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AN EMBEDDING THEOREM FOR HYPERINTEGRAL DOMAINS

BY

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Abstract. It is well-known that every integral domain D can be embedded in a field F and F is constructed so that F is (up to isomorphism) the smallest field containing D. We extend this result to hyperintegral domains and hyperfields. A commutative Krasner hyperring $(A, +, \cdot)$ is said to be a *hyperintegral domain* if $(A \setminus \{0\}, \cdot)$ is a semigroup and a *hyperfield* if $(A \setminus \{0\}, \cdot)$ is a group. It is shown that every hyperintegral domain $(D, +, \cdot)$ can be embedded in a hyperfield $(F, \oplus, \odot)$ and the constructed hyperfield $(F, \oplus, \odot)$ is (up to isomorphism) the smallest hyperfield containing D.

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1. Introduction

A *hyperoperation* on a nonempty set H is a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ where $\mathcal{P}(H)$ is the power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$, and $(H, \circ)$ is called a *hypergroupoid*. For nonempty sets A, B of H and $x \in H$, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \quad A \circ x = \bigcup_{a \in A} a \circ x \quad \text{and} \quad x \circ A = \bigcup_{a \in A} x \circ a.$$

An *identity* of $(H, \circ)$ is an element $e \in H$ such that $x \in x \circ e \cap e \circ x$ for all $x \in H$. An element $e \in H$ is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$ for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is the only one identity of $(H, \circ)$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$.

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$. A semihypergroup is called a *hypergroup* if $H \circ x = x \circ H = H$ for all $x \in H$.

An element y of a semihypergroup $(H, \circ)$ is called an *inverse* of an element $x \in H$ if $(x \circ y) \cap (y \circ x)$ contains at least one identity of $(H, \circ)$. A hypergroup $(H, \circ)$ is called *regular* if every element of $(H, \circ)$ has at least one inverse. A regular hypergroup $(H, \circ)$ is said to be *reversible* if

$$\text{for all } x, y, z \in H, x \in y \circ z \Rightarrow y \in x \circ z' \text{ and } z \in y' \circ x$$
$$\text{for some inverse } z' \text{ of } z \text{ and}$$
$$\text{some inverse } y' \text{ of } y.$$

A hypergroup $(H, \circ)$ is called a *canonical hypergroup* if

1. $(H, \circ)$ is commutative,
2. $(H, \circ)$ has a scalar identity,
3. every element of H has a unique inverse in $(H, \circ)$,
4. $(H, \circ)$ is reversible.

A *Krasner hyperring* is a triple $(A, +, \cdot)$ where

1. $(A, +)$ is a canonical hypergroup,
2. $(A, \cdot)$ is a semigroup with zero 0 where 0 is the scalar identity of $(A, +)$ and
3. $\cdot$ is distributive over $+$, that is, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$, for all $x, y, z \in H$.

For $x \in A$, let $-x$ denote the unique inverse of x in $(A, +)$. Then $-(-x) = x$ for all $x \in A$. In addition,

$$\text{for all } x, y, z, w \in A, \ (x + y) \cdot (z + w) \subseteq x \cdot z + x \cdot w + y \cdot z + y \cdot w,$$
$$x \cdot (-y) = (-x) \cdot y = -(x \cdot y)$$

([1], page 167). For $x, y \in A$, xy stands for $x \cdot y$.

A *subhyperring* of a Krasner hyperring $(A, +, \cdot)$ is a nonempty subset of A which forms a Krasner hyperring containing 0 under the hyperoperation $+$ and the operation $\cdot$ on A.

By an *identity* of a Krasner hyperring $(A, +, \cdot)$ we mean an element $1 \in A$ such that $x1 = 1x = x$ for all $x \in A$. We say that a Krasner hyperring $(A, +, \cdot)$ is *commutative* if $xy = yx$ for all $x, y \in A$.

A *hyperintegral domain* is a commutative Krasner hyperring $(A, +, \cdot)$ containing more than one element such that for all $x, y \in A$, $xy = 0$ implies $x = 0$ or $y = 0$, that is, $(A \setminus \{0\}, \cdot)$ is a semigroup. Notice that by the given definition, a hyperintegral domain need not have an identity. This is analogous to the definition of an integral domain given in [2, page 88]. A *hyperfield* is a commutative Krasner hyperring $(A, +, \cdot)$ with identity 1 such that $(A \setminus \{0\}, \cdot)$ is a group. Then every hyperfield is a hyperintegral domain. In addition, hyperintegral domains and hyperfields are generalizations of integral domains and fields, respectively.

Example 1.1 ([3])**.** Let $0 < p \leq 1$ and $q \geq 1$. Then $([0, p], \cdot)$ and $(\{0\} \cup [q, \infty), \cdot)$ are commutative semigroups with zero 0 where $\cdot$ is the usual multiplication. For $x, y \in [0, p]$, define

$$x \oplus y = \begin{cases} \{\max\{x, y\}\} & \text{if } x \neq y, \\ [0, x] & \text{if } x = y \end{cases}$$

and for $x, y \in \{0\} \cup [q, \infty)$, define

$$x \boxplus y = \begin{cases} y \boxplus x = \{x\} & \text{if } y = 0, \\ \{\min\{x, y\}\} & \text{if } x \neq 0, y \neq 0 \text{ and } x \neq y, \\ \{0\} \cup [x, \infty) & \text{if } x = y \neq 0. \end{cases}$$

Then $([0, p], \oplus, \cdot)$ and $(\{0\} \cup [q, \infty), \boxplus, \cdot)$ are Krasner hyperrings. Moreover, both of them are hyperintegral domains which are not hyperfields since $((0, p], \cdot)$ and $([q, \infty), \cdot)$ are semigroups which are not groups.

Example 1.2 ([1], page 170)**.** Let $(A, \cdot)$ be a commutative semigroup with zero 0 such that $(A \setminus \{0\}, \cdot)$ is a group. Define the hyperoperation $+$ on A by

$$x + y = \begin{cases} y + x = \{x\} & \text{if } y = 0, \\ A \setminus \{x\} & \text{if } x = y \neq 0, \\ \{x, y\} & \text{if } x, y \in A \setminus \{0\} \text{ and } x \neq y. \end{cases}$$

Then $(A, +, \cdot)$ is a hyperfield.

Let $(A,+,\cdot)$ and $(A',+',\cdot')$ be Krasner hyperrings. A mapping $\varphi : A \to A'$ is called a *good homomorphism* from $(A,+,\cdot)$ into $(A',+',\cdot')$ if

$$\varphi(x+y) = \varphi(x) +' \varphi(y) \text{ and } \varphi(x\cdot y) = \varphi(x)\cdot' \varphi(y) \text{ for all } x,y \in A.$$

A 1-1 good homomorphism from $(A,+,\cdot)$ onto $(A',+',\cdot')$ is called an *isomorphism*. We say that $(A,+,\cdot)$ can be *embedded* in $(A',+',\cdot')$ if there is a 1-1 good homomorphism from $(A,+,\cdot)$ into $(A',+',\cdot')$.

It is well-known that every integral domain D can be embedded in a field F and F is constructed so that F is (up to isomorphism) the smallest field containing D, that is, for every field K containing D as a subring, F can be embedded in K (or K contains a subfield isomorphic to F)([2], page 101-103).

The purpose of this paper is to extend this well-known result to hyperintegral domains and hyperfields. We show that every hyperintegral domain $(D,+,\cdot)$ can be embedded in a hyperfield $(F,\oplus,\odot)$ such that $(F,\oplus,\odot)$ is (up to isomorphism) the smallest hyperfield containing D.

2. Main results

It is obviously seen that a commutative ring R containing more than one element is an integral domain if and only if R is multiplicatively cancellative. This is also true for Krasner hyperrings.

Lemma 2.1. *Let $(A,+,\cdot)$ be a commutative Krasner hyperring. Then $(A,+,\cdot)$ is a hyperintegral domain if and only if for all $x,y,z \in A$, $xy = xz$ and $x \neq 0$ implies $y = z$.*

Proof. Assume that $(A,+,\cdot)$ is a hyperintegral domain. Then $(A \setminus \{0\},\cdot)$ is a semigroup. Let $x,y,z \in A$ be such that $xy = xz$ and $x \neq 0$. Then

$$0 \in xy - xy = xy - xz = xy + x(-z) = x(y-z),$$

so $0 = xu$ for some $u \in y - z$. Since $x \neq 0$, we have $u = 0$. Thus $0 \in y - z$, so $-z$ is an inverse of y in $(A,+)$. It follows that $-y = -z$ which implies that $y = z$.

The converse is evident. □

Theorem 2.2. *Every hyperintegral domain can be embedded in a hyperfield.*

Proof. Let $(D, +, \cdot)$ be a hyperintegral domain. Define the relation $\sim$ on $D \times (D \setminus \{0\})$ by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc \text{ for all } (a, b), (c, d) \in D \times (D \setminus \{0\}).$$

Since $(D, \cdot)$ is commutative, $\sim$ is reflexive and symmetric. If $(a, b), (c, d)$, $(e, f) \in D \times (D \setminus \{0\})$ are such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $ad = bc$ and $cf = de$, so $adf = (ad)f = bcf = b(cf) = bde$. But since $d \neq 0$, by Lemma 2.1, $af = be$. Thus $(a, b) \sim (e, f)$. Therefore $\sim$ is transitive. Hence $\sim$ is an equivalence relation on $D \times (D \setminus \{0\})$. It is clearly seen that

$$\text{for } a \in D,\ b, x \in D \setminus \{0\}, (a, b) \sim (ax, bx). \tag{1}$$

Let $F = D \times (D \setminus \{0\})/\sim$ and $(\overline{a, b})$ the equivalence class of $\sim$ containing $(a, b) \in D \times (D \setminus \{0\})$. Then

$$F = \left\{(\overline{a, b}) \mid (a, b) \in D \times (D \setminus \{0\})\right\}.$$

Define the hyperoperation $\oplus$ and the operation $\odot$ on F by

$$\begin{aligned}(\overline{a, b}) \oplus (\overline{c, d}) &= \left\{(\overline{x, bd}) \mid x \in ad + bc\right\}, \\ (\overline{a, b}) \odot (\overline{c, d}) &= (\overline{ac, bd}) \\ &\quad \text{for all } (a, b), (c, d) \in D \times (D \setminus \{0\}).\end{aligned}$$

Claim that $(F, \oplus, \odot)$ is a hyperfield. To show that $\oplus$ and $\odot$ are well-defined, let $(a, b), (a', b'), (c, d), (c', d') \in D \times (D \setminus \{0\})$ be such that $(\overline{a, b}) = (\overline{a', b'})$ and $(\overline{c, d}) = (\overline{c', d'})$. Then

$$ab' = ba' \text{ and } cd' = dc'. \tag{2}$$

Hence

$$\begin{aligned}(ad + bc)b'd' &= adb'd' + bcb'd' \\ &= ba'dd' + bb'dc' \quad \text{from (2)} \\ &= (a'd' + b'c')bd. \qquad (3)\end{aligned}$$

Let $y \in ad + bc$ be given, that is, $(\overline{y, bd}) \in (\overline{a, b}) \oplus (\overline{c, d})$. Then $yb'd' \in (ad + bc)b'd'$. By (3), $yb'd' = zbd$ for some $z \in a'd' + b'c'$. This implies that $(\overline{y, bd}) = (\overline{z, b'd'}) \in (\overline{a', b'}) \oplus (\overline{c', d'})$. This shows that $(\overline{a, b}) \oplus (\overline{c, d}) \subseteq (\overline{a', b'}) \oplus (\overline{c', d'})$. We have analogously that $(\overline{a', b'}) \oplus (\overline{c', d'}) \subseteq (\overline{a, b}) \oplus (\overline{c, d})$. Thus $(\overline{a, b}) \oplus (\overline{c, d}) =$

$(\overline{a',b'}) \oplus (\overline{c',d'})$. It follows from (2) that $(ac)(b'd') = ba'dc' = (bd)(a'c')$, so $(\overline{ac,bd}) = (\overline{a'c',b'c'})$. Hence $(\overline{a,b}) \odot (\overline{c,d}) = (\overline{a',b'}) \odot (\overline{c',d'})$.

Clearly, $(\overline{a,b}) \oplus (\overline{c,d}) = (\overline{c,d}) \oplus (\overline{a,b})$ and $(\overline{a,b}) \odot (\overline{c,d}) = (\overline{c,d}) \odot (\overline{a,b})$ for all $(a,b),(c,d) \in D \times (D \setminus \{0\})$.

To show that $(F, \oplus)$ is a semihypergroup, $(F, \odot)$ is a semigroup and $\odot$ is distributive over $\oplus$, let $(a,b),(c,d),(e,f) \in D \times (D \setminus \{0\})$. Then

$$\begin{aligned}
&((\overline{a,b}) \oplus (\overline{c,d})) \oplus (\overline{e,f}) = \{(\overline{x,bd}) \mid x \in ad+bc\} \oplus (\overline{e,f}) \\
&= \bigcup_{x \in ad+bc} (\overline{x,bd}) \oplus (\overline{e,f}) = \bigcup_{x \in ad+bc} \{(\overline{y,bdf}) \mid y \in xf+bde\} \\
&= \{(\overline{y,bdf}) \mid y \in (ad+bc)f+bde\} = \{(\overline{y,bdf}) \mid y \in adf+b(cf+de)\} \\
&= \bigcup_{x \in cf+de} \{(\overline{y,bdf}) \mid y \in adf+bx\} = \bigcup_{x \in cf+de} (\overline{a,b}) \oplus (\overline{x,df}) \\
&= (\overline{a,b}) \oplus \{(\overline{x,df}) \mid x \in cf+de\} = (\overline{a,b}) \oplus ((\overline{c,d}) \oplus (\overline{e,f})),
\end{aligned}$$

$$\begin{aligned}
&((\overline{a,b}) \odot (\overline{c,d})) \odot (\overline{e,f}) = (\overline{ac,bd}) \odot (\overline{e,f}) = (\overline{ace,bdf}) \\
&= (\overline{a(ce),b(df)}) = (\overline{a,b}) \odot (\overline{ce,df}) = (\overline{a,b}) \odot ((\overline{c,d}) \odot (\overline{e,f})),
\end{aligned}$$

$$\begin{aligned}
&(\overline{a,b}) \odot ((\overline{c,d}) \oplus (\overline{e,f})) = (\overline{a,b}) \odot \{(\overline{x,df}) \mid x \in cf+de\} \\
&= \{(\overline{a,b}) \odot (\overline{x,df}) \mid x \in cf+de\} = \{(\overline{ax,bdf}) \mid x \in cf+de\} \\
&= \{(\overline{abx,bdbf}) \mid x \in cf+de\} \qquad \text{from (1)} \\
&= \{(\overline{y,bdbf}) \mid y \in ab(cf+de)\} \\
&= \{(\overline{y,(bd)(bf)}) \mid y \in (ac)(bf)+(bd)(ae)\} \\
&= (\overline{ac,bd}) \oplus (\overline{ae,bf}) = ((\overline{a,b}) \odot (\overline{c,d})) \oplus ((\overline{a,b}) \odot (\overline{e,f})).
\end{aligned}$$

If $u \in D \setminus \{0\}$, then for all $(a,b) \in D \times (D \setminus \{0\})$,

$$(\overline{a,b}) \oplus (\overline{0,u}) = \{(\overline{x,bu}) \mid x \in au+b0\} = \{(\overline{au,bu})\} = \{(\overline{a,b})\} \qquad \text{from (1)},$$

$$\begin{aligned}
(\overline{a,b}) \oplus (\overline{-a,b}) &= \{(\overline{x,b^2}) \mid x \in ab+b(-a)\} \\
&= \{(\overline{x,b^2}) \mid x \in ab-ab\} \ni (\overline{0,b^2}) = (\overline{0,u}).
\end{aligned}$$

This shows that for $u \in D \setminus \{0\}$, $(\overline{0,u})$ is the scalar identity of the semihypergroup $(F, \oplus)$ and for all $(a,b) \in D \times (D \setminus \{0\})$, $(\overline{-a,b})$ is an inverse of $(\overline{a,b})$

in $(F, \oplus)$. If $(a, b), (c, d) \in D \times (D \setminus \{0\})$ are such that $\overline{(c, d)}$ is an inverse of $\overline{(a, b)}$ in $(F, \oplus)$, then $\overline{(0, b)} \in \overline{(a, b)} \oplus \overline{(c, d)} = \{\overline{(x, bd)} \mid x \in ad + bc\}$. Thus $\overline{(0, b)} = \overline{(y, bd)}$ for some $y \in ad + bc$. Then $y = 0$ since $b \neq 0$, so $0 \in ad + bc$. It follows that $bc = -(ad) = d(-a)$. Hence $\overline{(c, d)} = \overline{(-a, b)}$. Therefore we have that for every $(a, b) \in D \times (D \setminus \{0\}), \overline{(-a, b)}$ is the unique inverse of $\overline{(a, b)}$ in $(F, \oplus)$.

Next, to show that $(F, \oplus)$ is reversible, let $(a, b), (c, d), (e, f) \in D \times (D \setminus \{0\})$ be such that $\overline{(a, b)} \in \overline{(c, d)} \oplus \overline{(e, f)}$. Then $\overline{(a, b)} = \overline{(y, df)}$ for some $y \in cf + de$. Thus $adf = by$. Since $(D, +)$ is reversible and $y \in cf + de$, it follows that, $de \in -(cf) + y$. Hence

$$edb \in ((-c)f + y)b = (-c)fb + by = (-c)fb + adf = f((-c)b + da)$$

which implies that $edb = fz$ for some $z \in (-c)b + da$. It follows that

$$\overline{(e, f)} = \overline{(z, db)} \in \{(x, db) \mid x \in (-c)b + da\} = \overline{(-c, d)} \oplus \overline{(a, b)}.$$

If $u \in D \setminus \{0\}$ and $(a, b) \in D \times (D \setminus \{0\})$, then $\overline{(a, b)} \odot \overline{(0, u)} = \overline{(0, bu)} = \overline{(0, u)}$. Therefore we conclude that $(F, \oplus, \odot)$ is a commutative Krasner hyperring where $\overline{(0, u)}$ is the scalar identity of $(F, \oplus)$ for $u \in D \setminus \{0\}$. It follows directly from (1) that for $x \in D \setminus \{0\}$, $\overline{(x, x)}$ is the identity of $(F, \odot)$. If $(a, b) \in D \times (D \setminus \{0\})$ is such that (a, b) is not the scalar identity of $(F, \oplus)$, then $a \neq 0$ and $\overline{(a, b)} \odot \overline{(b, a)} = \overline{(ab, ba)} = \overline{(ab, ab)}$ which is the identity of $(F, \odot)$, so $\overline{(b, a)}$ is the inverse of $\overline{(a, b)}$ in the semigroup $(F, \odot)$. Thus $(F \setminus \{\overline{(0, u)}\}, \odot)$ is a group where $u \in D \setminus \{0\}$. Hence we have the claim, that is, $(F, \oplus, \odot)$ is a hyperfield.

Finally, to show that $(D, +, \cdot)$ can be embedded in $(F, \oplus, \odot)$, let $u \in D \setminus \{0\}$ be fixed and define $\varphi : D \to F$ by

$$\varphi(x) = \overline{(xu, u)} \text{ for all } x \in D.$$

If $a, b \in D$, then

$$\begin{aligned}
\varphi(a + b) &= \{\varphi(x) \mid x \in a + b\} = \{\overline{(xu, u)} \mid x \in a + b\} \\
&= \{\overline{(xu^2, u^2)} \mid x \in a + b\} \\
&= \{\overline{(y, u^2)} \mid y \in (a + b)u^2\} = \{\overline{(y, u^2)} \mid y \in au^2 + bu^2\} \\
&= \overline{(au, u)} \oplus \overline{(bu, u)} = \varphi(a) \oplus \varphi(b),
\end{aligned}$$

$$\varphi(ab) = \overline{(abu, u)} = \overline{(abu^2, u^2)} = \overline{(au, u)} \odot \overline{(bu, u)} = \varphi(a) \odot \varphi(b).$$

Thus φ is a good homomorphism from $(D,+,\cdot)$ into $(F,\oplus,\odot)$. If $a,b\in D$ are such that $\varphi(a)=\varphi(b)$, then $\overline{(au,u)}=\overline{(bu,u)}$, so $au^2=bu^2$. Since $u^2\neq 0$, by Lemma 2.1, $a=b$. Hence φ is 1-1.

Therefore the proof is completed. □

Theorem 2.3. *The hyperfield $(F,\oplus,\odot)$ contructed from the hyperintegral domain $(D,+,\cdot)$ in the proof of Theorem 2.2 is (up to isomorphism) the smallest hyperfield containing D.*

$(F,\oplus,\odot)$ may be called the *quotient hyperfield* of $(D,+,\cdot)$.

Proof. Let $(K,+,\cdot)$ be a hyperfield containing D as a subhyperring. Then $(K\setminus\{0\},\cdot)$ is a group where 0 is the scalar identity of $(K,+)$. To show that there is a 1-1 good homomorphism from $(F,\oplus,\odot)$ into $(K,+,\cdot)$, define $\theta: F\to K$ by

$$\theta(\overline{(a,b)})=ab^{-1} \text{ for all } (a,b)\in D\times(D\setminus\{0\}).$$

If $(a,b),(c,d)\in D\times(D\setminus\{0\})$ are such that $\overline{(a,b)}=\overline{(c,d)}$, then $ad=bc$. Thus $ab^{-1}=cd^{-1}$ in $(K,\cdot)$. Hence $\theta(\overline{(a,b)})=\theta(\overline{(c,d)})$, so θ is well-defined.

To show that θ is a good homomorphism, let $(a,b),(c,d)\in D\times(D\setminus\{0\})$. Then

$$\begin{aligned}&\theta(\overline{(a,b)}\oplus\overline{(c,d)})=\theta(\{\overline{(x,bd)}\mid x\in ad+bc\})=\{\theta(\overline{(x,bd)})\mid x\in ad+bc\}\\&=\{x(bd)^{-1}\mid x\in ad+bc\}=(ad+bc)(d^{-1}b^{-1})=add^{-1}b^{-1}+bcd^{-1}b^{-1}\\&=ab^{-1}+cd^{-1}=\theta(\overline{(a,b)})+\theta(\overline{(c,d)}),\end{aligned}$$

$$\begin{aligned}\theta(\overline{(a,b)}\odot\overline{(c,d)})&=\theta(\overline{(ac,bd)})=(ac)(bd)^{-1}=acd^{-1}b^{-1}\\&=(ab^{-1})(cd^{-1})=\theta(\overline{(a,b)})\theta(\overline{(c,d)}).\end{aligned}$$

Finally, to show that θ is 1-1, let $(a,b),(c,d)\in D\times(D\setminus\{0\})$ be such that $\theta(\overline{(a,b)})=\theta(\overline{(c,d)})$. Then $ab^{-1}=cd^{-1}$ which implies that $ad=bc$. Thus $\overline{(a,b)}=\overline{(c,d)}$.

Hence the desired result follows. □

Example 2.4. Let $(D_1,\oplus,\cdot)=([0,p],\oplus,\cdot)$ and $(D_2,\boxplus,\cdot)=(\{0\}\cup[q,\infty),\boxplus,\cdot)$ where $0<p\leq 1$ and $1\leq q$, be the hyperintegral domains defined in Example 1.1 and let $(F_1,\hat{\oplus},\odot)$ and $(F_2,\hat{\boxplus},\odot)$ be the hyperfields constructed, respectively from $(D_1,\oplus,\cdot)$ and $(D_2,\boxplus,\cdot)$ as in the proof of

Theorem 2.2. Define $\oplus$ and $\boxplus$ on $[0,\infty)$ exactly the same as $\oplus$ and $\boxplus$ in Example 1.1. As analogous results of Example 1.1, $([0,\infty),\oplus,\cdot)$ and $([0,\infty),\boxplus,\cdot)$ are commutative Krasner hyperrings. Since $((0,\infty),\cdot)$ is a group, we deduce that $([0,\infty),\oplus,\cdot)$ and $([0,\infty),\boxplus,\cdot)$ are hyperfields containing $(D_1,\oplus,\cdot)$ and $(D_2,\boxplus,\cdot)$ as subhyperrings, respectively.

Claim that $([0,\infty),\oplus,\cdot)$ and $([0,\infty),\boxplus,\cdot)$ are (up to isomorphism) the quotient hyperfields of $(D_1,\oplus,\cdot)$ and $(D_2,\boxplus,\cdot)$, respectively. From the proof of Theorem 2.3, we have that the mappings $\theta_1 : (F_1,\hat{\oplus},\odot) \to ([0,\infty),\oplus,\cdot)$ and $\theta_2 : (F_2,\hat{\boxplus},\odot) \to ([0,\infty),\boxplus,\cdot)$ defined by

$$\theta_1(\overline{(a,b)}) = \frac{a}{b} \qquad \text{for all } (a,b) \in [0,p] \times (0,p]$$

and

$$\theta_2(\overline{(c,d)}) = \frac{c}{d} \qquad \text{for all } (c,d) \in (\{0\} \cup [q,\infty)) \times [q,\infty)$$

are 1-1 good homomorphisms. If $e \in [0,\infty)$, then

$$e \in [0,p] \Rightarrow (ep,p) \in [0,p] \times (0,p] \text{ and } \theta_1(\overline{(ep,p)}) = \frac{ep}{p} = e,$$

$$e \in (p,\infty] \Rightarrow \left(p^2,\frac{p^2}{e}\right) \in (0,p] \times (0,p] \text{ and } \theta_1\left(\overline{\left(p^2,\frac{p^2}{e}\right)}\right) = \frac{p^2}{\frac{p^2}{e}} = e,$$

$$e \in \{0\} \cup [q,\infty) \Rightarrow (eq,q) \in (\{0\} \cup [q,\infty)) \times [q,\infty) \text{ and } \theta_2(\overline{(eq,q)}) = \frac{eq}{q} = e,$$

$$e \in (0,q) \Rightarrow \left(q^2,\frac{q^2}{e}\right) \in [q,\infty) \times [q,\infty) \text{ and } \theta_2\left(\overline{\left(q^2,\frac{q^2}{e}\right)}\right) = \frac{q^2}{\frac{q^2}{e}} = e.$$

Hence θ_1 and θ_2 are isomorphisms.

This proves that, up to isomorphism, $([0,\infty),\oplus,\cdot)$ and $([0,\infty),\boxplus,\cdot)$ are the quotient hyperfields of the hyperintegral domains $(D_1,\oplus,\cdot)$ and $(D_2,\boxplus,\cdot)$, respectively.

In particular, $([0,\infty),\oplus,\cdot)$ and $([0,\infty),\boxplus,\cdot)$ are the quotient hyperfields of the hyperintegral domains $([0,1],\oplus,\cdot)$ and $(\{0\}\cup[1,\infty),\boxplus,\cdot)$, respectively.

REFERENCES

1. Corsini, P. – *Prolegomena of Hypergroup Theory,* Supplement to Riv. Mat. Pura Appl. Aviani Editore, Tricesimo, 1993.
2. Herstein, I.N. – *Topics in Algebra,* Blaisdell Publishing Co. Ginn and Co., New York-Toronto-London, 1964.
3. Kemprasit, Y. – *Multiplicative interval semigroups on* $\mathbb{R}$ *admitting hyperring structure*, Ital. J. Pure Appl. Math., 11 (2002), 205–212.

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