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A SPATIAL DECAY ESTIMATE IN THERMOVISCOELASTIC COMPOSITE CYLINDERS

BY

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Abstract. This work concerns the study of time–harmonic vibrations in a right finite cylinder made of an isotropic and homogeneous mixture consisting of two components: an elastic solid and a Kelvin-Voigt material. The both thermal and viscoelastic effects are used to introduce an adequate measure of the amplitude of vibrations and to establish an exponential decay estimate of Saint-Venant type that holds for every frequency of vibrations and for mixtures for which the constitutive coefficients are supposed to satisfy some mild positive definiteness conditions.

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1. Introduction

The continuum theory of mixtures has been a subject of intensive study in literature. Various theories of mixtures developed in Eulerian or Lagrangian description have been proposed in the last decades in order to describe the thermomechanical behavior of interacting continua. A detailed discussion on the basic formulations and the progress in the field can be found in the review articles by Bowen [5], Atkin and Craine [2, 3], Bedford and Drumheller [4] and the books of Samohyl [25] and Rajagopal and Tao [24].

A great attention has been paid to the theories taking into account the viscoelastic effects. In this connection, Ieşan [19] has developed a continuum theory for a viscoelastic composite as a mixture of a porous elastic solid and a Kelvin-Voigt material. In this theory the independent constitutive variables are: the displacement fields, velocity fields, volume fractions, displacement gradients, velocity gradient, volume fraction gradients, temperature and temperature gradient. The theory takes into account the effects of porosity [18, 21], by considering the volume fraction of each constituent as an independent kinematic variable. We note that in the linear dynamic theory some uniqueness and continuous results have been established by Ieşan [19], the spatial behavior problem has been investigated by Galeş [13], some existence and exponential stability results have been derived by Quintanilla [22], while the problem of existence of weak solutions and the asymptotic partition of total energy have been studied by Galeş [14].

The purpose of this paper is to investigate the spatial behavior of solutions describing harmonic vibrations of a right cylinder filled by an isotropic and homogeneous viscoelastic mixture. Our analysis includes thermal effects. But as in [22], we do not consider the effect of porosity. We consider a finite cylinder subject to boundary data varying harmonically in time on one end, while the other end and lateral surface are subjected to null boundary data. The heat supply and the body forces are supposed to be absent.

Initial boundary value problems of this type have been treated by Flavin and Knops [10] in the context of the linearly damped wave equation and the linearly elastic damped cylinder. They proved that in both cases the existence of damping gives rise ultimately to a steady-state oscillation, whose amplitude decays exponentially from the excited end provided the exciting frequency is less than a certain critical value. The later case has been investigated under positive definiteness assumption upon elasticity tensor. This work has been followed by further developments (see [1], [6], [7], [11], [12], [16], [20], [23] and references therein). Within the framework of the theory of mixtures, the spatial behavior of solutions describing harmonic vibrations has been investigated previously in [12], where the case of swelling porous elastic soils has been considered, and in [23], where some exponential decay estimates for thermoelastic mixtures have been derived. The results established in all these papers, including those devoted to the theory of mixtures, suggest exponential decay of activity away from the excited end, provided the frequency of vibration is lower than a critical value and the constitutive coefficients satisfy some positive definiteness conditions.

In a recent paper [8] we considered a homogeneous and isotropic Kelvin-Voigt material, and we pointed out how the dissipative mechanism may be used to obtain decay estimates valid for every value of the frequency of vibrations and without any restriction conditions on the Lamé coefficients. This result was extended in [15] for thermoviscoelastic cylinders consisting of two anisotropic Kelvin-Voigt materials.

Here we consider a thermoviscoelastic composite cylinder consisting of an elastic solid and a Kelvin–Voigt material. Although one component of the mixture is elastic, we prove that the viscoelastic effect of the second constituent is so strong that for any value of the frequency of vibrations, the activity decays exponentially away from the excited end of the cylinder. In this sense, we construct a measure by combining two type of functions. One function is suggested by the results obtained in the case of classical (or generalized) elastic media (see [1], [6], [7], [10], [11], [12], [16], [20], [23]), while the other one is obtained as in [8]. Then, we follow the well known method, namely we derive a first–order differential inequality for this measure, which by an integration leads to a spatial decay estimate of Saint–Venant type. This estimate holds for mixtures for which the dissipation energy density is a positive definite quadratic form and some elastic constitutive coefficients are supposed to satisfy mild positive definiteness conditions.

2. Formulation of the problem

Throughout this paper, we refer the motion of a continuum to a fixed system of rectangular Cartesian axes $0x_k$ $(k = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3)$ whereas Greek subscripts are confined to the range $(1, 2)$, summation over repeated subscripts is implied, subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, and a superposed dot denotes time differentiation.

Let B denote the interior of a right cylinder of length $L > 0$ whose cross section is bounded by one or more piecewise smooth curves. Choose the Cartesian coordinates such that the origin lies in one end of the cylinder and such that the x_3-axis is parallel to the generators. Let $D(x_3)$ denote the cross section of the cylinder corresponding to the axial coordinate x_3, and let $\partial D(x_3)$ denote the cross-sectional boundary. We denote by π the lateral surface of the cylinder, that is $\pi = \partial D \times [0, L]$.

We assume that a chemically inert mixture consisting of two constituents, a Kelvin–Voigt material s_1 and an elastic solid s_2, fills B. According to the linear theory, the fundamental equations that govern the motion of an isotropic and homogeneous mixture, in the absence of the body forces and heat supply, are (see [19, 22]):

–the equations of motion

$$t_{rl,r} - p_l = \rho_1^0 \ddot{u}_l, \quad s_{rl,r} + p_l = \rho_2^0 \ddot{w}_l, \tag{2.1}$$

where ρ_1^0 and ρ_2^0 are the densities at time $t = 0$ of s_1 and s_2, t_{rl} and s_{rl} are the partial stress tensors, u_l and w_l are the displacement vector fields and p_l is the internal body force characterizing the mechanical interaction between constituents;

–the energy equation

$$\rho_0 T_0 \dot{\eta} = q_{l,l}, \tag{2.2}$$

where $\rho_0 = \rho_1^0 + \rho_2^0$, T_0 is the constant absolute temperature of the body in the reference configuration, q_i is the heat flux vector and η is the entropy;

–the constitutive equations

$$\begin{aligned} t_{rl} &= (\lambda + \nu)e_{nn}\delta_{rl} + 2(\mu + \zeta)e_{rl} + (\alpha + \nu)g_{nn}\delta_{rl} + (2\kappa + \zeta)g_{rl} \\ &\quad + (2\gamma + \zeta)g_{lr} - (\beta^{(1)} + \beta^{(2)})T\delta_{rl} + \lambda^* \dot{e}_{nn}\delta_{rl} + 2\mu^* \dot{e}_{rl}, \\ s_{rl} &= \nu e_{nn}\delta_{rl} + 2\zeta e_{rl} + \alpha g_{nn}\delta_{rl} + 2\kappa g_{lr} + 2\gamma g_{rl} - \beta^{(2)} T\delta_{rl}, \\ p_l &= \xi d_l + \xi^* \dot{d}_l + b^* T_{,l}, \\ \rho_0 \eta &= \beta^{(1)} e_{nn} + \beta^{(2)} g_{nn} + aT, \\ q_l &= k^* T_{,l} + f^* \dot{d}_l, \end{aligned} \tag{2.3}$$

where T is the temperature variation from the uniform reference temperature T_0, δ_{rl} is the Kronecker delta, λ, ν, μ, ζ, α, κ, γ, $\beta^{(1)}$, $\beta^{(2)}$, ξ, a, λ^*,μ^*, ξ^*, b^*, k^*, f^* are the constitutive coefficients and e_{rl}, g_{rl} and d_l are defined by

–the geometrical equations

$$e_{rl} = \frac{1}{2}(u_{r,l} + u_{l,r}), \qquad g_{rl} = w_{r,l} + u_{l,r}, \qquad d_l = u_l - w_l \ . \tag{2.4}$$

The Clausius-Duhem inequality implies that the dissipation energy density

$$\Phi \equiv \lambda^* \dot{e}_{rr}\dot{e}_{ll} + 2\mu^* \dot{e}_{rl}\dot{e}_{rl} + \xi^* \dot{d}_l \dot{d}_l + \frac{1}{T_0} k^* T_{,l} T_{,l} + \Big(b^* + \frac{1}{T_0} f^*\Big)\dot{d}_l T_{,l}, \tag{2.5}$$

is positive. This implies that

$$\mu^* \geq 0,\ 3\lambda^* + 2\mu^* \geq 0,\ k^* \geq 0,\ \xi^* \geq 0,\ 4\xi^* k^* \geq T_0\Big(b^* + \frac{1}{T_0} f^*\Big)^2. \tag{2.6}$$

We consider the initial boundary value problem defined by the equations of motion (2.1), the energy equation(2.2), the constitutive equations (2.3), the geometrical equations (2.4), the initial conditions

$$\begin{aligned} &u_l = a_l^{(1)}(x_1, x_2, x_3), \quad w_l = a_l^{(2)}(x_1, x_2, x_3), \quad T = \theta^0(x_1, x_2, x_3),\\ &\dot{u}_l = b_l^{(1)}(x_1, x_2, x_3), \quad \dot{w}_l = b_l^{(2)}(x_1, x_2, x_3), \qquad \text{in } B \text{ at } t = 0, \end{aligned} \tag{2.7}$$

and the lateral boundary conditions

$$u_l = 0, \qquad w_l = 0, \qquad T_l = 0 \qquad \text{on } \pi \times [0, t_0) \tag{2.8}$$

together with the end boundary conditions

$$\begin{aligned} &u_l = \tilde{u}_l(x_1, x_2)\exp(-i\omega t), \qquad w_l = \tilde{w}_l(x_1, x_2)\exp(-i\omega t),\\ &T = \tilde{T}(x_1, x_2)\exp(-i\omega t), \qquad \text{on } D(0) \times [0, t_0), \end{aligned} \tag{2.9}$$

and

$$u_l = 0, \quad w_l = 0, \quad T = 0, \qquad \text{on } D(L) \times [0, t_0), \tag{2.10}$$

where ω is a positive constant and represents the frequency of vibrations and $i = \sqrt{-1}$.

We can consider the decomposition

$$\begin{aligned} u_l &= U_l(x_1, x_2, x_3, t) + v_l(x_1, x_2, x_3)\exp(-i\omega t),\\ w_l &= W_l(x_1, x_2, x_3, t) + \psi_l(x_1, x_2, x_3)\exp(-i\omega t),\\ T &= \Theta(x_1, x_2, x_3, t) + \theta(x_1, x_2, x_3)\exp(-i\omega t), \end{aligned} \tag{2.11}$$

where (U_l, W_l, Θ) absorbs the initial conditions and satisfies the null boundary conditions and the equations (2.1),(2.2), (2.3) and (2.4), while (v_l, ψ_l, θ) satisfies the boundary value problem consisting of

$$\begin{aligned} T_{rl,r} - P_l &= -\rho_1^0 \omega^2 v_l,\\ S_{rl,r} + P_l &= -\rho_2^0 \omega^2 \psi_l, \end{aligned} \tag{2.12}$$

$$Q_{l,l} + i\omega T_0\Big[(\beta^{(1)} + \beta^{(2)})v_{l,l} + \beta^{(2)}\psi_{l,l} + a\theta\Big] = 0, \tag{2.13}$$

where

$$\begin{aligned}
T_{rl} &= (\lambda + \alpha + 2\nu - i\omega\lambda^*)v_{n,n}\delta_{rl} + (\mu + 2\zeta + 2\gamma - i\omega\mu^*)v_{r,l}\\
&\quad + (\mu + 2\zeta + 2\kappa - i\omega\mu^*)v_{l,r} + (\alpha + \nu)\psi_{n,n}\delta_{rl} + (2\kappa + \zeta)\psi_{r,l}\\
&\quad + (2\gamma + \zeta)\psi_{l,r} - (\beta^{(1)} + \beta^{(2)})\theta\delta_{rl},\\
S_{rl} &= (\alpha + \nu)v_{n,n}\delta_{rl} + (2\kappa + \zeta)v_{r,l} + (2\gamma + \zeta)v_{l,r}\\
&\quad + \alpha\psi_{n,n}\delta_{rl} + 2\kappa\psi_{l,r} + 2\gamma\psi_{r,l} - \beta^{(2)}\theta\delta_{rl},\\
P_l &= (\xi - i\omega\xi^*)(v_l - \psi_l) + b^*\theta_{,l},\\
Q_l &= k^*\theta_{,l} - i\omega f^*(v_l - \psi_l),
\end{aligned} \tag{2.14}$$

subjected to the homogeneous boundary conditions

$$v_l = 0, \qquad \psi_l = 0, \qquad \theta = 0 \qquad \text{on } \pi \tag{2.15}$$

and the end boundary conditions

$$v_l = \tilde{u}_l(x_1, x_2), \qquad \psi_l = \tilde{w}_l(x_1, x_2), \qquad \theta = \tilde{T}(x_1, x_2) \qquad \text{on } D(0) \tag{2.16}$$

$$v_l = 0, \qquad \psi_l = 0, \qquad \theta = 0 \qquad \text{on } D(L). \tag{2.17}$$

As in [10, 8], it may be established that (U_l, W_l, Θ) tends to zero as time tends to infinity. So that, (U_l, W_l, Θ) represents the transient and $(v_l, \psi_l, \theta)\exp(-i\omega t)$ the forced oscillation (see [10]).

From (2.12), (2.13) and (2.14) we deduce the following system of partial differential equations for the functions v_l, ψ_l and θ:

$$\begin{aligned}
&(\alpha_1 - i\omega\mu^*)v_{l,rr} + [\alpha_2 - i\omega(\lambda^* + \mu^*)]v_{r,rl} + \beta_1\psi_{l,rr} + \beta_2\psi_{r,rl}\\
&\quad - (a_1 + b^*)\theta_{,l} - (\xi - i\omega\xi^*)(v_l - \psi_l) + \rho_1^0\omega^2 v_l = 0\ ,\\
&\beta_1 v_{l,rr} + \beta_2 v_{r,rl} + \gamma_1\psi_{l,rr} + \gamma_2\psi_{r,rl}\\
&\quad - (a_2 - b^*)\theta_{,l} + (\xi - i\omega\xi^*)(v_l - \psi_l) + \rho_2^0\omega^2\psi_l = 0,\\
&k^*\theta_{,ll} + i\omega(T_0 a_1 - f^*)v_{l,l} + i\omega(T_0 a_2 + f^*)\psi_{l,l} + i\omega T_0 a\theta = 0,
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
&\alpha_1 = \mu + 2\zeta + 2\kappa, \quad \alpha_2 = \lambda + \mu + \alpha + 2\nu + 2\gamma + 2\zeta,\\
&\beta_1 = 2\gamma + \zeta, \quad \beta_2 = \alpha + \nu + 2\kappa + \zeta, \quad \gamma_1 = 2\kappa,
\end{aligned} \tag{2.19}$$

$$\gamma_2 = 2\gamma + \alpha, \quad a_1 = \beta^{(1)} + \beta^{(2)}, \quad a_2 = \beta^{(2)}.$$

Let us use the notation $(\mathcal{P})$ for the boundary value problem consisting of the field equations (2.18) in B and the boundary conditions (2.15), (2.16) and (2.17).

We note that, except the terms μ^*, λ^*, ξ^*, b^* and f^* the equations (2.18) are the same as those investigated in [23]. So, it is easy to extend the results established in [23] to the above model. However, based on these new terms, we develop a different technique to obtain more information on the behavior of amplitude of harmonic vibrations. If the estimates given in [23] hold for harmonic vibrations whose frequency is lower than a certain critical value, the result presented here holds for every value of the frequency of vibrations. Moreover, we use milder positive definiteness conditions on the elastic constitutive coefficients.

3. Hypotheses and some preliminary results

Throughout this paper we shall assume that the elastic coefficients of the mixture satisfy the following mild positive definiteness conditions:

$$\gamma_1 > 0\,, \quad \gamma_1 + 3\gamma_2 > 0\,. \tag{3.1}$$

Moreover, suggested by the dissipation (2.5), we suppose that the dissipation coefficients satisfy the conditions:

$$\mu^* > 0\,, \quad k^* > 0\,, \quad 4\xi^* k^* > T_0\Big(b^* + \frac{1}{T_0} f^*\Big)^2. \tag{3.2}$$

Clearly, (2.6) and (3.2) assure that the dissipation potential Φ (see (2.5)) is a positive definite quadratic form in terms of $\dot{e}_{ij}$, $\dot{d}_l$ and T_l. Then, there exist the positive constants k_m^* and ξ_m^* such that the following inequality holds true

$$\begin{aligned}\xi^*(v_l - \psi_l)(\overline{v}_l - \overline{\psi}_l) + \frac{1}{2}\Big(b^* + \frac{1}{T_0} f^*\Big)\Big[(v_l - \psi_l)\frac{\overline{i\theta_{,l}}}{\omega} + (\overline{v}_l - \overline{\psi}_l)\frac{i\theta_{,l}}{\omega}\Big] \\ + \frac{1}{T_0} k^* \frac{i\theta_{,l}}{\omega}\frac{\overline{i\theta_{,l}}}{\omega} \geq \xi_m^*(v_l - \psi_l)(\overline{v}_l - \overline{\psi}_l) + \frac{1}{T_0\omega^2} k_m^* \theta_{,l}\overline{\theta}_{,l},\end{aligned} \tag{3.3}$$

where the superposed bar denotes complex conjugate.

Now, let us consider two constants $\alpha_1^{\circledcirc}$ and $\alpha_2^{\circledcirc}$, having the same dimension as the constitutive constants α_1, α_2, γ_1, γ_2, such that

$$\alpha_1^{\circledcirc} > \frac{\beta_1^2}{\gamma_1}, \qquad \alpha_1^{\circledcirc} + 3\alpha_2^{\circledcirc} > \frac{(\beta_1+3\beta_2)^2}{\gamma_1+3\gamma_2}. \tag{3.4}$$

For example, we may set

$$\alpha_1^{\circledcirc} = \frac{\beta_1^2}{\gamma_1} + \gamma_1 \ \text{ and } \alpha_1^{\circledcirc} + 3\alpha_2^{\circledcirc} = \frac{(\beta_1+3\beta_2)^2}{\gamma_1+3\gamma_2} + \gamma_1 + 3\gamma_2.$$

Then, from (3.1) and (3.4) it follows that

$$\begin{aligned}\sigma = {} & \alpha_1^{\circledcirc} v_{r,l}\overline{v}_{r,l} + \alpha_2^{\circledcirc} v_{r,r}\overline{v}_{l,l} + \beta_1(v_{r,l}\overline{\psi}_{r,l} + \overline{v}_{r,l}\psi_{r,l}) \\ & + \beta_2(v_{l,l}\overline{\psi}_{r,r} + \overline{v}_{l,l}\psi_{r,r}) + \gamma_1\psi_{r,l}\overline{\psi}_{r,l} + \gamma_2\psi_{l,l}\overline{\psi}_{r,r},\end{aligned} \tag{3.5}$$

is a positive definite quadratic form in terms of $v_{r,l}$ and $\psi_{r,l}$. The distinct eigenvalues of the corresponding linear transformation can be calculated rapidly and easily by using a mathematical software. We note that these eigenvalues have been derived in [9] (see the relation (103)). In our notations, they are

$$\begin{aligned}\pi_{1;2} &= \frac{1}{2}\Big[\alpha_1^{\circledcirc} + \gamma_1 \mp \sqrt{4\beta_1^2 + (\alpha_1^{\circledcirc} - \gamma_1)^2}\Big], \\ \pi_{3;4} &= \frac{1}{2}\Big[\alpha_1^{\circledcirc} + \gamma_1 + 3(\alpha_2^{\circledcirc} + \gamma_2) \\ &\quad \mp \sqrt{4(\beta_1+3\beta_2)^2 + (\alpha_1^{\circledcirc} + 3\alpha_2^{\circledcirc} - \gamma_1 - 3\gamma_2)^2}\Big].\end{aligned} \tag{3.6}$$

Introducing the notations

$$\pi_m = \min\{\pi_1,\ \pi_3\}, \qquad \pi_M = \max\{\pi_2,\ \pi_4\}, \tag{3.7}$$

then we have

$$\pi_m(v_{r,l}\overline{v}_{r,l} + \psi_{r,l}\overline{\psi}_{r,l}) \le \sigma \le \pi_M(v_{r,l}\overline{v}_{r,l} + \psi_{r,l}\overline{\psi}_{r,l}). \tag{3.8}$$

We associate with the amplitude $\mathcal{V} = \{v_l, \psi_l, \theta\}$ of the steady-state vibration the following cross-sectional functionals

$$
\begin{aligned}
K(x_3) = &-\int_{D(x_3)}\Big[\alpha_1(v_{l,3}\overline{v}_l+\overline{v}_{l,3}v_l)+\alpha_2(v_{r,r}\overline{v}_3+\overline{v}_{r,r}v_3)\\
&+\beta_1(\psi_{l,3}\overline{v}_l+\overline{\psi}_{l,3}v_l+v_{l,3}\overline{\psi}_l+\overline{v}_{l,3}\psi_l)\\
&+\beta_2(\psi_{r,r}\overline{v}_3+\overline{\psi}_{r,r}v_3+v_{r,r}\overline{\psi}_3+\overline{v}_{r,r}\psi_3)+\gamma_1(\psi_{l,3}\overline{\psi}_l+\overline{\psi}_{l,3}\psi_l)\\
&+\gamma_2(\psi_{r,r}\overline{\psi}_3+\overline{\psi}_{r,r}\psi_3)-a_1(v_3\overline{\theta}+\overline{v}_3\theta)-a_2(\psi_3\overline{\theta}+\overline{\psi}_3\theta)\\
&+\frac{f^*}{T_0}[(v_3-\psi_3)\overline{\theta}+(\overline{v}_3-\overline{\psi}_3)\theta]-\frac{i}{\omega T_0}k^*(\theta\overline{\theta}_{,3}-\overline{\theta}\theta_{,3})\\
&-i\omega\mu^*(v_{l,3}\overline{v}_l-\overline{v}_{l,3}v_l)-i\omega(\lambda^*+\mu^*)(v_{r,r}\overline{v}_3-\overline{v}_{r,r}v_3)\Big]dA,
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
H(x_3) = &-\int_{D(x_3)}\Big[i\alpha_1(v_{l,3}\overline{v}_l-\overline{v}_{l,3}v_l)+i\alpha_2(v_{r,r}\overline{v}_3-\overline{v}_{r,r}v_3)\\
&+i\beta_1(\psi_{l,3}\overline{v}_l-\overline{\psi}_{l,3}v_l+v_{l,3}\overline{\psi}_l-\overline{v}_{l,3}\psi_l)\\
&+i\beta_2(\psi_{r,r}\overline{v}_3-\overline{\psi}_{r,r}v_3+v_{r,r}\overline{\psi}_3-\overline{v}_{r,r}\psi_3)\\
&+i\gamma_1(\psi_{l,3}\overline{\psi}_l-\overline{\psi}_{l,3}\psi_l)+i\gamma_2(\psi_{r,r}\overline{\psi}_3-\overline{\psi}_{r,r}\psi_3)\\
&+ia_1(v_3\overline{\theta}-\overline{v}_3\theta)+ia_2(\psi_3\overline{\theta}-\overline{\psi}_3\theta)-\frac{if^*}{T_0}[(v_3-\psi_3)\overline{\theta}-(\overline{v}_3-\overline{\psi}_3)\theta]\\
&+\frac{1}{\omega T_0}k^*(\theta\overline{\theta}_{,3}+\overline{\theta}\theta_{,3})+\omega\mu^*(v_{l,3}\overline{v}_l+\overline{v}_{l,3}v_l)\\
&+\omega(\lambda^*+\mu^*)(v_{r,r}\overline{v}_3+\overline{v}_{r,r}v_3)\Big]dA.
\end{aligned}
\tag{3.10}
$$

The field equations (2.18), the boundary conditions (2.15) and the divergence theorem may be used to deduce the following:

Lemma 1. *The first derivatives of the functions $K(\cdot)$ and $H(\cdot)$ associated with the solution $\mathcal{V}=\{v_l,\psi_l,\theta\}$ of the boundary value problem $(\mathcal{P})$ are*

$$
\begin{aligned}
\frac{dK}{dx_3}(x_3) = -2\int_{D(x_3)}\Big[&(\alpha_1-\alpha_1^{\circledcirc})v_{l,r}\overline{v}_{l,r}+(\alpha_2-\alpha_2^{\circledcirc})v_{r,r}\overline{v}_{l,l}+\sigma\\
&+\xi(v_l-\psi_l)(\overline{v}_l-\overline{\psi}_l)-\rho_1^0\omega^2v_l\overline{v}_l-\rho_2^0\omega^2\psi_l\overline{\psi}_l\\
&+\frac{1}{2}\Big(b^*+\frac{f^*}{T_0}\Big)[\theta_{,l}(\overline{v}_l-\overline{\psi}_l)+\overline{\theta}_{,l}(v_l-\psi_l)]+a|\theta|^2\Big]dA,
\end{aligned}
\tag{3.11}
$$

and

$$\begin{aligned}\frac{dH}{dx_3}(x_3) = -2\omega \int_{D(x_3)} \Big[&\mu^* v_{l,r}\overline{v}_{l,r} + (\lambda^* + \mu^*) v_{r,r}\overline{v}_{l,l} \\ &+ \xi^*(v_l - \psi_l)(\overline{v}_l - \overline{\psi}_l) + \frac{1}{2}\Big(b^* + \frac{f^*}{T_0}\Big)\Big[\frac{i\theta_{,l}}{\omega}(\overline{v}_l - \overline{\psi}_l) \\ &+ \frac{\overline{i\theta}_{,l}}{\omega}(v_l - \psi_l)\Big] + \frac{1}{T_0}k^*\frac{i\theta_{,l}}{\omega}\frac{\overline{i\theta}_{,l}}{\omega}\Big] dA,\end{aligned} \tag{3.12}$$

where σ is given by (3.5).

Proof. We give the details for the derivative dH/dx_3. The relation (3.11) follows in a similar manner.

Differentiating (3.10), we get

$$\begin{aligned}\frac{dH}{dx_3}(x_3) = -\int_{D(x_3)} \Big[&i\alpha_2(v_{r,r}\overline{v}_{3,3} - \overline{v}_{r,r}v_{3,3}) \\ &+ i\beta_2(\psi_{r,r}\overline{v}_{3,3} - \overline{\psi}_{r,r}v_{3,3} + v_{r,r}\overline{\psi}_{3,3} - \overline{v}_{r,r}\psi_{3,3}) \\ &+ i\gamma_2(\psi_{r,r}\overline{\psi}_{3,3} - \overline{\psi}_{r,r}\psi_{3,3}) + ia_1(v_{3,3}\overline{\theta} + v_3\overline{\theta}_{,3} - \overline{v}_{3,3}\theta - \overline{v}_3\theta_{,3}) \\ &+ ia_2(\psi_{3,3}\overline{\theta} + \psi_3\overline{\theta}_{,3} - \overline{\psi}_{3,3}\theta - \overline{\psi}_3\theta_{,3}) \\ &- \frac{if^*}{T_0}[(v_{3,3} - \psi_{3,3})\overline{\theta} + (v_3 - \psi_3)\overline{\theta}_{,3} - (\overline{v}_{3,3} - \overline{\psi}_{3,3})\theta - (\overline{v}_3 - \overline{\psi}_3)\theta_{,3}] \\ &+ \frac{2}{\omega T_0}k^*\theta_{,3}\overline{\theta}_{,3} + 2\omega\mu^* v_{l,3}\overline{v}_{l,3} + \omega(\lambda^* + \mu^*)(v_{r,r}\overline{v}_{3,3} + \overline{v}_{r,r}v_{3,3})\Big] dA \\ - \int_{D(x_3)} \Big[&i\alpha_1(v_{l,33}\overline{v}_l - \overline{v}_{l,33}v_l) + i\alpha_2(v_{r,r3}\overline{v}_3 - \overline{v}_{r,r3}v_3) \\ &+ i\beta_1(\psi_{l,33}\overline{v}_l - \overline{\psi}_{l,33}v_l + v_{l,33}\overline{\psi}_l - \overline{v}_{l,33}\psi_l) \\ &+ i\beta_2(\psi_{r,r3}\overline{v}_3 - \overline{\psi}_{r,r3}v_3 + v_{r,r3}\overline{\psi}_3 - \overline{v}_{r,r3}\psi_3) \\ &+ i\gamma_1(\psi_{l,33}\overline{\psi}_l - \overline{\psi}_{l,33}\psi_l) \\ &+ i\gamma_2(\psi_{r,r3}\overline{\psi}_3 - \overline{\psi}_{r,r3}\psi_3) + \frac{1}{\omega T_0}k^*(\theta\overline{\theta}_{,33} + \overline{\theta}\theta_{,33}) \\ &+ \omega\mu^*(v_{l,33}\overline{v}_l + \overline{v}_{l,33}v_l) + \omega(\lambda^* + \mu^*)(v_{r,r3}\overline{v}_3 + \overline{v}_{r,r3}v_3)\Big] dA.\end{aligned} \tag{3.13}$$

On the basis of the field equations (2.18), the square bracket of the second integral may be written in the form

$$
\begin{aligned}
& i\alpha_1(v_{l,33}\overline{v}_l - \overline{v}_{l,33}v_l) + i\alpha_2(v_{r,r3}\overline{v}_3 - \overline{v}_{r,r3}v_3) + i\beta_1(\psi_{l,33}\overline{v}_l - \overline{\psi}_{l,33}v_l \\
& + v_{l,33}\overline{\psi}_l - \overline{v}_{l,33}\psi_l) + i\beta_2(\psi_{r,r3}\overline{v}_3 - \overline{\psi}_{r,r3}v_3 + v_{r,r3}\overline{\psi}_3 - \overline{v}_{r,r3}\psi_3) \\
& + i\gamma_1(\psi_{l,33}\overline{\psi}_l - \overline{\psi}_{l,33}\psi_l) + i\gamma_2(\psi_{r,r3}\overline{\psi}_3 - \overline{\psi}_{r,r3}\psi_3) \\
& + \frac{1}{\omega T_0}k^*(\theta\overline{\theta}_{,33} + \overline{\theta}\theta_{,33}) \\
& + \omega\mu^*(v_{l,33}\overline{v}_l + \overline{v}_{l,33}v_l) + \omega(\lambda^* + \mu^*)(v_{r,r3}\overline{v}_3 + \overline{v}_{r,r3}v_3) \\
& = -i\alpha_1(v_{l,\rho\rho}\overline{v}_l - \overline{v}_{l,\rho\rho}v_l) - i\alpha_2(v_{r,r\rho}\overline{v}_\rho - \overline{v}_{r,r\rho}v_\rho) \\
& - i\beta_1(\psi_{l,\rho\rho}\overline{v}_l - \overline{\psi}_{l,\rho\rho}v_l + v_{l,\rho\rho}\overline{\psi}_l - \overline{v}_{l,\rho\rho}\psi_l) \\
& - i\beta_2(\psi_{r,r\rho}\overline{v}_\rho - \overline{\psi}_{r,r\rho}v_\rho + v_{r,r\rho}\overline{\psi}_\rho - \overline{v}_{r,r\rho}\psi_\rho) \\
& - i\gamma_1(\psi_{l,\rho\rho}\overline{\psi}_l - \overline{\psi}_{l,\rho\rho}\psi_l) - i\gamma_2(\psi_{r,r\rho}\overline{\psi}_\rho - \overline{\psi}_{r,r\rho}\psi_\rho) \\
& - \omega\mu^*(v_{l,\rho\rho}\overline{v}_l + \overline{v}_{l,\rho\rho}v_l) - \omega(\lambda^* + \mu^*)(v_{r,r\rho}\overline{v}_\rho + \overline{v}_{r,r\rho}v_\rho) \\
& + 2\omega\xi^*(v_l - \psi_l)(\overline{v}_l - \overline{\psi}_l) - \frac{1}{\omega T_0}k^*(\theta_{,\rho\rho}\overline{\theta} + \overline{\theta}_{,\rho\rho}\theta) \\
& - ia_1(v_{l,l}\overline{\theta} + v_l\overline{\theta}_{,l} - \overline{v}_{l,l}\theta - \overline{v}_l\theta_{,l}) - ia_2(\psi_{l,l}\overline{\theta} + \psi_l\overline{\theta}_{,l} - \overline{\psi}_{l,l}\theta - \overline{\psi}_l\theta_{,l}) \\
& + ib^*[\theta_{,l}(\overline{v}_l - \overline{\psi}_l) - \overline{\theta}_{,l}(v_l - \psi_l)] + \frac{if^*}{T_0}[(v_{l,l} - \psi_{l,l})\overline{\theta} - (\overline{v}_{l,l} - \overline{\psi}_{l,l})\theta].
\end{aligned}
\tag{3.14}
$$

By using the relation (3.14), the divergence theorem and the boundary conditions (2.15) in the second integral of (3.13), we deduce the relation (3.12). □

Lemma 2. *Let $\mathcal{V} = \{v_l, \psi_l, \theta\}$ be solution of the boundary value problem $(\mathcal{P})$. Then, the functions $K(\cdot)$ and $H(\cdot)$ associated with $\mathcal{V} = \{v_l, \psi_l, \theta\}$ satisfy the estimates:*

$$
\begin{aligned}
|K(x_3)| \leq \int_{D(x_3)} \Big[& \mathcal{D}_1 v_{l,r}\overline{v}_{l,r} + \mathcal{D}_2 v_{r,r}\overline{v}_{l,l} + \mathcal{E}_1\psi_{l,r}\overline{\psi}_{l,r} \\
& + \mathcal{E}_2\psi_{r,r}\overline{\psi}_{l,l} + \mathcal{K}^*\theta_{,l}\overline{\theta}_{,l}\Big] dA,
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
|H(x_3)| \leq \int_{D(x_3)} \Big[& \mathcal{D}_1 v_{l,r}\overline{v}_{l,r} + \mathcal{D}_2 v_{r,r}\overline{v}_{l,l} + \mathcal{E}_1\psi_{l,r}\overline{\psi}_{l,r} \\
& + \mathcal{E}_2\psi_{r,r}\overline{\psi}_{l,l} + \mathcal{K}^*\theta_{,l}\overline{\theta}_{,l}\Big] dA,
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
&\mathcal{D}_1 = \frac{|\alpha_1| + |\alpha_2| + |\beta_1| + |\beta_2| + \omega\mu^* + \omega(\lambda^* + \mu^*) + T_0|a_1| + |f^*|}{\sqrt{\chi_1}},\\
&\mathcal{D}_2 = \frac{|\alpha_2| + |\beta_2| + \omega(\lambda^* + \mu^*)}{\sqrt{\chi_1}},\\
&\mathcal{E}_1 = \frac{\gamma_1 + |\gamma_2| + |\beta_1| + |\beta_2| + T_0|a_2| + |f^*|}{\sqrt{\chi_1}},\\
&\mathcal{E}_2 = \frac{|\gamma_2| + |\beta_2|}{\sqrt{\chi_1}},\\
&\mathcal{K}^* = \frac{|a_1| + |a_2| + 2T_0^{-1}|f^*| + k^*\chi_1\omega^{-1}}{T_0\chi_1\sqrt{\chi_1}}.
\end{aligned} \tag{3.17}
$$

Proof. Let us note that, on the basis of the boundary conditions (2.15), the following inequalities hold:

$$
\begin{aligned}
&\int_{D(x_3)} v_{l,\rho}\bar{v}_{l,\rho}dA \geq \chi_1 \int_{D(x_3)} v_l\bar{v}_l dA\ ,\\
&\int_{D(x_3)} \psi_{l,\rho}\bar{\psi}_{l,\rho}dA \geq \chi_1 \int_{D(x_3)} \psi_l\bar{\psi}_l dA\ ,\\
&\int_{D(x_3)} \theta_{,\rho}\bar{\theta}_{,\rho}dA \geq \chi_1 \int_{D(x_3)} \theta\bar{\theta} dA\ ,
\end{aligned} \tag{3.18}
$$

where χ_1 is the lowest eigenvalue in the two-dimensional clamped membrane problem for the cross section D. It should be mentioned that the dimension of χ_1 is $[\chi_1] = [l^{-2}]$ (l denotes length unit).

Then, the arithmetic-geometric and Schwarz's inequalities, and the inequalities (3.18) may be used to estimate the terms in the right hand of (3.9) and (3.10). We give the details for four of them. Thus, we have

$$
\begin{aligned}
&-\int_{D(x_3)} \alpha_1(v_{l,3}\bar{v}_l + \bar{v}_{l,3}v_l)dA\\
&\leq 2|\alpha_1|\left(\int_{D(x_3)} v_{l,3}\bar{v}_{l,3}dA\right)^{1/2}\left(\frac{1}{\chi_1}\int_{D(x_3)} v_{l,\rho}\bar{v}_{l,\rho}dA\right)^{1/2}\\
&\leq \frac{|\alpha_1|}{\sqrt{\chi_1}}\int_{D(x_3)} v_{l,r}\bar{v}_{l,r}dA\ ;
\end{aligned} \tag{3.19}
$$

$$\begin{aligned}(3.20)\qquad &\int_{D(x_3)} i\omega(\lambda^* + \mu^*)(v_{r,r}\bar{v}_3 - \bar{v}_{r,r}v_3)dA\\ &\leq \frac{\omega(\lambda^* + \mu^*)}{\sqrt{\chi_1}} \int_{D(x_3)} (v_{r,r}\bar{v}_{l,l} + v_{3,\rho}\bar{v}_{3,\rho})dA;\end{aligned}$$

$$\begin{aligned}(3.21)\qquad &\int_{D(x_3)} \frac{f^*}{T_0}[(v_3 - \psi_3)\bar{\theta} + (\bar{v}_3 - \bar{\psi}_3)\theta]dA\\ &\leq \frac{|f^*|}{T_0\sqrt{\chi_1}}\Big[T_0 \int_{D(x_3)} (v_{3,\rho}\bar{v}_{3,\rho} + \psi_{3,\rho}\bar{\psi}_{3,\rho})dA\\ &+ \frac{2}{T_0\chi_1} \int_{D(x_3)} \theta_{,\rho}\bar{\theta}_{,\rho}dA\Big] \ ;\end{aligned}$$

$$\begin{aligned}(3.22)\qquad &\int_{D(x_3)} \frac{ik^*}{\omega T_0}(\theta\bar{\theta}_{,3} - \bar{\theta}\theta_{,3})dA\\ &\leq \frac{2k^*}{\omega T_0\sqrt{\chi_1}}\left(\int_{D(x_3)} \theta_{,3}\bar{\theta}_{,3}dA\right)^{1/2}\left(\int_{D(x_3)} \theta_{,\rho}\bar{\theta}_{,\rho}dA\right)^{1/2}\\ &\leq \frac{k^*}{\omega T_0\sqrt{\chi_1}} \int_{D(x_3)} \theta_{,l}\bar{\theta}_{,l}dA.\end{aligned}$$

The others estimates can be obtained in a similar manner. Collecting the results, we deduce (3.15) and (3.16). □

4. A spatial decay estimate for the amplitude of the steady-state vibration

In this section, by using the properties of the functions $K(\cdot)$ and $H(\cdot)$, we construct a measure for the solution $\mathcal{V}$ of the boundary value problem $(\mathcal{P})$ that decays more rapidly than an exponential function of the distance from the excited end $D(0)$ of the cylinder.

In this sense, we consider the function $I(\cdot) \equiv K(\cdot) + \varkappa H(\cdot)$, where $\varkappa$ is chosen in such a way that $\frac{dI}{dx_3}(x_3) \leq 0$ for all $x_3 \in [0, L]$. Then, we prove that $I(\cdot)$ is an acceptable measure (namely it is positive for all v_r, ψ_r and θ and it vanishes only when $v_r = \psi_r = 0$ and $\theta = 0$) that satisfies a first–order differential inequality. This differential inequality leads to an estimate which holds for every frequency ω of vibrations.

In order to introduce the parameter $\varkappa$ we need first to give some estimates for the derivatives dK/dx_3 and dH/dx_3. Thus, using the arithmetic–

geometric inequality

$$z^1\overline{z^2}+\overline{z^1}z^2 \le \epsilon z^1\overline{z^1}+\frac{1}{\epsilon}z^2\overline{z^2}, \qquad z^1,\ z^2\in\mathbb{C}, \quad \epsilon\in(0,\infty), \tag{4.1}$$

we have

$$\begin{aligned}\xi(v_l-\psi_l)(\overline{v}_l-\overline{\psi}_l) &= \xi v_l\overline{v}_l+\xi\psi_l\overline{\psi}_l-\xi(v_l\overline{\psi}_l+\overline{v}_l\psi_l)\\ &\ge(\xi-|\xi|)(v_l\overline{v}_l+\psi_l\overline{\psi}_l),\end{aligned} \tag{4.2}$$

$$\frac{1}{2}\Big(b^*+\frac{f^*}{T_0}\Big)(\theta_{,l}\overline{v}_l+\overline{\theta}_{,l}v_l)\ge-\frac{1}{2}\Big|b^*+\frac{f^*}{T_0}\Big|\Big(T_0\chi_1 v_l\overline{v}_l+\frac{1}{T_0\chi_1}\theta_{,l}\overline{\theta}_{,l}\Big), \tag{4.3}$$

$$-\frac{1}{2}\Big(b^*+\frac{f^*}{T_0}\Big)(\theta_{,l}\overline{\psi}_l+\overline{\theta}_{,l}\psi_l)\ge-\frac{1}{2}\Big|b^*+\frac{f^*}{T_0}\Big|\Big(T_0\chi_1 \psi_l\overline{\psi}_l+\frac{1}{T_0\chi_1}\theta_{,l}\overline{\theta}_{,l}\Big). \tag{4.4}$$

Moreover, the inequality (3.18) yields

$$\int_{D(x_3)}a\theta\overline{\theta}dA\ge-\frac{|a|}{\chi_1}\int_{D(x_3)}\theta_{,\rho}\overline{\theta}_{,\rho}dA\ge-\frac{|a|}{\chi_1}\int_{D(x_3)}\theta_{,l}\overline{\theta}_{,l}dA. \tag{4.5}$$

From (3.11) and the estimates (4.2)–(4.5) we obtain

$$\begin{aligned}\frac{dK}{dx_3}\le -2\int_{D(x_3)}\Big[&(\alpha_1-\alpha_1^{\circledcirc})v_{l,r}\overline{v}_{l,r}+(\alpha_2-\alpha_2^{\circledcirc})v_{r,r}\overline{v}_{l,l}+\sigma\\ &-\Lambda v_l\overline{v}_l-\Omega\psi_l\overline{\psi}_l-\Upsilon\theta_{,l}\overline{\theta}_{,l}\Big]dA,\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}\Lambda&=\rho_1^0\omega^2+|\xi|-\xi+\frac{1}{2}\Big|b^*+\frac{f^*}{T_0}\Big|T_0\chi_1,\\ \Omega&=\rho_2^0\omega^2+|\xi|-\xi+\frac{1}{2}\Big|b^*+\frac{f^*}{T_0}\Big|T_0\chi_1,\\ \Upsilon&=\frac{|a|}{T_0}+\frac{1}{T_0\chi_1}\Big|b^*+\frac{f^*}{T_0}\Big|.\end{aligned} \tag{4.7}$$

Clearly, Λ and Ω are positive (since $\rho_1^0>0$ and $\rho_2^0>0$) and Υ is non–negative.

As regards the function $H(\cdot)$, we note that the inequality

$$(v_l-\psi_l)(\overline{v}_l-\overline{\psi}_l)\ge\Big(1-\frac{1}{\varepsilon}\Big)v_l\overline{v}_l+(1-\varepsilon)\psi_l\overline{\psi}_l, \tag{4.8}$$

where ε is a positive constant that will be chosen later, the hypotheses (3.2) and the inequality (3.3) imply that dH/dx_3 (see the Lemma 1), satisfies the following inequality:

$$\begin{aligned}\frac{dH}{dx_3}(x_3) \leq -2\omega \int_{D(x_3)} \Big[&\mu^* v_{l,r}\overline{v}_{l,r} + (\lambda^* + \mu^*) v_{r,r}\overline{v}_{l,l} \\ &+ \xi_m^* \Big(1 - \frac{1}{\varepsilon}\Big) v_l \overline{v}_l + \xi_m^* (1-\varepsilon)\psi_l \overline{\psi}_l + \frac{1}{T_0\omega^2} k_m^* \theta_{,l}\overline{\theta}_{,l}\Big] dA.\end{aligned} \tag{4.9}$$

Now, since $\xi_m^*(1 - 1/\varepsilon)$ and $\xi_m^*(1-\varepsilon)$ are not both positive, following an idea introduced in [17], we use the first term in (4.9) and the inequality (3.18) in order to obtain

$$\begin{aligned}\frac{dH}{dx_3}(x_3) \leq -2\omega \int_{D(x_3)} \Big\{&\frac{\mu^*}{2} v_{l,r}\overline{v}_{l,r} + (\lambda^* + \mu^*) v_{r,r}\overline{v}_{l,l} \\ &+ \Big[\frac{\mu^*\chi_1}{2} + \xi_m^* \Big(1 - \frac{1}{\varepsilon}\Big)\Big] v_l \overline{v}_l + \xi_m^* (1-\varepsilon)\psi_l \overline{\psi}_l + \frac{1}{T_0\omega^2} k_m^* \theta_{,l}\overline{\theta}_{,l}\Big\} dA.\end{aligned} \tag{4.10}$$

We choose ε such that

$$\frac{2\xi_m^*}{2\xi_m^* + \mu^*\chi_1} < \varepsilon < 1. \tag{4.11}$$

All the preliminaries are set to introduce the function

$$I(x_3) = K(x_3) + \varkappa H(x_3), \tag{4.12}$$

where $\varkappa$ is sufficiently large such that the following inequalities hold:

$$\begin{aligned}&\alpha_1 - \alpha_1^{\circledcirc} + \frac{\varkappa\omega\mu^*}{2} \geq 0, \qquad \alpha_2 - \alpha_2^{\circledcirc} + \varkappa\omega(\lambda^* + \mu^*) \geq 0, \\ &-\Lambda + \varkappa\omega\Big[\frac{\mu^*\chi_1}{2} + \xi_m^*\Big(1 - \frac{1}{\varepsilon}\Big)\Big] \geq 0, \qquad -\Omega + \varkappa\omega\xi_m^*(1-\varepsilon) \geq 0, \\ &-\Upsilon + \frac{\varkappa k_m^*}{T_0\omega} > 0.\end{aligned} \tag{4.13}$$

In the following, for simplicity we set

$$\begin{aligned}\varkappa = \max\Big\{&\frac{2(\alpha_1^{\circledcirc} - \alpha_1)}{\omega\mu^*}, \frac{\alpha_2^{\circledcirc} - \alpha_2}{\omega(\lambda^* + \mu^*)}, \frac{2\Lambda}{\omega\mu^*\chi_1 + 2\omega\xi_m^*(1 - \frac{1}{\varepsilon})}, \\ &\frac{\Omega}{\omega\xi_m^*(1-\varepsilon)}, \frac{2\Upsilon T_0\omega}{k_m^*}\Big\}.\end{aligned} \tag{4.14}$$

We can state the following:

Lemma 3. *If ρ_1^0 and ρ_2^0 are positive and the conditions* (3.1) *and* (3.2) *hold true, then the function $I(\cdot)$ associated with the solution $\mathcal{V} = \{v_l, \psi_l, \theta\}$ of the problem $(\mathcal{P})$ is a non–increasing function on $[0, L]$. Moreover, we have*

$$\frac{dI}{dx_3}(x_3) \leq -2 \int_{D(x_3)} \Big[\pi_m (v_{r,l}\overline{v}_{r,l} + \psi_{r,l}\overline{\psi}_{r,l}) + \frac{\varkappa k_m^*}{2T_0\omega^2}\theta_{,l}\overline{\theta}_{,l}\Big] dA, \tag{4.15}$$

where π_m is defined by (3.7).

Proof. From (4.6), (4.10), (4.12) and (4.14) we obtain

$$\frac{dI}{dx_3}(x_3) \leq -2 \int_{D(x_3)} \Big[\sigma + \frac{\varkappa k_m^*}{2T_0\omega^2}\theta_{,l}\overline{\theta}_{,l}\Big] dA. \tag{4.16}$$

(3.8) and (4.16) lead to (4.15) and the proof is complete. □

From Lemma 2 we obtain

Lemma 4. *Let $\mathcal{V} = \{v_l, \psi_l, \theta\}$ be solution of the boundary value problem $(\mathcal{P})$. Then, the functions $I(\cdot)$ associated with $\mathcal{V} = \{v_l, \psi_l, \theta\}$ satisfies the estimate:*

$$|I(x_3)| \leq (1 + \varkappa) \int_{D(x_3)} \Big[\mathcal{M}(v_{l,r}\overline{v}_{l,r} + \psi_{l,r}\overline{\psi}_{l,r}) + \mathcal{K}^*\theta_{,l}\overline{\theta}_{,l}\Big] dA, \tag{4.17}$$

where

$$\mathcal{M} = \max\{(\mathcal{D}_1 + 3\mathcal{D}_2), (\mathcal{E}_1 + 3\mathcal{E}_2)\}. \tag{4.18}$$

Proof. The lemma follows from the inequalities (3.15), (3.16) and the fact that the distinct eigenvalues of the linear transformations associated with the positive definite quadratic forms

$$\sigma^{(1)} = \mathcal{D}_1 v_{l,r}\bar{v}_{l,r} + \mathcal{D}_2 v_{l,l}\bar{v}_{r,r}, \qquad \sigma^{(2)} = \mathcal{E}_1 \psi_{l,r}\bar{\psi}_{l,r} + \mathcal{E}_2 \psi_{l,l}\bar{\psi}_{r,r}, \tag{4.19}$$

are

$$\pi_1^{(1)} = \mathcal{D}_1, \qquad \pi_2^{(1)} = \mathcal{D}_1 + 3\mathcal{D}_2, \tag{4.20}$$

and

$$\pi_1^{(2)} = \mathcal{E}_1, \qquad \pi_2^{(2)} = \mathcal{E}_1 + 3\mathcal{E}_2, \tag{4.21}$$

respectively. □

We are ready now to prove the main result of the paper.

Theorem 1. *In the context of a finite composite cylinder made of a mixture consisting of an elastic solid and a Kelvin–Voigt material, whose constitutive coefficients satisfies the mild positive definiteness conditions* (3.1) *and* (3.2)*, the cross–sectional functional* $I(\cdot)$ *represents an acceptable measure of the solution* $\mathcal{V} = \{v_r, \psi_r, \theta\}$ *of the problem* $(\mathcal{P})$*. Moreover, it satisfies the following exponential decay estimate*

$$0 \leq I(x_3) \leq I(0) \exp\Big(-\frac{x_3}{C}\Big), \qquad x_3 \in [0, L], \tag{4.22}$$

where

$$C = (1+\varkappa) \max\Big\{\frac{\mathcal{M}}{2\pi_m}, \frac{\mathcal{K}^* T_0 \omega^2}{\varkappa k_m^*}\Big\}. \tag{4.23}$$

Proof. It follows immediately from (4.15) and (4.17) that

$$|I(x_3)| \leq -C \frac{dI}{dx_3}(x_3), \qquad x_3 \in [0, L]. \tag{4.24}$$

Since $I(\cdot)$ is a non–increasing function and $I(L) = 0$ (see (2.17), (3.9), (3.10) and (4.12)), it follows that $I(x_3) \geq 0$ for every $x_3 \in [0, L]$. Moreover, integrating (4.15) on $[x_3, L]$, it follows that

$$I(x_3) \geq 2 \int_{B(x_3,L)} \Big[\pi_m (v_{r,l}\overline{v}_{r,l} + \psi_{r,l}\overline{\psi}_{r,l}) + \frac{\varkappa k_m^*}{2T_0\omega^2}\theta_{,l}\overline{\theta}_{,l}\Big] dV, \tag{4.25}$$

where $B(x_3, L) = D(x_3) \times [x_3, L]$. This relation together with the boundary conditions prove that $I(x_3) = 0$ implies $v_r = \psi_r = 0$ and $\theta = 0$ on $B(x_3, L)$, so that $I(\cdot)$ is an acceptable measure of the amplitude of the steady–state vibration. Therefore, (4.24) became

$$\frac{1}{C} I(x_3) + \frac{dI}{dx_3}(x_3) \leq 0, \qquad x_3 \in [0, L]. \tag{4.26}$$

By integration one obtains the estimate (4.22) and the proof is complete.□

5. Conclusions

In this paper we have studied the time–harmonic vibrations in a cylinder filled by a binary mixture consisting of an elastic solid and a Kelvin–Voigt material. The dissipative mechanism has been used to derive a measure which leads to the decay estimate of Saint–Venant type (4.22).

The estimate (4.22) holds for every value of the frequency of vibrations. Moreover, the assumptions (3.2) represent a natural extension of the consequences (2.6) assured by the dissipation inequality, while the conditions (3.1) are milder than those utilized in [23].

The result obtained here may be extended to a semi–infinite cylinder to obtain an appropriate alternative of Phragmèn–Lindelöf type.

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