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EXISTENCE OF EQUILIBRIUM PAIRS FOR GENERALIZED GAMES

BY

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Abstract. In this paper, we propose the concept of equilibrium pair for an abstract economy and prove several theorems of equilibrium existence for abstract economies with different types of correspondences.

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Key words: Q-majorized correspondences, abstract economy, equilibrium pair.

1. Introduction

Debreu's works on the existence of equilibrium in a generalized N-person game [5] were extended by several authors. In [10] Shafer and Sonnenschein proved the existence of equilibrium of an economy with finite dimensional commodity space and irreflexive preferences reprezented as correspondences with open graph. Yannelis and Prahbakar [15] developed new techniques based on selection theorems and fixed-point theorems. Their main result concerns the existence of equilibrium when the constraint and preference correspondences have open lower sections. They work within different framework (countable infinite number of agents, infinite dimensional strategy spaces).

Borglin and Keiding used for their existence results new concepts of K.F.-correspondences and KF-majorized correspondences. The second notion was extended by Yannelis and Prabhakar [15] to L-majorized correspondences. Different types of majorized correspondences was introduced by Ding and Tan [4], Tan and Yuan [11], Tulcea [12], Yannelis [14], Yannelis and Prabhakar [15]. In [17] Yuan and Taradfar proposed the notion of U-majorized correspondences and proved several equilibrium theorems. LIU and CAI [8] introduced the notion of Q-majorized correspondences and gave a new fixed point theorem. As its applications, they obtained some new existence theorems of an abstract economy.

In [16], YUAN proposed a model of abstract economy more general then that introduced by BORGLIN and KEING in [1] in the sens that the constraint mapping has been split into two parts A and B due to the fact the "small" constraint correspondences may have not enough fixed points but a "big" constraint correspondences B does so.

In this paper we define a model of abstract economy in which the preference correspondence is also split in two parts P_i and F_i and describe the equilibrium pair. We prove the existence of equilibrium pair of abstract economies in four cases. For example, the correspondences P_i have open lower sections and the correspondences B_i are upper semicontinuous or, other case treat the abstract economies with Q-majorized correspondences P_i and upper semicontinuous correspondences B_i.

The paper is organized in the following way: Section 2 contains preliminaries and notation. The equilibrium theorems are stated in section 3.

2. Preliminaries and notation

Throughout this paper, we shall use the following notations and definitions:

Let A be a subset of a topological space X.

1. $\mathcal{F}(A)$ denotes the family of all non-empty finite subset of A.
2. 2^A denotes the family of all subsets of A.
3. cl A denotes the closure of A in X.
4. If A is a subset of a vector space, co$A$ denotes the convex hull of A.
5. If $F, T : A \to 2^X$ are correspondences, then co$T$, cl T, $T \cap F : A \to 2^X$ are correspondences defined by $(\mathrm{co}T)(x) = \mathrm{co}T(x)$, $(\mathrm{cl}T)(x) = \mathrm{cl}T(x)$ and $(T \cap F)(x) = T(x) \cap F(x)$ for each $x \in A$, respectively.
6. The graph of $T : X \to 2^Y$ is the set $\mathrm{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$

7. The correspondence $\overline{T}$ is defined by

$$\overline{T}(x) = \{y \in Y : (x, y) \in \mathrm{cl}_{X\times Y}\mathrm{Gr}(T)\}$$

(the set $\mathrm{cl}_{X\times Y}\mathrm{Gr}(T)$ is called the adherence of the graph of T).

It is easy to see that $\mathrm{cl}T(x) \subset \overline{T}(x)$ for each $x \in X$.

Definition 1. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence.

1. T is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$.

2. T is said to be *lower semicontinuous* if for each x$\in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.

3. T is said to have *open lower sections* if $T^{-1}(y) := \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

Lemma 1 ([16])**.** *Let X and Y be two topological spaces and let A be a closed (resp. open) subset of X. Suppose $F_1 : X \to 2^Y$, $F_2 : X \to 2^Y$ are lower semicontinuous (resp. upper semicontinuous) such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the correspondence $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & \text{if } x \notin A, \\ F_2(x), & \text{if } x \in A \end{cases}$$

is also lower semicontinuous (resp. upper semicontinuous).

Theorem 1 ([13])**.** *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex topological space E_i, D_i a nonempty compact metrizable subset of X_i* and $S_i, T_i : X = \prod_{i\in I} X_i \to 2^{D_i}$ *be correspondences such that:*

1. *$S_i(x)$ is non-empty and $\mathrm{clco}S_i(x) \subset T_i(x)$ for each $x \in X$;*

2. *S_i is lower semi-continuous.*

Then there exists $\overline{x} \in D := \prod_{i \in I} D_i$ *such that* $\overline{x}_i \in T_i(\overline{x})$ *for each* $i \in I$.

Definition 2 ([8])**.** Let X be a topological space and Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a mapping and $T : X \to 2^Y$ be a correspondence.

1. T is said to be *of class* Q_θ (or Q) if
   (a) for each $x \in X$, $\theta(x) \notin \mathrm{cl}T(x)$ and
   (b) T is lower semicontinuous with open and convex values in Y;
2. A correspondence T_x is said to be a Q_θ*-majorant of* T *at* x if there exists an open neighborhood $N(x)$ of x such that $T_x : N(x) \to 2^Y$ and
   (a) For each $z \in N(x)$, $T(z) \subset T_x(z)$ and $\theta(z) \notin \mathrm{cl}T_x(z)$
   (b) T_x is l.s.c. with open and convex values;
3. T is said to be Q_θ*-majorized* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists a Q_θ-majorant T_x of T at x.

We need the following Lemma to prove the existence theorems in the next section.

Lemma 2 ([8])**.** *Let* X *be a paracompact topological space and* Y *be a non-empty subset of a vector space* E*. Let* $\theta : X \to E$ *be a single-valued function and* $P : X \to 2^Y \setminus \{\emptyset\}$ *be* Q*-majorized. Then there exists a correspondence* $S : X \to 2^Y$ *of class* Q *such that* $P(x) \subset S(x)$ *for each* $x \in X$.

Definition 3 ([17])**.** Let X be a topological space and Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a mapping and $T : X \to 2^Y$ be a correspondence.

1. T is said to be *of class* U_θ (or U) if
   (a) for each $x \in X$, $\theta(x) \notin T(x)$ and
   (b) T is upper semicontinuous with closed and convex values in Y;
2. A correspondence $T_x : X \to 2^Y$ is said to be a U_θ*-majorant of* T *at* x if there exists an open neighborhood $N(x)$ of x such that

(a) For each $z \in N(x)$, $T(z) \subset T_x(z)$ and $\theta(z) \notin T_x(z)$

(b) T_x is u.s.c. with closed and convex values;

3. T is said to be $U_\theta-$*majorized* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists an U-majorant T_x of T at x.

We need the following Lemma to prove the existence theorems in the next section.

Lemma 3 ([17]). *Let X be a paracompact normal space and Y be a non-empty subset of a vector space E. Let $\theta : X \to E$ be a single-valued function and $P : X \to 2^Y \setminus \{\emptyset\}$ be U_θ-majorized. Then there exists a correspondence $S : X \to 2^Y$ of class U_θ such that $P(x) \subset S(x)$ for each $x \in X$.*

Definition 4. Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. An element $x \in X$ is named *maximal element* for T if $T(x) = \Phi$.

3. Existence of equilibrium pairs for abstract economies

In [16], Yuan proposed a model of abstract economy more general then that introduced by Borglin and Keing in [1] in the sens that the constraint mapping has been split into two parts A and B due to the fact the "small" constraint correspondence may have not enough fixed points but a "big" constraint correspondence B does so.

In this section we define a model of abstract economy in which the preference correspondence is also split in two parts P_i and F_i and describe the equilibrium pair. We prove the existence of equilibrium of abstract economies in four cases.

Let I be a nonempty set (the set of agents). For each $i \in I$, let X_i be a non-empty topological vector space representing the set of actions and define $X := \prod_{i \in I} X_i$; let A_i, $B_i : X \to 2^{X_i}$ be the constraint correspondences and P_i, F_i the preference correspondences.

Definition 5. An *abstract economy* (*or a generalized game*) $\Gamma = (X_i, A_i, B_i, P_i, F_i)_{i \in I}$ is defined as a family of ordered quintuplets $(X_i, A_i, B_i, P_i, F_i)$.

When $P_i = F_i$ for each $i \in I$, then the previous definitions can be reduced to the standard definition of abstract economy due to Yuan [16].

Definition 6. An *equilibrium pair* for Γ is defined as a pair of points $(\overline{x},\overline{y}) \in X \times X$ such that $\overline{x_i} \in \overline{B}_i(\overline{x})$, $\overline{y_i} \in \overline{F}_i(\overline{x})$ with $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$ for each $i \in I$.

We state some new equilibrium existence theorems for abstract economies.

Theorem 3 is an existence theorem of equilibrium pair for an abstract economy with correspondences P_i having open lower sections and upper semicontinuous correspondences B_i. To prove this theorem we need the following existence theorem for maximal elements. The result below is a consequence of Corollary 2.4 in [16] (we take $B_i(x) = A_i(x) = X$ for each $x \in X$).

Theorem 2 ([16])**.** *Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game where I is an index set such that for each $i \in I$,*

1) X_i be a nonempty convex compact subset of a Hausdorff locally convex topological vector space E,

2) the correspondence P_i has open lower sections;

3) for all $y \in Y$, $y_i \notin \mathrm{co}P_i(y)$.

Then there exists a point $\overline{x} \in X$ such that $P_i(\overline{x}) = \emptyset$ for all $i \in I$,i.e. $\overline{x}$ is a maximal element of Γ.

Theorem 3. *Let I be any index set and $\Gamma = (X_i, A_i, B_i, P_i, F_i)_{i\in I}$ an abstract economy such that for each $i \in I$:*

(1) X_i be a nonempty compact and convex subset of a locally convex Hausdorff topological vector space E;

(2) $P_i : X := \prod_{i\in I} X_i \to 2^{X_i}$ has lower open sections on X and nonempty, convex values;

(3) $A_i, B_i : X \to 2^{X_i}$ are such that B_i is upper semicontinuous, each $B_i(x)$ is a closed and convex subset of X_i, $A_i(x)$ is nonempty convex and $A_i(x) \subset B_i(x)$ for each $x \in X$;

(4) $F_i : X \to 2^{X_i}$ is such that each $F_i(x)$ is a nonempty, closed subset of X_i and $P_i(x) \subset F_i(x)$ for each $x \in X$;

(5) $x_i \notin P_i(x)$ for each $x \in X$.

Then there exists an equilibrium pair $(\overline{x},\overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ with $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$.

Proof. For each $i \in I$, B_i is upper semicontinuous and it has non-empty, convex and closed values. We define $B : X \to 2^X$, by $B(x) = \prod_{i\in I} B_i(x)$. Then B is upper semicontinuous with nonempty, convex and closed values.

By Fan's fixed-point theorem [6], there exists $\overline{x} \in X$ a fixed point for B, i.e. $\overline{x} \in B(\overline{x})$, then $\overline{x}_i \in B_i(\overline{x})$ for each $i \in I$. It remains to show that there exists a point $\overline{y} \in X$ such that $\overline{y}_i \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$ for each $i \in I$.

For each $i \in I$ define $T_i : X \to 2^{X_i}$,

$$T_i(y) = \begin{cases} A_i(\overline{x}) \cap P_i(y), & \text{if } y_i \in F_i(\overline{x}); \\ P_i(y), & \text{if } y_i \notin F_i(\overline{x}). \end{cases}$$

$$\begin{aligned} T_i^{-1}(y_i) = \{z \in X : y_i \in T_i(z)\} &= \{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in T_i(z)\} \\ &\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in T_i(z)\} \\ &= \{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x}) \\ &\cap P_i(z)\} \cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\} \\ &= (\{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x})\} \cap \{z \in \pi_i^{-1}(F_i(\overline{x}) : y_i \in P_i(z)\}) \\ &\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\} \\ &= P_i^{-1}(y_i) \cap (\{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x})\} \\ &\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}). \end{aligned}$$

The projection map $\pi_i : X \to X_i$ is an open map. Since $F_i(\overline{x})$ is a closed subset in X_i, $\pi_i^{-1}(F_i(\overline{x}))$ is a closed subset in X.

If $y_i \in A_i(\overline{x})$ it follows that

$$\begin{aligned} T_i^{-1}(y_i) = P_i^{-1}(y_i) &\cap (\{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x})\} \\ &\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}) \\ &= P_i^{-1}(y_i) \cap (\pi_i^{-1}(F_i(\overline{x})) \cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}) \\ &= P_i^{-1}(y_i). \end{aligned}$$

If $y_i \notin A_i(\overline{x})$ it follows that

$$\begin{aligned} T_i^{-1}(y_i) = P_i^{-1}(y_i) &\cap (\{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x})\} \\ &\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}) \\ &= P_i^{-1}(y_i) \cap (\{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}) \\ &= P_i^{-1}(y_i) \cap (X \diagdown \pi_i^{-1}(F_i(\overline{x}))). \end{aligned}$$

It follows that T_i has lower open sections on X, it has convex values and $x_i \notin T_i(x)$ for each $x \in X$.

By Theorem 2 of existence of maximal elements, there exists $\overline{y} \in X$ such that $T_i(\overline{y}) = \emptyset$ for each $i \in I$.

For each $y_i \notin F_i(\overline{x})$, $P_i(y)$ is a non-empty subset of X_i. We have $\overline{y_i} \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$. Hence, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$ for each $i \in I$, and then $(\overline{x}, \overline{y})$ is an equilibrium pair for Γ.

Theorem 4 is an existence theorem of equilibrium pair for an abstract economy with correspondences P_i having open lower sections and lower semicontinuous correspondences B_i.

Theorem 4. *Let I be any index set and $\Gamma = (X_i, A_i, B_i, P_i, F_i)_{i \in I}$ an abstract economy such that for each $i \in I$:*

(1) *X_i be a nonempty compact metrizable and convex subset of a locally convex Hausdorff topological vector space E;*

(2) *$P_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ has lower open sections on X and nonempty, convex values;*

(3) *$A_i, B_i : X \to 2^{X_i}$ are such that B_i is lower semicontinuous on X such that each $B_i(x)$ is a closed convex subset of X_i, $A_i(x)$ is nonempty convex and $A_i(x) \subset B_i(x)$ for each $x \in X$;*

(4) *$F_i : X \to 2^{X_i}$ is such that each $F_i(x)$ is a nonempty closed subset of X_i and $P_i(x) \subset F_i(x)$ for each $x \in X$;*

(5) *$x_i \notin P_i(x)$ for each $x \in X$.*

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ with $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$.

Proof. For each $i \in I$ B_i is lower semicontinuous and it has non-empty, convex and closed values. By Theorem 1 there exists $\overline{x} \in X$ such that $\overline{x}_i \in \overline{B}_i(\overline{x})$ for each $i \in I$. It remains to show that there exists a point $\overline{y} \in X$ such that $\overline{y}_i \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$ for each $i \in I$.

Define $T_i : X \to 2^{X_i}$, $T_i(y) = \begin{cases} A_i(\overline{x}) \cap P_i(y), & \text{if } y_i \in F_i(\overline{x}); \\ P_i(y), & \text{if } y_i \notin F_i(\overline{x}). \end{cases}$

$$\begin{aligned} T_i^{-1}(y_i) &= \{z \in X : y_i \in T_i(z)\} = \{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in T_i(z)\} \\ &\cup \{z \in X \setminus \pi_i^{-1}(F_i(\overline{x})) : y_i \in T_i(z)\} \\ &= \{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x}) \cap P_i(z)\} \\ &\cup \{z \in X \setminus \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\} \end{aligned}$$

$$= (\{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x})\} \cap \{z \in \pi_i^{-1}(F_i(\overline{x}) : y_i \in P_i(z)\}$$
$$\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}$$
$$= P_i^{-1}(y_i) \cap (\{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x})\}$$
$$\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}).$$

The projection map $\pi_i : X \to X_i$ is an open map. Since $F_i(\overline{x})$ is a closed subset in X_i, $\pi_i^{-1}(F_i(\overline{x}))$ is a closed subset in X.

If $y_i \in A_i(\overline{x})$ it follows that

$$\begin{aligned} T_i^{-1}(y_i) = P_i^{-1}(y_i) &\cap (\{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x})\} \\ &\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}) \\ &= P_i^{-1}(y_i) \cap (\pi_i^{-1}(F_i(\overline{x})) \cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}) \\ &= P_i^{-1}(y_i). \end{aligned}$$

If $y_i \notin A_i(\overline{x})$ it follows that

$$\begin{aligned} T_i^{-1}(y_i) = P_i^{-1}(y_i) &\cap (\{z \in \pi_i^{-1}(F_i(\overline{x})) : y_i \in A_i(\overline{x})\} \\ &\cup \{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}) \\ &= P_i^{-1}(y_i) \cap (\{z \in X \diagdown \pi_i^{-1}(F_i(\overline{x})) : y_i \in P_i(z)\}) = P_i^{-1}(y_i) \\ &\cap (X \diagdown \pi_i^{-1}(F_i(\overline{x}))). \end{aligned}$$

It follows that T_i has lower open sections on X, it has convex values and $x_i \notin T_i(x)$ for each $x \in X$.

By Theorem 2 of existence of maximal elements, there exists $\overline{y} \in X$ such that $T_i(\overline{y}) = \emptyset$ for each $i \in I$.

For each $y_i \notin F_i(\overline{x})$, $P_i(y)$ is a non-empty subset of X_i. We have $\overline{y_i} \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$. Hence, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$ for each $i \in I$, and then $(\overline{x}, \overline{y})$ is an equilibrium pair for Γ.

Theorem 6 is an existence theorem of equilibrium pair for an abstract economy with U-majorized correspondences P_i and upper semicontinuous correspondences B_i. To prove this theorem we need the following result that is Theorem 4.2 in [17].

Theorem 5 ([17])**.** *Let X be a nonempty convex subset of a Hausdorff locally convex topological vector space E and let D be a nonempty compact subset of X.* Let $P : X \to 2^D$ *be $U_\theta-$majorized. Then there exists a point $\overline{x} \in$co$D$ such that $P(\overline{x}) = \emptyset$.*

Theorem 6. *Let I be any index set and $\Gamma = (X_i, A_i, B_i, P_i, F_i)_{i\in I}$ an abstract economy such that for each $i \in I$:*

(1) X_i be a nonempty compact and convex subset of a locally convex Hausdorff topological vector space E;

(2) $P_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ has nonempty values and it is U_{π_i}-majorized on X;

(3) $A_i, B_i : X \to 2^{X_i}$ are such that B_i is upper semicontinuous, each $B_i(x)$ is a closed convex subset of X_i, A_i has nonempty closed convex values and $A_i(x) \subset B_i(x)$ for each $x \in X$;

(4) $F_i : X \to 2^{X_i}$ is such that each $F_i(x)$ is a nonempty open convex subset of X_i and $P_i(x) \subset F_i(x)$ for each $x \in X$;

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ with $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$.

Proof. For each $i \in I$, B_i is upper semicontinuous and it has non-empty, convex and closed values. We define $B : X \to 2^X$, by $B(x) = \prod_{i\in I} B_i(x)$. Then B is upper semicontinuous with nonempty, convex and closed values. By Fan's fixed-point theorem [6], there exists $\overline{x} \in X$ a fixed point for B, i.e. $\overline{x} \in B(\overline{x})$, i.e. $\overline{x}_i \in B_i(\overline{x})$ for each $i \in I$. It remains to show that there exists a point $\overline{y} \in X$ such that $\overline{y}_i \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$ for each $i \in I$.

Since X is paracompact and P_i is U_{π_i}-majorized, by Lemma 3 there exists a correspondence $\varphi_i : X \to 2^{X_i}$ of class U_{π_i} such that $P_i(x) \subset \varphi_i(x)$ for each $x \in X$. Then, φ_i is upper semicontinuous with nonempty closed, convex values and $x_i \notin \varphi_i(x)$ for each $x \in X$.

For each $i \in I$ define $T_i : X \to 2^{X_i}$,

$$T_i(y) = \begin{cases} A_i(\overline{x}) \cap \varphi_i(y), & \text{if } y_i \in F_i(\overline{x}); \\ \varphi_i(y), & \text{if } y_i \notin F_i(\overline{x}). \end{cases}$$

By Lemma 1, it follows that T_i is upper semicontinuous on X, it has convex closed values and $y_i \notin T_i(y)$. Define $T : X \to 2^X$, $T(y) = \prod_{i\in I} T_i(y)$. T is upper semicontinuous on X and it has convex closed values and $y \notin T(y)$. Therefore, it is U-majorized.

By Theorem 5 of existence of maximal elements, there exists $\overline{y} \in X$ such that $T(\overline{y}) = \emptyset$, i.e. $T_i(\overline{y}) = \emptyset$ for each $i \in I$.

For each $y_i \notin F_i(\overline{x})$, $\varphi_i(y)$ is a non-empty subset of X_i. We have $\overline{y_i} \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap \varphi_i(\overline{y}) = \emptyset$. Since $P_i(\overline{y}) \subset \varphi_i(\overline{y})$ we have that $A_i(\overline{x}) \cap$

$P_i(\overline{y}) = \emptyset$. Hence, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$ for each $i \in I$, and then $(\overline{x}, \overline{y})$ is an equilibrium pair for Γ. $\square$

Corollary 1. *Let I be any index set and $\Gamma = (X_i, A_i, B_i, P_i, F_i)_{i\in I}$ an abstract economy such that for each $i \in I$:*

(1) *X_i be a nonempty compact and convex subset of a locally convex Hausdorff topological vector space E;*

(2) *$P_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ is upper semicontinuous on X, has nonempty convex values and $x_i \notin P_i(x)$ for each $x \in X$;*

(3) *$A_i, B_i : X \to 2^{X_i}$ are such that B_i is upper semicontinuous, each $B_i(x)$ is a closed and convex subset of X_i, A_i has nonempty closed convex values and $A_i(x) \subset B_i(x)$ for each $x \in X$;*

(4) *$F_i : X \to 2^{X_i}$ is such that each $F_i(x)$ is a nonempty open convex subset of X_i and $P_i(x) \subset F_i(x)$* for each $x \in X$;

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ such that for each $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ with $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$.

Theorem 8 is an existence theorem of equilibrium pair for an abstract economy with Q-majorized correspondences P_i and lower semicontinuous correspondences B_i. To prove this theorem we need the following result that is Theorem 7 in [13].

Theorem 7 ([13])**.** *Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game where I is an index set such that for each $i \in I$, the following conditions hold:*

1) *X_i is a nonempty convex compact metrizable subset of a Hausdorff locally convex topological vector space E and $X := \prod_{i\in I} X_i$,*

2) *$P_i : X \to 2^{X_i}$ is lower semi-continuous;*

4) *for each $x \in X$, $x_i \notin \mathrm{clco} P_i(x)$*

Then there exists a point $\overline{x} \in X$ such that $P_i(\overline{x}) = \emptyset$ for all $i \in I$, i.e. $\overline{x}$ is a maximal element of Γ.

Theorem 8. *Let I be any index set and $\Gamma = (X_i, A_i, B_i, P_i, F_i)_{i\in I}$ an abstract economy such that for each $i \in I$:*

(1) *X_i be a nonempty compact convex metrizable subset of a locally convex Hausdorff topological vector space E;*

(2) *$P_i : X = \prod_{i\in I} X_i \to 2^{X_i}$ is $Q_{\pi_i}-$majorized on X and has nonempty values;*

(3) *$A_i, B_i : X \to 2^{X_i}$ are such that B_i is lower semicontinuous, each $B_i(x)$ is a closed convex subset of X_i, $A_i(x)$ is nonempty convex and $A_i(x) \subset B_i(x)$ for each $x \in X$;*

(4) $F_i : X \to 2^{X_i}$ *is such that each* $F_i(x)$ *is a nonempty closed subset of* X_i *and* $P_i(x) \subset F_i(x)$ *for each* $x \in X$;

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ *such that for each* $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ *with* $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$.

Proof. For each $i \in I$, B_i is lower semicontinuous and it has non-empty, convex and closed values. By Theorem 1, there exists $\overline{x} \in X$ with $\overline{x}_i \in B_i(\overline{x})$ for each $i \in I$. It remains to show that there exists a point $\overline{y} \in X$ such that $\overline{y}_i \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$ for each $i \in I$.

Since X is paracompact and P_i is Q_{π_i}-majorized, by Lemma 2 there exists a correspondence $\varphi_i : X \to 2^{X_i}$ of class Q_{π_i} such that $P_i(x) \subset \varphi_i(x)$ for each $x \in X$. Then, φ_i is lower semicontinuous with nonempty open convex values and $x_i \notin \mathrm{cl}\varphi_i(x)$ for each $x \in X$.

For each $i \in I$ define $T_i : X \to 2^{X_i}$,

$$T_i(y) = \begin{cases} A_i(\overline{x}) \cap \varphi_i(y), & \text{if } y_i \in F_i(\overline{x}); \\ \varphi_i(y), & \text{if } y_i \notin F_i(\overline{x}). \end{cases}$$

By Lemma 1, it follows that T_i is lower semicontinuous on X. Then $\mathrm{cl}T_i$ is lower semicontinuous, it has convex values and $x_i \notin \mathrm{cl}T_i(x)$.

By Theorem 7 of existence of maximal elements, there exists $\overline{y} \in X$ such that $\mathrm{cl}T_i(\overline{y}) = \emptyset$ for each $i \in I$.

For each $y_i \notin F_i(\overline{x})$, $\varphi_i(y)$ is a non-empty subset of X_i. We have $\overline{y_i} \in F_i(\overline{x})$ and $\mathrm{cl}(A_i(\overline{x}) \cap \varphi_i(\overline{y})) = \emptyset$. It follows that $A_i(\overline{x}) \cap \varphi_i(\overline{y}) = \emptyset$. Since $P_i(\overline{y}) \subset \varphi_i(\overline{y})$ we have that $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$. Hence, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ and $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$, for each $i \in I$, and then $(\overline{x}, \overline{y})$ is an equilibrium pair for Γ. $\square$

Corollary 2. *Let* I *be any index set and* $\Gamma = (X_i, A_i, B_i, P_i, F_i)_{i \in I}$ *an abstract economy such that for each* $i \in I$:

(1) X_i *be a nonempty compact convex metrizable subset of a locally convex Hausdorff topological vector space E;*

(2) $P_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ *is lower semicontinuous on* X, *has nonempty convex values and* $x_i \notin P_i(x)$ *for each* $x \in X$;

(3) $A_i, B_i : X \to 2^{X_i}$ *are such that* B_i *is lower semicontinuous, each* $B_i(x)$ *is a closed convex subset of* X_i, $A_i(x)$ *is nonempty convex and* $A_i(x) \subset B_i(x)$ *for each* $x \in X$;

(4) $F_i : X \to 2^{X_i}$ *is such that each* $F_i(x)$ *is a nonempty closed and convex subset of* X_i *and* $P_i(x) \subset F_i(x)$ for each $x \in X$;

Then there exists an equilibrium pair $(\overline{x}, \overline{y}) \in X \times X$ *such that for each* $i \in I$, $\overline{x}_i \in B_i(\overline{x})$, $\overline{y}_i \in F_i(\overline{x})$ *with* $A_i(\overline{x}) \cap P_i(\overline{y}) = \emptyset$.

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