimal harvesting problem associated to a limit problem is investigated. Existence of an optimal control and necessary optimality conditions are established.

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1. Formulation of the problem. The starting point is the following nonlinear age-structured population model

$$(1)\qquad \begin{cases} y_t + y_a + \mu(a)y + \Phi(Y(t))y = -u(a)y, & (a,t) \in Q \\ Y(t) = \displaystyle\int_0^A y(a,t)\ da, & t \in (0,+\infty) \\ y(0,t) = \displaystyle\int_0^A \beta(a)y(a,t)\ da, & t \in (0,+\infty) \\ y(0,a) = y_0(a), & a \in (0,A), \end{cases}$$

where $Q = (0,A) \times (0,+\infty)$. Here $A \in (0,+\infty)$ represents the maximal age for the population species.

$y(a,t)$ is the population density of age $a \in [0,A]$ at time $t \in [0,+\infty)$ and by y_0 we have denoted the initial distribution of densities.

The third equation in (1) describes the birth process and it is known as the *renewal law*; $y(0,t)$ gives the density of the newborn population at time t, and β is the fertility rate which depends in this case only on the age a. Therefore $\beta(a)y(a,t)$ stands for the density of newborns at time t, with parents of age a.

μ is the mortality rate and depends also only on age. System (1) describes the evolution of an age-structured population which includes a logistic term depending on the total population density at moment t, $Y(t)$. $\Phi(Y(t))$ represents an additional mortality rate caused by overpopulation. u is the harvesting effort (and depends on age) and $u(a)y(a,t)$ gives the harvested population of age a at the moment t, while $\int_0^A u(a)y(a,t)da$ gives the total harvest at the moment t.

The goal of the paper is to find the harvesting effort (control)

$$u \in L^\infty(0,A), \qquad 0 \leq u(a) \leq L \quad \text{a.e. in} \quad (0,A),$$

$(L \in (0,+\infty))$ which maximizes

$$\lim_{t\to\infty} \int_0^A u(a)y^u(a,t)\ da,$$

where y^u is the solution to (1) corresponding to u. In fact, we will show that this problem is equivalent to the following optimal harvesting problem:

$$\max_{u\in\mathcal{U}} \int_0^A u(a)\tilde{y}^u(a)\ da, \tag{2}$$

where $\mathcal{U} = \{v \in L^\infty(0,A);\ 0 \leq v(a) \leq L \text{ a.e. in } (0,A)\}$ denotes the set of admissible controls, and $\tilde{y}^u$ is the nontrivial nonnegative solution to

$$\begin{cases} y'(a) + \mu(a)y(a) + \Phi(Y_0)y(a) = -u(a)y(a), \quad a \in (0,A) \\ Y_0 = \displaystyle\int_0^A y(a)\ da \\ y(0) = \displaystyle\int_0^A \beta(a)y(a)\ da. \end{cases} \tag{3}$$

The paper is structured as follows: in the next section we present an asymptotic behavior result; it will be shown that, under proper assumptions, the solution to (1) is stabilized toward the nontrivial nonnegative solution to (3).

Then we consider the optimal harvesting problem (2)-(3); we prove the existence of an optimal control and provide the necessary optimality conditions.

We mention that in the case without control, it has been proved that the system (1) has a unique nonnegative nontrivial solution (see e.g. [4]).

A large amount of papers has been devoted to optimal control problems for age-structured population dynamics, see e.g. [1], [5], [8], [7], [9], [12].

For the asymptotic behavior of the solutions to age-structured systems we refer to [2], [3], [6] and [11].

Here are the assumptions of this paper:

(A1) $\beta \in L^\infty(0, A)$, $\beta(a) \geq 0$ a.e. $a \in (0, A)$;

(A2) $\mu \in L^1_{loc}([0, A))$, $\mu(a) \geq 0$ a.e. $a \in (0, A)$, $\int_0^A \mu(a)\ da = +\infty$;

(A3) $y_0 \in L^\infty(0, A)$, $y_0(a) \geq 0$ a.e. $a \in (0, A)$;

(A4) $\Phi : [0, +\infty) \mapsto [0, +\infty)$ is an increasing continuously differentiable function which satisfies $\Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = +\infty$;

(A5) $\int_0^A \beta(a) \exp\{-\int_0^a (\mu(s) + L)\ ds\}\ da > 1.$

2. Large time behavior of the solution. Let us notice that the solution to (1) is a separable one, i.e., $y(a, t) = x(t)p(a, t)$, $(a, t) \in Q$, where p is the solution to

$$\begin{cases} p_t + p_a + \mu(a)p = -u(a)p, & (a, t) \in Q \\ p(0, t) = \int_0^A \beta(a)p(a, t)\ da, & t \in (0, +\infty) \\ p(a, 0) = y_0(a), & a \in (0, A) \end{cases} \tag{4}$$

and x is the solution to

$$\begin{cases} x'(t) + \Phi(x(t)P(t))x(t) = 0, & t \in (0, +\infty) \\ P(t) = \int_0^A p(a, t)\ da \\ x(0) = 1. \end{cases} \tag{5}$$

It is obvious that y is a solution to (1). Note that also the solution x depends on the control u trough the dependence of the function $\Phi(x(t)P(t))$. For every $u \in \mathcal{U}$ fixed, (4) has a unique nonnegative solution and (5) has a unique Carathéodory solution (which is also nonnegative). Actually the solution p to (4) is given by

$$p(a,t) = \begin{cases} b(t-a)\exp\{-\int_0^a (\mu(s)+u(s))\ ds\}, & a < t \\ y_0(a-t)\exp\{-\int_0^t (\mu(a-t+s)+u(a-t+s))\ ds\}, & a > t. \end{cases}$$

Therefore, for $a < t$, the solution is written as

$$p(a,t) = b(t-a)\exp\{-\int_0^a (\mu(s)+u(s))\ ds\},$$

where b satisfies the following Volterra equation

$$b(t) = F(t) + \int_0^t K(t-s)b(s)\ ds. \tag{6}$$

Here

$$F(t) = \begin{cases} \int_t^A \beta(a)e^{-\int_0^t(\mu(a-t+s)+u(a-t+s))\ ds} y_0(a-t)\ da, t < A \\ 0, \qquad \text{otherwise} \end{cases}$$

and

$$K(t) = \begin{cases} \beta(t)\exp\{-\int_0^t (\mu(s)+u(s))\ ds\}, & t < A \\ 0, \qquad \text{otherwise.} \end{cases}$$

We have that $F \in C(R^+)$, $K \in L^\infty(R^+)$ and, consequently, $b \in C(R^+)$.

The following result holds (see e.g. [10]):

Theorem 1. *The solution b to* (6) *satisfies $b(t) = e^{\alpha^* t}b_0(t)$, $\forall t \in R^+$, where $\lim_{t\to\infty} b_0(t) = \bar{b}_0 \geq 0$ and α^* is the solution to the following equation:*

$$\int_0^A \beta(a)\exp\{-\int_0^a (\mu(s)+u(s)+\alpha)\ ds\}\ da = 1.$$

Taking into account the previous result, we obtain that, for $a < t$, the solution p can be written as

$$p(a,t) = e^{\alpha^*(t-a)}b_0(t-a)e^{-\int_0^a(\mu(s)+u(s))\ ds},$$

and we can state the next result:

Theorem 2. *In the case of non-trivial datum, i.e.,* $\bar{b}_0 > 0$*, we infer that*

$$\lim_{t\to\infty} \|p(t)\|_{L^\infty(0,A)} = 0 \text{ if } \alpha^* < 0,$$

$$\lim_{t\to\infty} \|p(t)\|_{L^1(0,A)} = +\infty \text{ if } \alpha^* > 0,$$

$$\lim_{t\to\infty} \|p(t) - \tilde{p}\|_{L^2(0,A)} = 0 \text{ if } \alpha^* = 0,$$

where $\tilde{p} = \bar{b}_0 e^{-\int_0^a (\mu(s)+u(s))\ ds}$ *is a nontrivial steady state solution to* (4)*.*

For more details and complete proofs of the previous theorems see [4]. Denoting by

$$R = \int_0^A \beta(a) e^{-\int_0^a (\mu(s)+u(s))\ ds}\ da,$$

the statements of Theorem 2 are equivalent to:

- If $R < 1$, then $\|p(t)\|_{L^\infty(0,A)} \to 0$, as $t \to +\infty$;
- If $R > 1$, then $\|p(t)\|_{L^1(0,A)} \to +\infty$, as $t \to +\infty$;
- If $R = 1$, then $\|p(t) - \tilde{p}\|_{L^2(0,A)} \to 0$, as $t \to +\infty$, where $\tilde{p}$ is the same as in Theorem 2.

Let now x be the unique Carathéodory solution to (5); obviously x is nonincreasing and nonnegative, therefore there exists $\lim_{t\to\infty} x(t) \in [0, 1]$.

Analyzing the previous results, we can see that in the case $R < 1$, the solution converges to 0 as $t \to \infty$, even without the additional mortality rate expressed by $\Phi(Y(t))$. Therefore, in what follows we shall consider the assumption (A5) (which implies $R > 1$).

Theorem 3. *If* p_0 *is a non-trivial datum and assumption* (A5) *is satisfied, then the solution* y *to* (1) *satisfies*

$$\lim_{t\to\infty} \|y(t) - \tilde{y}\|_{L^\infty(0,A)} = 0,$$

where

$$\tilde{y}(a) = \exp\{-\alpha^* a\} \bar{b}_0 \exp\{-\int_0^a (\mu(s) + u(s))\ ds\} \frac{1}{h_0} \Phi^{-1}(\alpha^*),$$

$a \in [0, A]$*, is a stationary solution to* (1) *and* $h_0 = \bar{b}_0 \int_0^A e^{-\int_0^a (\mu(s)+u(s))\ ds}\ da$*.*

Proof. By (A5) we have that $P(t){=}e^{\alpha^* t}h(t)$, $t \geq 0$, where $\lim_{t\to\infty} h(t){=}$ $h_0 > 0$. Then, the corresponding equation in x takes the form

$$x'(t) + \Phi(x(t)h(t)e^{\alpha^* t})x(t) = 0, \quad t > 0,$$

$$x(0) = 1,$$

or, equivalently, taking $z(t) = e^{\alpha^* t}x(t)$ $(t \geq 0)$:

$$z'(t) = (\alpha^* - \Phi(h(t)z(t)))z(t), \quad t > 0,$$

$$z(0) = 1.$$

Obviously, $z(t) > 0$, $\forall t \in R^+$; in fact

$$\lim_{t\to\infty} z(t) = z_0, \quad z_0 > 0,$$

where z_0 is the unique solution to $\alpha^* - \Phi(h_0 z_0) = 0$. Indeed, if we denote by z_h the unique solution to $\alpha^* - \Phi(hz) = 0$, and let $0 < h_1 < h_0 < h_2$; it follows that $z_{h_1} > z_0 > z_{h_2}$; as $\lim_{t\to\infty} h(t) = h_0$, it can be shown that

$$\lim_{t\to\infty} \mathrm{dist}(z(t), [z_{h_2}, z_{h_1}]) = 0,$$

which implies that

$$\lim_{t\to\infty} z(t) = z_0,$$

where

$$z_0 = \frac{\Phi^{-1}(\alpha^*)}{h_0}.$$

Therefore we have obtained that

$$\lim_{t\to\infty} y(t) = \tilde{y} \quad in \quad L^\infty(0, A),$$

where

$$\tilde{y}(a) = \exp\{-\alpha^* a\}\bar{b}_0 \exp\{-\int_0^a (\mu(s) + u(s))\ ds\}\frac{1}{h_0}\Phi^{-1}(\alpha^*),$$

$a \in [0, A]$, is a stationary solution to $(2)_{1-3}$.

In fact (3) has only two nonnegative solutions, one of them being the trivial one (see [4] and [6]).

We mention that this asymptotic behavior result for the linear age structured population dynamics has been proved first in [6]. □

3. Optimal harvesting. Assume in addition that $y_0(a) > 0$ a.e. in $(0, A)$. We shall consider next the optimal harvesting problem (2)-(3). First note that for every fixed $u \in \mathcal{U}$, the nontrivial and nonnegative solution to (3) is given by $\tilde{y}(a) = \tilde{y}(0)e^{-\int_0^a (u(s)+\mu(s)+\Phi(Y_0))\, ds}$, $a \in [0, A]$ and substituting this in the third equation in (3) we get

$$1 = \int_0^A \beta(a)e^{-\int_0^a (u(s)+\mu(s)+\Phi(Y_0))\, ds}\, da. \tag{7}$$

We can solve equation (7) for the single variable Y_0, and from

$$Y_0 = \int_0^A \tilde{y}(a)\, da = \tilde{y}(0)\int_0^A e^{-\int_0^a (u(s)+\mu(s)+\Phi(Y_0))\, ds}\, da,$$

we get the initial value $\tilde{y}(0)$.

Existence of an optimal solution

Theorem 4. *Problem* (2)-(3) *admits at least one optimal control.*

Proof. Denote by $d = \sup_{u\in\mathcal{U}} J(u)$, where $J(u) = \int_0^A u(a)\tilde{y}^u(a)\, da$ and $\tilde{y}^u$ denotes the nontrivial and nonnegative solution to (3) corresponding to the control $u \in \mathcal{U}$. By a comparison result we get that $0 \le J(u) \le L\int_0^A \overline{y}(a)\, da$, where $\overline{y}$ is the solution to (3) corresponding to $u \equiv 0$. Let now a sequence $\{u_n\}_{n\in N^*} \subset \mathcal{U}$ be such that $d - \frac{1}{n} < J(u_n) \le d$, $\forall n \in \mathbf{N}^*$. Since $\{u_n\}$ is bounded in $L^\infty(0, A)$ it follows that is also bounded in $L^2(0, A)$, so we can take a subsequence (also denoted by $\{u_n\}$) such that $u_n \to u^*$ weakly in $L^2(0, A)$. Since $\mathcal{U}$ is a convex and closed subset of $L^2(0, A)$, it is also weakly closed, therefore $u^* \in \mathcal{U}$. We have that

$$0 \le \tilde{y}^{u_n} \le \overline{y} \quad \text{a.e} \quad a \in (0, A), \tag{8}$$

which implies that, on a subsequence, $\tilde{y}^{u_n} \to \tilde{y}^*$ weakly in $L^2(0, A)$ and $\tilde{y}^{u_n}(0) \to \gamma$ in $\mathbf{R}$. Using a corollary to Mazur's theorem, we get that there exists a sequence $\tilde{y}_n$ such that

$$\tilde{y}_n = \sum_{i=n+1}^{k_n} \lambda_i^n \tilde{y}^{u_i}, \qquad \lambda_i^n \ge 0, \qquad \sum_{i=n+1}^{k_n} \lambda_i^n = 1,$$

and $\tilde{y}_n \to \tilde{y}^*$ in $L^2(0, A)$. Let $\tilde{u}_n$ be defined as

$$\tilde{u}_n(a) = \begin{cases} \dfrac{\sum_{i=n+1}^{k_n} \lambda_i^n y^{u_i}(a) u_i(a)}{\sum_{i=n+1}^{k_n} \lambda_i^n y^{u_i}(a)}, & \text{if } \displaystyle\sum_{i=n+1}^{k_n} \lambda_i^n y^{u_i}(a) \neq 0 \\ 0, & \text{if } \displaystyle\sum_{i=n+1}^{k_n} \lambda_i^n y^{u_i}(a) = 0. \end{cases}$$

For these controls we have $\tilde{u}_n \in \mathcal{U}$. In the following, we shall denote by

$$Y_0^{u_i} = \int_0^A \tilde{y}^{u_i}(a)\, da,$$

and

$$Y_0^* = \int_0^A \tilde{y}^*(a)\, da.$$

Since $\tilde{y}^{u_n} \to \tilde{y}^*$ weakly in $L^2(0, A)$ it follows that $Y_0^{u_n} \to Y_0^*$. Let us consider now the system (3) corresponding to the controls u_i:

$$\text{(9)} \qquad \begin{cases} (\tilde{y}^{u_i})' + \mu(a)\tilde{y}^{u_i} + \Phi(Y_0^{u_i})\tilde{y}^{u_i} = -u_i \tilde{y}^{u_i}, \quad a \in (0, A) \\ \tilde{y}^{u_i}(0) = \displaystyle\int_0^A \beta(a)\tilde{y}^{u_i}(a)\, da. \end{cases}$$

Multiplying the system (9) by λ_i^n and summarizing from $n+1$ to k_n, we obtain:

$$\begin{cases} (\tilde{y}_n)' + \mu(a)\tilde{y}_n + \displaystyle\sum_{n+1}^{k_n} \lambda_i^n \Phi(Y_0^{u_i})\tilde{y}^{u_i} = -u_n \tilde{y}_n, \quad a \in (0, A) \\ \tilde{y}_n(0) = \displaystyle\int_0^A \beta(a)\tilde{y}_n(a)\, da. \end{cases}$$

The solution to the above system, $\tilde{y}_n$, is given by

$$\begin{aligned} \tilde{y}_n(a) &= \tilde{y}_n(0) e^{-\int_0^a (\mu(s) + \tilde{u}_n(s))\, ds} \\ &\quad - \int_0^a e^{-\int_s^a (\mu(\tau) + \tilde{u}_n(\tau))\, d\tau} \sum_{n+1}^{k_n} \lambda_i^n \Phi(Y_0^{u_i})\tilde{y}^{u_i}(s)\, ds, \end{aligned}$$

for $a \in [0, A]$. Passing to the limit, one obtain that

$$\tilde{y}^*(a) = \gamma e^{-\int_0^a (\mu(s) + u^*(s))\, ds} - \int_0^a e^{-\int_s^a (\mu(\tau) + u^*(\tau))\, d\tau} \Phi(Y_0^*)\tilde{y}^*(s)\, ds,$$

a.e. $a \in (0, A)$. It follows immediately that there exists a representant of $\tilde{y}^*$ in $L^2(0, A)$ continuous on $[0, A]$ (also denoted by $\tilde{y}^*$) which verifies

$$\tilde{y}^*(a) = \tilde{y}^*(0)e^{-\int_0^a (\mu(s)+u^*(s))\ ds} - \int_0^a e^{-\int_s^a (\mu(\tau)+u^*(\tau))\ d\tau}\Phi(Y_0^*)\tilde{y}^*(s)\ ds,$$

$a \in [0, A]$, i.e. $\tilde{y}^*$ is a nonnegative solution to (3), corresponding to u^*. We will prove immediately that this solution is nontrivial (i.e. $\tilde{y}^* = \tilde{y}^{u^*}$). We will show in addition that $\tilde{y}_n \to \tilde{y}^*$ in $C([0, A])$. This yields

$$\begin{aligned}
|\tilde{y}_n(a) - \tilde{y}^*(a)| &\leq |\tilde{y}_n(0)e^{-\int_0^a (\mu(s)+\tilde{u}_n(s))\ ds} - \tilde{y}^*(0)e^{-\int_0^a (\mu(s)+u^*(s))\ ds}| \\
&+ \left|\int_0^a \left[e^{-\int_s^a (\mu(\tau)+\tilde{u}_n(\tau))\ d\tau}\sum_{n+1}^{k_n}\lambda_i^n\Phi(Y_0^{u_i})\tilde{y}^{u_i}(s)\right.\right. \\
&- \left.\left. e^{-\int_s^a (\mu(\tau)+u^*(\tau))\ d\tau}\Phi(Y_0^*)\tilde{y}^*(s)\right]\ ds\right| \\
&\leq e^{-\int_0^a (\mu(s)+u^*(s))\ ds}|\tilde{y}_n(0) - \tilde{y}^*(0)| \\
&+ |y^*(0)|\left|e^{-\int_0^a (\mu(s)+\tilde{u}_n(s))\ ds} - e^{-\int_0^a (\mu(s)+u^*(s))\ ds}\right| \\
&+ \int_0^a \left|e^{-\int_s^a (\mu(\tau)+\tilde{u}_n(\tau))\ d\tau}\sum_{n+1}^{k_n}\lambda_i^n\Phi(Y_0^{u_i})\tilde{y}^{u_i}(s)\right. \\
&- \left. e^{-\int_s^a (\mu(\tau)+u^*(\tau))\ d\tau}\Phi(Y_0^*)\tilde{y}^*(s)\right|\ ds,
\end{aligned}$$

for $a \in [0, A]$. Then taking $\sup_{a\in[0,A]}$ and having in mind the boundedness of the functions involved, we get that

$$\begin{aligned}
&\sup_{a\in[0,A]} |\tilde{y}_n(a) - \tilde{y}^*(a)| \\
&\leq C_1|\tilde{y}_n(0) - \tilde{y}^*(0)| + C_2\left|e^{-\int_0^a (\mu(s)+\tilde{u}_n(s))\ ds} - e^{-\int_0^a (\mu(s)+u^*(s))\ ds}\right| \\
&+ C_3\int_0^A \left(|\Phi(Y_0^*)|\left|\sum_{n+1}^{k_n}\lambda_i^n\tilde{y}^{u_i}(s) - \tilde{y}^*(s)\right|\right. \\
&+ \left.\left|\sum_{n+1}^{k_n}\lambda_i^n\tilde{y}^{u_i}(s)\right||\Phi(Y_0^{u_i}) - \Phi(Y_0^*)|\right)\ ds,
\end{aligned}$$

where C_1, C_2, C_3 are positive constants. Now, since $\tilde{u}_n \to u^*$, $\tilde{y}_n \to \tilde{y}^*$ in $L^2(0, A)$ and $Y_0^{u_n} \to Y_0^*$ in $\mathbf{R}$ (which implies $\Phi(Y_0^{u_n}) \to \Phi(Y_0^*)$ due to the continuity of Φ), it follows that $\tilde{y}_n \to \tilde{y}^*$ in $C([0,A])$. Actually, (3) has two nonnegative solutions: a trivial and a nontrivial one.

Since

$$J(\tilde{u}_n) = \sum_{i=n+1}^{k_n} \lambda_i^n J(u_i) \to d, \quad \text{as} \quad n \to +\infty,$$

we conclude that $J(u^*) = d > 0$ and consequently that $\tilde{y}^* = \tilde{y}^{u^*}$. $\square$

Necessary optimality conditions. Let us denote by q the adjoint state, i.e., q satisfies:

$$\begin{cases} q' - \mu(a)q - \Phi(Y_0^*)q + \Psi = u^*(a)(1+q), \quad a \in (0,A) \\ \Psi = \beta(a)q(0) - \Phi'(Y_0^*)\displaystyle\int_0^A q(a)\tilde{y}^*(a)\, da \\ q(A) = 0, \end{cases} \tag{10}$$

where $(u^*, \tilde{y}^*)$ is an optimal pair for (2)-(3) and $Y_0^* = \int_0^A \tilde{y}^*(a)\, da$.

Then the necessary optimality conditions are given by:

Theorem 5. *Let q the corresponding adjoint state* (*which exists and is unique*). *Then*

$$u^*(a) = \begin{cases} 0, & \text{if } 1 + q(a) < 0 \\ L, & \text{if } 1 + q(a) > 0. \end{cases} \tag{11}$$

Proof. The existence and uniqueness of the adjoint variable q can be proved via Banach's fixed point theorem.

Let $v \in L^\infty(0,A)$ such that $u^* + \varepsilon v \in \mathcal{U}$ for any $\varepsilon > 0$ small enough. From the optimality of u^* we get:

$$\int_0^A u^*(a)\tilde{y}^*(a) \geq \int_0^A (u^*(a) + \varepsilon v(a))\tilde{y}^{u^*+\varepsilon v}(a)\, da,$$

which implies that

$$\int_0^A u^*(a)\frac{\tilde{y}^{u^*+\varepsilon v}(a) - \tilde{y}^*(a)}{\varepsilon}\, da + \int_0^A v(a)\tilde{y}^{u^*+\varepsilon v}(a)\, da \leq 0. \tag{12}$$

In order to prove the maximum principle, we need to prove first the following convergencies:

$$\lim_{\varepsilon\to 0} \tilde{y}^{u^*+\varepsilon v} = \tilde{y}^* \quad \text{in} \quad C([0,A]), \tag{13}$$

and

$$\lim_{\varepsilon\to 0} \frac{\tilde{y}^{u^*+\varepsilon v} - \tilde{y}^*}{\varepsilon} = z \quad \text{in} \quad L^2(0,A), \tag{14}$$

where z is the solution to

$$\begin{cases} z' + [\mu(a) + \Phi(Y_0^*)]z + \Phi'(Y_0^*)\tilde{y}^* Z = -u^* z - v\tilde{y}^*, \quad a \in (0,A) \\ Z = \int_0^A z(a)\, da \\ z(0) = \int_0^A \beta(a) z(a)\, da. \end{cases} \tag{15}$$

Note that $\tilde{y}^{u^*+\varepsilon v}$ is the solution to

$$\begin{cases} (\tilde{y}^{u^*+\varepsilon v})' + r(a)\tilde{y}^{u^*+\varepsilon v} = -(\varepsilon v + \Phi(Y_0^{u^*+\varepsilon v}))\tilde{y}^{u^*+\varepsilon v}, \quad a \in (0,A) \\ Y_0^{u^*+\varepsilon v} = \int_0^A \tilde{y}^{u^*+\varepsilon v}(a)\, da \\ \tilde{y}^{u^*+\varepsilon v}(0) = \int_0^A \beta(a)\tilde{y}^{u^*+\varepsilon v}(a)\, da, \end{cases} \tag{16}$$

where $r(a) = \mu(a) + u^*(a)$. This implies $0 < \tilde{y}^{u^*+L} \le \tilde{y}^{u^*+\varepsilon v}(a) \le \overline{y}(a)$, $a \in [0,A]$, (for $\varepsilon \in (0,1)$ small enough) and, since the sequence is uniformly equicontinuous, we get that there exists a sequence ε_n such that $\tilde{y}^{u^*+\varepsilon_n v}$ converges uniformly to a limit $\bar{y}^*$. The uniform convergence of $\tilde{y}^{u^*+\varepsilon_n v}$ also implies

$$Y_0^{u^*+\varepsilon v} = \int_0^A \tilde{y}^{u^*+\varepsilon v}(a)\, da \to \bar{Y}_0^* = \int_0^A \bar{y}^*(a)\, da, \quad \text{in} \quad \mathbf{R}.$$

Writing the solution to (16), we get for every $a \in [0,A]$:

$$\begin{aligned} \tilde{y}^{u^*+\varepsilon v}(a) &= \tilde{y}^{u^*+\varepsilon v}(0)\exp\{-\int_0^a r(s)\, ds\} \\ &- \int_0^a \tilde{y}^{u^*+\varepsilon v}(s)[\Phi(Y_0^{u^*+\varepsilon v}) + \varepsilon v(s)]\exp\{-\int_s^a r(\tau)\, d\tau\}\, ds. \end{aligned}$$

Then, passing to the limit we obtain that:

$$\begin{aligned} \bar{y}^*(a) &= \bar{y}^*(0)\exp\{-\int_0^a r(s)\, ds\} \\ &- \int_0^a \bar{y}^*(s)\Phi(\bar{Y}_0^*)\exp\{-\int_s^a r(\tau)\, d\tau\}\, ds, \ a \in [0,A], \end{aligned}$$

(here $\bar{y}^*(a) \geq \tilde{y}^{u^*+L}(a)$, $\forall a \in [0, A]$), which is obviously the nontrivial non-negative solution to (3) corresponding to u^*, i.e. $\bar{y}^* = \tilde{y}^*$. In order to prove the second assertion, let us denote by $z^\varepsilon = \frac{1}{\varepsilon}\left[\tilde{y}^{u^*+\varepsilon v} - \tilde{y}^{u^*}\right]$. Subtracting (3) from (16) and dividing by ε, we obtain that z^ε is solution to:

$$\begin{cases} (z^\varepsilon)' + f(a)z^\varepsilon + \dfrac{1}{\varepsilon}\left[\Phi(Y_0^{u^*+\varepsilon v}) - \Phi(Y_0^*)\right]\tilde{y}^{u^*+\varepsilon v} = -v\tilde{y}^{u^*+\varepsilon v}, \\ \hfill a \in (0, A) \\ \Phi(Y_0^{u^*+\varepsilon v}) = \displaystyle\int_0^A \tilde{y}^{u^*+\varepsilon v}(a)\ da \\ z^\varepsilon(0) = \displaystyle\int_0^A \beta(a) z^\varepsilon(a)\ da, \end{cases} \tag{17}$$

where $f(a) = \mu(a) + u^*(a) + \Phi(Y_0^*)$, $a \in (0, A)$.

Let us take next $w_\varepsilon = z^\varepsilon - z$ and prove that $w_\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^2(0, A)$. Note that, due to the continuous differentiability of Φ, the following relation hold:

$$\frac{1}{\varepsilon}\left[\Phi(Y_0^{u^*+\varepsilon v}) - \Phi(Y_0^*)\right] = \Phi'(Y_0^*)\int_0^A z^\varepsilon(a)\ da + \varphi(\varepsilon), \tag{18}$$

where $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Subtracting now (15) from (17), we obtain that w_ε is the solution to

$$\begin{cases} w_\varepsilon' + f(a)w_\varepsilon + \Phi'(Y_0^*)\tilde{y}^* \displaystyle\int_0^A w_\varepsilon(a)\ da = F_\varepsilon(a), \quad a \in (0, A) \\ w_\varepsilon(0) = \displaystyle\int_0^A \beta(a) w_\varepsilon(a)\ da, \end{cases} \tag{19}$$

where we have denoted by

$$F_\varepsilon(a) = -(v(a) + \Phi'(Y_0^*)\int_0^A z(a)\ da)[\tilde{y}^{u^*+\varepsilon v}(a) - \tilde{y}^*(a)] - \varphi(\varepsilon)\tilde{y}^{u^*+\varepsilon v}(a).$$

Since $\tilde{y}^{u^*+\varepsilon v} \to \tilde{y}^*$ in $C([0, A])$, it is obvious that $F_\varepsilon \to 0$ in $L^\infty(0, A)$, as $\varepsilon \to 0$. Then we get that $w_\varepsilon \to w$ as $\varepsilon \to 0$, where w is the solution to:

$$\begin{cases} w' + f(a)w + \Phi'(Y_0^*)\tilde{y}^* \displaystyle\int_0^A w(a)\ da = 0, \quad a \in (0, A) \\ w(0) = \displaystyle\int_0^A \beta(a) w(a)\ da. \end{cases} \tag{20}$$

Actually, via Banach's fixed point theorem we infer that the above system has a unique solution. Therefore we have that $w_\varepsilon \to 0$ in $L^\infty(0, A)$, which concludes the proof of the second assertion.

Making now $\varepsilon \to 0$ in (12), and using that $\tilde{y}^{u^*+\varepsilon v} \to \tilde{y}^*$ in $C([0, A])$, and that $z^\varepsilon \to z$ in $L^2(0, A)$, we obtain that

$$\int_0^A u^*(a)z(a)\ da + \int_0^A v(a)\tilde{y}^*(a)\ da \leq 0,$$

for all $v \in L^\infty(0, A)$ such that $u^* + \varepsilon v \in \mathcal{U}$ for any $\varepsilon > 0$ small enough. Multiplying the first equation in (10) by z and integrating over $(0, A)$ we get:

$$\int_0^A z(a) \left[q'(a) - \mu(a)q(a) - \Phi(Y_0^*)q(a) + \Psi\right] da = \int_0^A z(a)u^*(a)(1+q(a))da.$$

Integrating by parts, replacing z' and eliminating the identical terms, it follows

$$\int_0^A z(a)u^*(a)\ da = \int_0^A v(a)\tilde{y}^*(a)q(a)\ da.$$

Therefore, we obtain that

$$\int_0^A \tilde{y}^*(a)(1 + q(a))v(a)\ da \leq 0,$$

for any $v \in L^\infty(0, A)$ such that $u^* + \varepsilon v \in \mathcal{U}$ for any $\varepsilon > 0$ small enough, which is equivalently to (11). □

Corollary 6. *By* (10) *and* (11) *it follows that the adjoint state satisfies*

$$\begin{cases} q' - \mu(a)q - \Phi(Y_0^*)q + \beta(a)q(0) \\ \qquad\qquad -\Phi'(Y_0^*)\displaystyle\int_0^A q(a)\tilde{y}^*(a)\ da = L(1+q)^+, \quad a \in (0, A) \\ q(A) = 0. \end{cases}$$

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