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ANOTHER GOULD TYPE INTEGRAL WITH RESPECT TO A MULTISUBMEASURE

BY

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Abstract. The aim of this paper is to present a new kind of a Gould type integral with respect to a closed bounded interval-valued multisubmeasure using an order relation considered by Guo and Zhang on the class of all nonempty,compact and convex sets of $\mathbb{R}_+$and to investigate some basic properties of this integral.

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1. Introduction

In [4], Gould introduced an integral for real functions with respect to finite additive measures taking values in a Banach space and studied several properties of this kind of integral .

Using a Gould type procedure, Precupanu and Croitoru introduced in [9] a multivalued integral for real bounded functions with respect to finite additive multimeasures of finite variation and in [10] a Gould set-valued integral in the general case in which the function may be not bounded and the multimeasure may be not of finite variation.

In [1], [2] Gavriluţ, using the construction from [9] and [10] , defined and studied a Gould type integral for real bounded functions with respect to multisubmeasures of finite variation in which the multisubmeasure takes its values in $\mathcal{P}_{bf}(X)$ the family of all nonempty, bounded and closed subsets of X, where X is a Banach space.

Different from [1] and [2], we present in this paper a Gould type integral with respect to a multisubmeasure taking values in $\mathcal{P}_{kc}(\mathbb{R}_+)$,the class of

all nonempty, compact and convex sets of $\mathbb{R}_+$, using in definition of multisubmeasures instead of inclusion an order relation considered by Guo and Zhang in [5]. Section 2 presents notations and some basic concepts. In section 3 we introduce a Gould type integral for non-negative functions with respect to multisubmeasures of finite variation taking values in $\mathcal{P}_{kc}(\mathbb{R}_+)$, and we present some basic properties of this kind of integral. In section 4 we disscus the general case of integral for real functions with respect to multisubmeasures of finite variation taking values in $\mathcal{P}_{kc}(\mathbb{R}_+)$.

2. Preliminaries

Let S be a nonempty set, $\mathcal{A}$ an algebra of subsets of S and $X = [0, \infty) = \mathbb{R}_+$.

We denote by: $\mathcal{P}_0(X)$, the family of all nonempty subsets of X, $\mathcal{P}_k(X)$, the family of all nonempty compact subsets of X, $\mathcal{P}_f(X)$, the family of all nonempty, closed subsets of X, $\mathcal{P}_b(X)$, the family of all nonempty, bounded subsets of X, $\mathcal{P}_{bf}(X)$, the family of all nonempty, bounded and closed subsets of X and by $\mathcal{P}_{kc}(X)$, the family of nonempty, compact convex subsets of X.

For every $A, B \in \mathcal{P}_0(X)$ we denote by $e(A, B) = \sup_{x\in A} \inf_{y\in B} |x - y|$ and by $h : \mathcal{P}_0(X) \times \mathcal{P}_0(X) \longrightarrow \mathbb{R}_+$ the Hausdorff pseudometric defined by $h(A, B) = \max\{e(A, B), e(B, A)\}$.

We observe that h becomes a metric on $\mathcal{P}_{bf}(X)$ ([6], ch.I.1). We also denote by $|A| = h(A, \{0\})$, for every $A \in \mathcal{P}_0(X)$.

By the definition of the Hausdorff metric, we have immediately the following:

Lemma 2.1. *For every $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$, with $A = [a, b]$, $B = [c, d]$ we have $h(A, B) = \max\{|a - c|, |b - d|\}$.*

In [5], Guo and Zhang introduced the following order relation on $\mathcal{P}_{kc}(\mathbb{R}_+)$:

Definition 2.2. Let $A, B \in \mathcal{P}_0(X)$.

1) By $A \lesssim B$ it means that

(i) for each $x_0 \in A$, there exists $y_0 \in B$ such that $x_0 \leqslant y_0$,

(ii) for each $y_0 \in B$ there exists $x_0 \in A$ such that $x_0 \leqslant y_0$.

2) By $A \lesssim B$ it means that

(i') for each $x_0 \in A$, there exists $y_0 \in B$ such that $x_0 < y_0$,

(ii') for each $y_0 \in B$ there exists $x_0 \in A$ such that $x_0 < y_0$.

It is easy to prove the following:

Proposition 2.3. *If $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$ where $A = [a, b]$ and $B = [c, d]$, $a, b, c, d \in \mathbb{R}_+$, we have that $A \lesssim B$ if and only if $a \leqslant c$ and $b \leqslant d$.*

Remark 2.4. It is easy to see that if A is singleton, $A \in \mathcal{P}_{kc}(\mathbb{R}_+)$, the relation $\lesssim$ becomes the usual order relation on $\mathbb{R}_+$.

Definition 2.5. For a net $(A_i)_{i\in I}$, where I is a filtering set, we define $\limsup_{i\in I} A_i = \{x \in \mathbb{R}_+ : x = \lim_k x_{i_k}, x_{i_k} \in A_{i_k}\}$ and $\liminf_{i\in I} A_i = \{x \in \mathbb{R}_+ : x = \lim_i x_i, x_i \in A_i\}$.

We say that A is the Kuratowski limit (briefly K-limit) of $\{A_i\}$ if $\limsup_{i\in I} A_i = \liminf_{i\in I} A_i = A$ and we denote it by $A_i \xrightarrow{K} A$.

This kind of convergence is called *Kuratowski convergence* (briefly K-convergence) ([6], §1.4; [8], ch.3)

Lemma 2.6. *If $A_i = [a_i, b_i] \subset \mathcal{P}_{kc}(\mathbb{R}_+)$, $i \in I$ then $A_i \xrightarrow{K} A = [a, b]$ if and only if $a_i \longrightarrow a, b_i \longrightarrow b$.*

Definition 2.7 ([7])**.** If $[a, b], [c, d] \in \mathcal{P}_{kc}(\mathbb{R}_+)$ and $k \in \mathbb{R}_+$ then we define $[a, b] + [c, d] = [a + c, b + d]$, $k[a, b] = [ka, kb]$, $[a, b][c, d] = [ac, bd]$, $[a, b] \wedge [c, d] = [a \wedge c, b \wedge d]$ and $[a, b] \vee [c, d] = [a \vee c, b \vee d]$.

Remark 2.8. 1) From [8], ch.3, [6], §1.4 we have that K-convergence of a net $(A_i)_{i\in I} \subset \mathcal{P}_{kc}(\mathbb{R})$ is equivalent to its convergence in Hausdorff metric. It is easy to prove that the same is true for $\mathcal{P}_{kc}(\mathbb{R}_+)$.

2) For all $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$ we have:

i) $A \lesssim A \vee B$ and $B \lesssim A \vee B$;

ii) $A \wedge B \lesssim A$ and $A \wedge B \lesssim B$.

Lemma 2.9. *With the previous notations we have the following statements:*

(i) if $A \lesssim B$ and $k > 0$, then $kA \lesssim kB$, for every $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$;

(ii) if $A \lesssim B$ *and* $A' \lesssim B'$*, then* $A+A' \lesssim B+B'$*, for every* $A, B, A', B' \in \mathcal{P}_{kc}(\mathbb{R}_+)$*;*

(iii) if $A_i = [a_i, b_i] \lesssim B_i = [c_i, d_i]$*,* $i \in I$*, are nets in* $\mathcal{P}_{kc}(\mathbb{R}_+)$ *with* $a_i \longrightarrow a, b_i \longrightarrow b, c_i \longrightarrow c, d_i \longrightarrow d$*, then* $A = [a, b] \lesssim B = [c, d]$*;*

(iv) if $A \lesssim B$ *and* $A \lesssim B'$ *then* $A \lesssim B + B'$*, for every* $A, B, B' \in \mathcal{P}_{kc}(\mathbb{R}_+)$*;*

(v) for $m \leq n, m, n \in \mathbb{R}_+$ *we have* $mA \lesssim nA, \forall A \in \mathcal{P}_{kc}(\mathbb{R}_+)$*;*

(vi) if $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$*,* $A \cap B = \emptyset$*, then* $A \cup B \lesssim A + B$*;*

(vii) if $A \lesssim B + C$ *and* $k > 0$ *then* $kA \lesssim kB + kC$*,* $A, B, C \in \mathcal{P}_{kc}(\mathbb{R}_+)$*.*

Proof. To prove (i), let us take $A = [a, b]$, $B = [c, d]$, $A, B \in \mathcal{P}_{kc}(\mathbb{R}_+)$ with $A \lesssim B$ and $k > 0$. Since $ka \leq kb$ and $kc \leq kd$ using proposition 2.3 we obtain that $kA \lesssim kB$.

To prove (ii), we consider $A = [a, b]$, $B = [c, d]$, $A' = [a', b']$, $B' = [c', d'] \in \mathcal{P}_{kc}(\mathbb{R}_+)$ with $A \lesssim B$ and $A' \lesssim B'$. Since $a \leq c, b \leq d$, $a' \leq c'$, $b' \leq d'$, we have $a + a' \leq c + c'$ and $b + b' \leq d + d'$. From definition 2.7 we obtain that $A + A' = [a + a', b + b']$, $B + B' = [c + c', d + d']$ and again from proposition 2.3 we have that $A + A' \lesssim B + B'$.

To prove iii) let us take the nets $A_i, B_i \in \mathcal{P}_{kc}(\mathbb{R}_+)$ with $A_i = [a_i, b_i]$, $B_i = [c_i, d_i]$, $i \in I$, such that $A_i \lesssim B_i$; then $a_i \leq c_i$ and $b_i \leq d_i$. Since $a_i \longrightarrow a$, $b_i \longrightarrow b, c_i \longrightarrow c$, $d_i \longrightarrow d$, using lemma 2.6 and proposition 2.3. we obtain that $[a, b] \lesssim [c, d]$, so (iii) is true.

To prove (iv), it is sufficient to see that for $A = [a, b]$, $B = [c, d]$, $B' = [c', d']$ with $A \lesssim B$ and $A \lesssim B'$ we have $a \leq c$, $a \leq c'$, $b \leq d, b \leq d'$ Then $2a \leq c + c'$ and $2b \leq d + d'$ and we obtain that $a \leq \frac{1}{2}(c + c') \leq c + c'$ and $b \leq \frac{1}{2}(d + d') \leq d + d'$, hence (iv).

(vi) Let be $A = [a, b]$, $B = [c, d]$ with $a, b, c, d \in \mathbb{R}_+$ such that $A \cap B = \emptyset (a < b, c < d,\ a \neq b \neq c \neq d)$. Let be $x_0 \in [a, b] \cup [c, d]$ arbitrarily chosen. Then $x_0 \in [a, b]$ or $x_0 \in [c, d]$. If $x_0 \in [a, b]$ we have $a \leq x_0 \leq b$ which implies $a + c \leq x_0 + c \leq b + c$ and hence $a < a + c \leq x_0 + c \leq b + c < b + d$. Then we have two cases:

(1) either $a < a + c \leq x_0 \leq x_0 + c \leq b + c < b + d$ or

(2) $a \leq x_0 < a + c \leq x_0 + c \leq b + c < b + d$.

Hence in two cases there exists $y_0 \in [a + c, b + d]$ such that $x_0 \leq y_0$. Analogously, for $x_0 \in [c, d]$, we obtain the existence of an element y_0 such that $x_0 \leq y_0$ and thus we have the first condition from definition 2.1.

Similarly, we can prove the second condition from definition 2.1.

The statements (v), (vii) are evident from definition 2.2 and the properties of the usual order relation on $\mathbb{R}_+$. □

Remark 2.10. The order relation "$\lesssim$" can be extended to $\mathcal{P}_{kc}(\mathbb{R})$ and the statements (i), (ii), (iii), (iv), (vi) and (vii) from previous lemma are also valid for this case.

Definition 2.11. The set function $m : \mathcal{A} \longrightarrow \mathbb{R}_+$ is called:
1) a *submeasure in Drewnowski sense* ([1]) if the following statements hold:

(i) $m(\emptyset) = 0$;

(ii) $m(A \cup B) \leq m(A) + m(B)$ for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$;

(iii) $m(A) \leq m(B)$ for every $A, B \in \mathcal{A}$ with $A \subseteq B$.

2) a *strict submeasure* if hold (i), and

(ii') $m(A \cup B) < m(A) + m(B)$ for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$,

(iii') $m(A) < m(B)$ for every $A, B \in \mathcal{A}$ with $A \subseteq B$.

We also consider the following set functions associated to the submeasure $m : \mathcal{A} \longrightarrow \mathbb{R}_+$:

Definition 2.12. 1) $\overline{m}(A) = \sup\{\sum_{i=1}^{n} m(A_i); (A_i) \subset \mathcal{A}, A_i \cap A_j = \emptyset, i \neq j, \bigcup_{i=1}^{n} A_i = A\}$, for every $A \in \mathcal{A}$, called *the total variation* of m and
2) $\widetilde{m}(A) = \inf\{\overline{m}(B); A \subseteq B, B \in \mathcal{A}\}$, for every $A \subseteq S$, called *the semivariation* of m.

Remark 2.13. We observe that for every $A \in \mathcal{A}$ we have that $m(A) \leq \overline{m}(A)$ and if m is a submeasure, $\overline{m}$ is a finite additive set function.

Definition 2.14. A multifunction $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is said to be:
1) a *multisubmeasure* if the following conditions are satisfied:

(i) $\mu(\emptyset) = \{0\}$;

(ii) $\mu(A \cup B) \lesssim \mu(A) + \mu(B)$ for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$;

(iii) $\mu(A) \lesssim \mu(B)$ for every $A, B \in \mathcal{A}$ with $A \subseteq B$.

2) a *strict multisubmeasure* if hold (i), and

(ii') $\mu(A\cup B)\lesssim \mu(A)+\mu(B)$ for every $A,B\in\mathcal{A}$ with $A\cap B=\emptyset$;

(iii') $\mu(A)\lesssim \mu(B)$ for every $A,B\in\mathcal{A}$ with $A\subseteq B$.

Now, we point out some examples of multisubmeasures:

Example 2.15. 1) Let be $S=\{a,b\},\mathcal{A}=\{\emptyset,\{a\},\{b\},\{a,b\}\}$ and $\mu:\mathcal{A}\longrightarrow\mathcal{P}_{kc}(\mathbb{R}_+)$ defined by $\mu(\emptyset)=[0,0],\mu(\{a\})=[1,\frac{5}{3}],\mu(\{b\})=[\frac{3}{2},\frac{5}{3}]$, $\mu(\{a,b\})=[\frac{3}{2},2]$. Then μ is a multisubmeasure.

2) Let $m:\mathcal{A}\longrightarrow\mathbb{R}_+$ be a strict submeasure and let be $\mu:\mathcal{A}\longrightarrow\mathcal{P}_{kc}(\mathbb{R}_+)$ defined by $\mu(A)=[0,m(A)]$. Then μ is a strict multisubmeasure, called *the strict multisubmeasure induced by the submeasure* m.

Indeed, it is evident that $\mu(\emptyset)=\{0\}$.

We easily see that $\mu(A)\lesssim\mu(B)$ for $A,B\in\mathcal{A}$ with $A\subseteq B$. Indeed if $\mu(A)=[0,m(A)]$, $\mu(B)=[0,m(B)]$ and $x_0\in[0,m(A)]$ then $0\leq x_0\leq m(A)$ and since $A\subseteq B$ and m is a strict submeasure we have $0\leq x_0\leq m(A)<m(B)$ so there exists $y_0\in[0,m(B)]$ such that $x_0<y_0$.

To prove the second condition from definition 2.2-2 let be any $y_0\in[0,m(B)]$. Since $A\subseteq B$ we have $m(A)<m(B)$ and we consider the following cases:

a) if $y_0=m(B)\neq 0$ then $0<m(B)=y_0$ and from $m(A)<m(B)$ there exists $x_0\in[0,m(A)]$ such that $x_0<y_0$;

b) if $0<y_0<m(B)$, from $m(A)<m(B)$ we have two situations:

$b_1)0<m(A)<y_0<m(B)$ and then there exists $0<x_0<m(A)$ such that $x_0<y_0$, or $b_2)0<y_0<m(A)<m(B)$ and then there exists $0<x_0<m(A)$ such that $x_0<y_0$.

Thus we obtain that $[0,m(A)]\lesssim[0,m(B)]$ which assures (iii') from definition 2.14.

To prove (ii') from the definition 2.14, let us take $A,B\in\mathcal{A}$ with $A\cap B=\emptyset$. Then we have $m(A)\leq m(A\cup B)$, $m(B)\leq m(A\cup B)$, $m(A\cup B)<m(A)+m(B)$. We see that $\mu(A\cup B)\lesssim\mu(A)+\mu(B)$.

Because $x_0\in[0,m(A\cup B)]$, then $0\leq x_0\leq m(A\cup B)$ which implies $0\leq x_0\leq m(A\cup B)<m(A)+m(B)$ which assures that there exists $y_0\in[0,m(A)+m(B)]$ such that $x_0<y_0$ and if $y_0\in[0,m(A)+m(B)]=[0,m(A)]+[0,m(B)]=\mu(A)+\mu(B)$ we have that for each $x_0\in\mu(A\cup B)$ there exists $y_0\in\mu(A)+\mu(B)$ such that $x_0<y_0$. Analogously we obtain (ii') from definition 2.2; hence $\mu(A\cup B)\lesssim\mu(A)+\mu(B)$.

Definition 2.16. Let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multimeasure and $\lambda : \mathcal{A} \longrightarrow \overline{\mathbb{R}}_+$ a finite additive measure. We say that μ is *absolutely continuous* with respect to λ, denoted by $\mu \ll \lambda$, if $\lambda(A) = 0$, $A \in \mathcal{A}$, implies $\mu(A) = \{0\}$.

Let $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ be a multisubmeasure. We associate to μ the following real set functions:

Definition 2.17. a) $\overline{\mu}(A) = \sup\{\sum_{i=1}^{n} |\mu(E_i)|\}$, where the supremum is taken over all finite partitions $\{E_i\}_{i=\overline{1,n}}$ of $A \in \mathcal{A}$, with $E_i \in \mathcal{A}$, $i \in \{1, \dots n\}$, $\forall A \in \mathcal{A}$, called *the total variation* of μ;

b) $\widetilde{\mu}(A) = \inf\{\overline{\mu}(B), A \subseteq B, B \in \mathcal{A}\}$, for each $A \subset S$, where $\overline{\mu}$ is the variation of μ, called *the semivariation* of μ.

We note that $\overline{\mu}(.)$ is a finite additive measure on $\mathcal{A}$ and $\widetilde{\mu}(.)$ is a subadditive set function.

Definition 2.18. If $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is a multisubmeasure we say that:

1) μ is of finite variation if $\overline{\mu}(S) < \infty$;

2) μ is of finite semivariation if $\widetilde{\mu}(S) < \infty$.

In the following we shall also use the following notions:

Definition 2.19. 1) A *partition* of S is a finite family $P = \{E_i\}_{i=\overline{1,n}} \subset \mathcal{A}$, $E_i \cap E_j = \emptyset$ $(i \neq j)$ such that $\bigcup_{i=1}^{n} E_i = S$.

2) Let be $P = \{E_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ two partitions of S. P' is *finer than* P, denoted $P \leq P'$, if for any $j \in \{1, \dots, m\}$ there is $i \in \{1, \dots n\}$ such that $B_j \subseteq E_{i_j}$.

3) The *common refinement* of two partitions $P = \{E_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ is the partition $P \wedge P' = \{E_i \cap B_j\}$.

We denote by $\mathcal{P}$ the family of all partitions of S.

Remark 2.20. The relation $\leq$ is an order relation on $\mathcal{P}$.

Definition 2.21. The function $f : S \longrightarrow \mathbb{R}$ is said to be *μ-totally measurable* over S if for every $\varepsilon > 0$ there exists a partition $P = \{E_i\}_{i=\overline{0,n}}$ of S such that

i) $\overline{\mu}(E_0) < \varepsilon$;

ii) $\sup_{t,s \in E_i} |f(t) - f(s)| < \varepsilon, \forall i = \overline{1,n}$.

Definition 2.22. A multifunction μ: $\mathcal{A} \longrightarrow \mathcal{P}_0(\mathbb{R})$ is called *weakly null-additive* if for any A,B$\in \mathcal{A}$ with $\mu(A) = \mu(B) = \{0\}$ we have $\mu(A \cup B) = \{0\}$.

3. Another multivalued Gould type integral

In this section, we introduce a new type of Gould multivalued integral for bounded functions with respect to multisubmeasures taking values in $\mathcal{P}_{kc}(\mathbb{R}_+)$ endowed with the order relation from definition 2.2 and we studie some properties of this integral.

In what follows, let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation and $f : S \longrightarrow \mathbb{R}_+$ a bounded function. For every partition $P = \{A_i\}_{i=\overline{1,n}}$ of S we associate the sum

$$\sigma_P(f, \mu; t_i) = \sigma_P(f) = \sigma_P = \sum_{i=1}^{n} f(t_i)\mu(A_i), \tag{1}$$

for any choice of $t_i \in A_i$. We see that $\{\sigma_P\}_{P\in(\mathcal{P},\leq)}$ is a net.

Definition 3.1. 1) We say that f is *Gould μ-integrable* on S (briefly μ-integrable) if the net $\{\sigma_P\}_{P\in(\mathcal{P},\leq)}$ is K-convergent in $\mathcal{P}_{kc}(\mathbb{R}_+)$ to I that is $\sigma_P(f) \xrightarrow{K} I$.

If this limit exists we denote it by $\int_S f d\mu$ or $\int f d\mu$ and we call it the *Gould $\mu-$ integral* of f on S with respect to μ (briefly the μ-integral).

2) We say that f is μ-integrable on $T \in \mathcal{A}$ if the restriction $f/_T$ of f to T is μ-integrable on $(T, \mathcal{A}_T, \mu_T)$, where $\mathcal{A}_T = \{A \cap T,\, A \in \mathcal{A}\}$and $\mu_T = \mu/_{\mathcal{A}_T}$.

Remark 3.2. Since from remark 2.8, the K-convergence and the Hausdorff convergence are equivalent, we observe that $\int_S f d\mu$ is, at the same time, the Hausdorff limit of the net $\{\sigma_P(f)\}$ that is for every $\varepsilon > 0$ there exists $P_\varepsilon \in \mathcal{P}$ such that for every $P \geq P_\varepsilon$ we have $h(\sigma_P(f), \int_S f d\mu) < \varepsilon$.

In what follows, we point out some properties of the integral considered in the previous definition:

Proposition 3.3. Let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation. If $f = 0$ then f is μ-integrable on S and $\int_S f d\mu = \{0\}$.

Proof. It is easy to see that for every $P = \{A_i\}_{i=\overline{1,n}}$ we have $\sigma_P = \sum_{i=1}^{n} f(s_i)\mu(A_i) = \{0\} \in \mathcal{P}_{kc}(\mathbb{R}_+)$ hence $(K)\text{-}\lim_P \sigma_P(f) = \{0\} = \int_S f d\mu$.□

Proposition 3.4. *Let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation and $f : S \longrightarrow \mathbb{R}_+$ a μ-integrable function on S.*

i) If $\alpha \in \mathbb{R}_+$ then αf is μ-integrable on S and $\int_S (\alpha f) d\mu = \alpha \int_S f d\mu$;

ii) If $\alpha \in \mathbb{R}_+$ then f is $\alpha\mu$-integrable on S and $\int_S f d(\alpha\mu) = \alpha \int_S f d\mu$.

Proof. i) Since f is μ-integrable on S then there exists the K-limit of the net $\{\sigma_P(f)\}_{P\in(\mathcal{P},\leq)}$ that is $(K)\text{-}\lim_P \sigma_P(f) = \int_S f d\mu$. Let be $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ and $\sigma_P(f) = \sum_{i=1}^n f(s_i)\mu(A_i)$, for any choice of $s_i \in A_i$. Then for $\alpha \in \mathbb{R}_+$ we have $\sigma_P(\alpha f) = \sum_{i=1}^n (\alpha f)(s_i)\mu(A_i) = \alpha \sum_{i=1}^n f(s_i)\mu(A_i) = \alpha\sigma_P(f)$ and passing to the K-limit we obtain that $\int_S (\alpha f) d\mu = \alpha \int_S f d\mu$.

Analogously, it is easy to see that ii) is true. □

Theorem 3.5. *Let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation and $f, g : S \longrightarrow \mathbb{R}_+ \mu$-integrable functions on S such that $f \leq g$ on S. Then $\int_S f d\mu \lesssim \int_S g d\mu$.*

Proof. Let $P = \{C_k\}_{k=\overline{1,p}}$ be a partition of S arbitrarily fixed. Since f is μ-integrable there exists the K-limit of the net $\{\sigma_P(f)\}_{P\in(\mathcal{P},\leq)}$ that is $(K)\text{-}\lim_P \sigma_P(f) = \int_S f d\mu$. Since $\sigma_P(f) = \sum_{k=1}^p f(t_k)\mu(C_k) \in \mathcal{P}_{kc}(\mathbb{R}_+)$, where $t_k \in C_k$, we can write it as $[\alpha_P, \beta_P]$ and, on the other hand, since $\int_S f d\mu \in \mathcal{P}_{kc}(\mathbb{R}_+)$ we can write it as $[\alpha, \beta]$.

Analogously, since g is μ-integrable there exists the K-limit of the net $\{\sigma_P(g)\}_{P\in(\mathcal{P},\leq)}$ that is $(K)\text{-}\lim_P \sigma_P(g) = \int_S g d\mu$.

Since $\sigma_P(g) = \sum_{k=1}^p g(t_k)\mu(C_k) \in \mathcal{P}_{kc}(\mathbb{R}_+)$,where $t_k \in C_k$, we can write it as $[\alpha'_P, \beta'_P]$ and, on the other hand, since $\int_S g d\mu \in \mathcal{P}_{kc}(\mathbb{R}_+)$ we can write it as $[\alpha', \beta']$.

Since $f \leq g$ on S we have that $f(t_k) \leq g(t_k)$, $t_k \in C_k$, and since $f(t_k) \geq 0$, $g(t_k) \geq 0$ for any $k \in \{1, \ldots, p\}$ we obtain that $\sigma_P(f) = \sum_{k=1}^p f(t_k)\mu(C_k) \lesssim \sigma_P(g) = \sum_{k=1}^p g(t_k)\mu(C_k)$.

Then $[\alpha_P, \beta_P] \lesssim [\alpha'_P, \beta'_P]$ and passing to the K-limit and using lemma 2.9, iii) we obtain that $[\alpha, \beta] \lesssim [\alpha', \beta']$; hence $\int_S f d\mu \lesssim \int_S g d\mu$. □

Theorem 3.6. *Let be $\mu_1 : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$, $\mu_2 : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ two multisubmeasures of finite variation and $f : S \longrightarrow \mathbb{R}_+$ a function simultaneously μ_1-integrable and μ_2-integrable on S. If $\mu_1(A) \lesssim \mu_2(A)$, for every $A \in \mathcal{A}$, then $\int_S f d\mu_1 \lesssim \int_S f d\mu_2$.*

Proof. Since f is μ_1-integrable there exists the K-limit of the net $\{\sigma_P(f,\mu_1)\}_{P\in(\mathcal{P},\leq)}$, that is $(K)\text{-}\lim_P \sigma_P(f,\mu_1) = \int_S f d\mu_1$. For an arbitrary $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ we have $\sigma_P(f,\mu_1) = \sum_{i=1}^n f(s_i)\mu_1(A_i) \in \mathcal{P}_{kc}(\mathbb{R}_+)$, where $s_i \in A_i$, and we can write it as $[\alpha_P, \beta_P]$. From μ_2-integrability of f there exists the K-limit of the net $\{\sigma_P(f,\mu_2)\}_{P\in(\mathcal{P},\leq)}$ and for the same arbitrary partition $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ we have $\sigma_P(f,\mu_2) = \sum_{i=1}^n f(s_i)\mu_2(A_i) \in \mathcal{P}_{kc}(\mathbb{R}_+)$, for any $s_i \in A_i$. We can write it as $[\alpha'_P, \beta'_P]$.

Since, from the hypothesis, $\mu_1(A) \lesssim \mu_2(A)$, for every $A \in \mathcal{A}$, we have that $\mu_1(A_i) \lesssim \mu_2(A_i)$, $\forall i \in \{1,\dots,n\}$. From lemma 2.9, i) and since $f \geq 0$ on S we obtain that $[\alpha_P, \beta_P] \lesssim [\alpha'_P, \beta'_P]$ for any $s_i \in A_i$. Using lemma 2.9, iii) and passing to the K-limit we obtain that $\int_S f d\mu_1 \lesssim \int_S f d\mu_2$. □

Proposition 3.7. *Let $f, g : S \longrightarrow \mathbb{R}_+$ be two μ-integrable functions. Then:*

i) $\int_S f d\mu \vee \int_S g d\mu \lesssim \int_S (f \vee g) d\mu$;

ii) $\int_S (f \wedge g) d\mu \lesssim \int_S f d\mu \wedge \int_S g d\mu$, *where* $f \vee g = \max\{f, g\}$ *and* $f \wedge g = \min\{f, g\}$.

Proof. Because f is-integrable on S then we have $(K)\text{-}\lim_P \sigma_P(f) = \int_S f d\mu$, where $\sigma_P(f) = \sum_{i=1}^n f(t_i)\mu(A_i) \in \mathcal{P}_{kc}(\mathbb{R}_+)$ for $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ and for any choice of $t_i \in A_i$. We observe that $\sigma_P(f)$ can be written as $[\alpha_P^f, \beta_P^f]$. Since g is μ-integrable on S we have $(K)\text{-}\lim_P \sigma_P(g) = \int_S g d\mu$, where $\sigma_P(g) = \sum_{i=1}^n g(t_i)\mu(A_i) \in \mathcal{P}_{kc}(\mathbb{R}_+)$ for every $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ and for any choice of $t_i \in A_i$. $\sigma_P(g)$ can be written as $[\lambda_P^g, \delta_P^g]$.

Then from definition 2.7 we have $\sigma_P(f) \vee \sigma_P(g) = \sum_{i=1}^n f(t_i)\mu(A_i) \vee \sum_{i=1}^n g(t_i)\mu(A_i) = [\alpha_P^f, \beta_P^f] \vee [\lambda_P^g, \delta_P^g] = [\alpha_P^f \vee \lambda_P^g, \beta_P^f \vee \delta_P^g]$. Since $\mu(A_i) \in \mathcal{P}_{kc}(\mathbb{R}_+)$ we have that $\mu(A_i) = [a_i, b_i] \in \mathcal{P}_{kc}(\mathbb{R}_+)$ and hence we obtain that $\sigma_P(f) = \sum_{i=1}^n f(t_i)\mu(A_i) = \sum_{i=1}^n f(t_i)[a_i, b_i] = \sum_{i=1}^n [f(t_i)a_i, f(t_i)b_i] = [\alpha_P^f, \beta_P^f]$ with $\alpha_P^f = \sum_{i=1}^n f(t_i)a_i$, $\beta_P^f = \sum_{i=1}^n f(t_i)b_i$ and

$$\sigma_P(g) = \sum_{i=1}^n g(t_i)\mu(A_i) = \sum_{i=1}^n g(t_i)[a_i, b_i] = \sum_{i=1}^n [g(t_i)a_i, g(t_i)b_i] = [\lambda_P^g, \delta_P^g],$$

where $\lambda_P^g = \sum_{i=1}^n g(t_i)a_i$, $\delta_P^f = \sum_{i=1}^n g(t_i)b_i$. Since

$$\alpha_P^f \vee \lambda_P^g = (\sum_{i=1}^n f(t_i)a_i) \vee (\sum_{i=1}^n g(t_i)a_i) \leq \sum_{i=1}^n (f \vee g)(t_i)a_i$$

and $\beta_P^f \vee \delta_P^f = (\sum_{i=1}^n f(t_i)b_i) \vee (\sum_{i=1}^n g(t_i)b_i) \leq \sum_{i=1}^n (f \vee g)(t_i)b_i$ we obtain that $[\alpha_P^f \vee \lambda_P^g, \beta_P^f \vee \delta_P^g] \lesssim [\sum_{i=1}^n (f \vee g)(t_i)a_i, \sum_{i=1}^n (f \vee g)(t_i)b_i]$ and $\sigma_P(f) \vee \sigma_P(g) \lesssim [\sum_{i=1}^n (f \vee g)(t_i)a_i, \sum_{i=1}^n (f \vee g)(t_i)b_i] = \sigma_P(f \vee g)$. Hence passing to the K-limit we obtain $\int_S f d\mu \vee \int_S g d\mu \lesssim \int_S (f \vee g) d\mu$. Analogously we can prove ii). □

Proposition 3.8. *Let be $f : S \longrightarrow \mathbb{R}_+$ a bounded function and $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation such that f is μ-integrable on S. Then for each $T \in \mathcal{A}$ f is μ-integrable on T.*

Proof. Since the function f is μ-integrable on S, then from remark 3.2 for every $\varepsilon > 0$ there exists $P_\varepsilon \in \mathcal{P}$ such that for every $P_1 \in \mathcal{P}$ with $P_1 \geq P_\varepsilon$ we have

$$h(\sigma_P(f), \int_S f d\mu) < \frac{\varepsilon}{2}. \tag{2}$$

Let be $T \in \mathcal{A}$ and $P_\varepsilon^T = \{A_i \cap T\}_{i=\overline{1,n}}$; then P_ε^T is a partition of T.

For every $P_1^T = \{E_j\}_{j=\overline{1,m}} \in \mathcal{P}_T$ and $P_2^T = \{C_k\}_{k=\overline{1,q}} \in \mathcal{P}_T$ with $P_1^T \geq P_\varepsilon^T$ and $P_2^T \geq P_\varepsilon^T$ we have $\sigma_{P_1}^T(f) = \sum_{j=1}^m f(t_j)\mu(E_j)$ written as $[\alpha_{P_1}^T, \beta_{P_1}^T]$ and $\sigma_{P_2}^T(f) = \sum_{k=1}^q f(c_k)\mu(C_k)$, written as $[\alpha_{P_2}^T, \beta_{P_2}^T]$, for any choice of $t_j \in E_j$, $j \in \{1, \dots m\}$, and $c_k \in C_k, k \in \{1, \dots q\}$.

Let us consider the partition $P' = P_1^T \cup \{A_i \cap T\}_{i=\overline{1,n}} \in \mathcal{P}$ and $P" = P_2^T \cup \{A_i \cap T\}_{i=\overline{1,n}} \in \mathcal{P}$. Since $P' \geq P_\varepsilon$, $P'' \geq P_\varepsilon$, for any choice of $a_i \in A_i - T$, we have by virtue of (2) that

$$\begin{aligned}
&h([\alpha_{P_1}^T, \beta_{P_1}^T], [\alpha_{P_2}^T, \beta_{P_2}^T]) = h(\sum_{j=1}^m f(t_j)\mu(E_j), \sum_{k=1}^q f(c_k)\mu(C_k)) \\
&= h(\sum_{j=1}^m f(t_j)\mu(E_j) + \sum_{i=1}^n f(a_i)\mu(A_i - T), \sum_{k=1}^q f(c_k)\mu(C_k) \\
&+ \sum_{i=1}^n f(a_i)\mu(A_i - T)) = h(\sigma_{P'}(f), \sigma_{P}"(f)) \leq h(\sigma_{P'}(f), \int_S f d\mu) \\
&+ h(\int_S f d\mu, \sigma_{P"}(f)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

But

$$\left|\alpha_{P_1}^T - \alpha_{P_2}^T\right| \leq \max\{\left|\alpha_{P_1}^T - \alpha_{P_2}^T\right|, \left|\beta_{P_1}^T - \beta_{P_2}^T\right|\} = h([\alpha_{P_1}^T, \beta_{P_1}^T], [\alpha_{P_2}^T, \beta_{P_2}^T]) < \varepsilon$$

and $\left|\beta_{P_1}^T - \beta_{P_2}^T\right| \leq \max\{\left|\alpha_{P_1}^T - \alpha_{P_2}^T\right|, \left|\beta_{P_1}^T - \beta_{P_2}^T\right|\} = h([\alpha_{P_1}^T, \beta_{P_1}^T],[\alpha_{P_2}^T, \beta_{P_2}^T])$ $< \varepsilon$.

Then for every $P_1^T, P_2^T \in \mathcal{P}_T$ with $P_1^T \geq P_\varepsilon, P_2^T \geq P_\varepsilon$ we have that $\left|\alpha_{P_1}^T - \alpha_{P_2}^T\right| < \varepsilon$ and $\left|\beta_{P_1}^T - \beta_{P_2}^T\right| < \varepsilon$; hence (α_P^T) and (β_P^T) are Cauchy nets in $[0, \infty)$ and then exists $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+$ such that $\alpha_P^T \longrightarrow \alpha$ and $\beta_P^T \longrightarrow \beta$. By virtue of lemma 2.6 and using lemma 2.9, iii) we have that $\sigma_P^T = [\alpha_P^T, \beta_P^T] \xrightarrow{K} [\alpha, \beta]$, hence f is μ-integrable on T. □

Theorem 3.9. *Let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation, $f : S \longrightarrow \mathbb{R}_+$ bounded on S and $B, C \in \mathcal{A}$ with $B \subseteq C$. If f is μ-integrable on C then $\int_B f d\mu \lesssim \int_C f d\mu$.*

Proof. Let us observe that since f is μ -integrable on C, using proposition 3.8 for $(C, \mathcal{A}_C, \mu_C)$ and $B \subseteq C$ we have that f is μ -integrable on B. Let be I_1, $I_2 \in \mathcal{P}_{kc}(\mathbb{R}_+)$, with $I_1 = \int_B f d\mu = [a, b], I_2 = \int_C f d\mu = [c, d]$.

Since f is μ-integrable on C, the net $\{\sigma_P^C\}_{P \in \mathcal{P}_C}$ is K-convergent to $\int_C f d\mu$, where $P = \{A_k\}_{k=\overline{1,n}} \in \mathcal{P}_C$ is arbitrary and $\sigma_P^C(f) = \sum_{k=1}^n f(t_k)\mu(A_k)$. We see that σ_P^C can be written as $[\alpha_P^C, \beta_P^C]$.

We observe that $P' = \{A_k \cap B\}_{k=\overline{1,n}}$ is a partition of B. Let be $\sigma_P^B(f) = \sum_{k=1}^n f(t_k)\mu(A_k \cap B)$ which can be written as $[\alpha_{P'}^B, \beta_{P'}^B]$. From $A_k \cap B \subseteq A_k$, for every $k \in \{1, \dots n\}$ we have that $\mu(A_k \cap B) \lesssim \mu(A_k)$, $k \in \{1, \dots n)$ and since $f(t_k) \geq 0$, from lemma 2.9, i) we obtain $[\alpha_{P'}^B, \beta_{P'}^B] \lesssim [\alpha_P^C, \beta_P^C]$.

Since $[\alpha_P^C, \beta_P^C] \xrightarrow{K} \int_C f d\mu$, $[\alpha_{P'}^B, \beta_{P'}^B] \xrightarrow{K} \int_B f d\mu$ from lemma 2.9, iii) we have that $\int_B f d\mu \lesssim \int_C f d\mu$. □

Proposition 3.10. *Let $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ be the multisubmeasure of finite variation induced by a strict submeasure $m : \mathcal{A} \longrightarrow \mathbb{R}_+$ (see example 2.15, (2)) and let $f : S \longrightarrow \mathbb{R}_+$ be a μ-totally measurable, μ-integrable function. Then f is Gould integrable in the sense of* [4] *with respect to $\overline{m}$ and $\int_S f d\mu \lesssim [0, \int_S f d\overline{m}]$, where $\overline{m}$ is the variation of m.*

Proof. First, we observe that from lemma 2.1 we have

$$\overline{\mu}(A) = \sup\{\sum_{i=1}^n \left|\mu(A_i)\right|, A_i \in \mathcal{A}, \bigcup_{i=1}^n A_i = A\}$$
$$= \sup\{\sum_{i=1}^n h(\{0\}, [0, m(A_i)]), A_i \in \mathcal{A}, \bigcup_{i=1}^n A_i = A\}$$

$$= \sup\{\sum_{i=1}^{n} |m(A_i)|, A_i \in \mathcal{A}, \bigcup_{i=1}^{n} A_i = A\} = \overline{m}(A).$$

Since m is a strict submeasure we have that $\widetilde{\overline{m}}(A) = \overline{m}(A)$, for every $A \in \mathcal{A}$ ([3]), hence $\overline{\mu}(A) = \widetilde{\overline{m}}(A) = \overline{m}(A)$, $\forall A \in \mathcal{A}$. From μ-totally measurability of f we have that for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of S such that $\overline{\mu}(A_0) < \varepsilon$ and $\sup_{t,s\in A_i} |f(t) - f(s)| < \varepsilon$, $\forall i = \overline{1,n}$. Then $\overline{m}(A_0) < \varepsilon$ because $\overline{\mu}(A) = \overline{m}(A)$, $\forall$A$\in \mathcal{A}$ and since $\sup_{t,s\in A_i} |f(t) - f(s)| < \varepsilon, \forall i = \overline{1,n}$we obtain that f is $\overline{m}$-totally measurable. From [3], theorem 2.16 we have that f is Gould integrable with respect to $\overline{m}$ and moreover $\int_S f d\overline{m} \in \mathbb{R}_+$.

Then for every $\varepsilon > 0$ there exists $\overline{P_\varepsilon} \in \mathcal{P}$ such that for every $P_1 = \{A_j\}_{j=\overline{1,p}} \in \mathcal{P}$ with $P_1 \geq \overline{P_\varepsilon}$ we have

$$|\sum_{j=1}^{p} f(t_j)\overline{m}(A_j) - \int_S f d\overline{m}| < \varepsilon. \tag{3}$$

Since f is μ-integrable on S, the net $\{\sigma_P(f,\mu)\}_{P\in\mathcal{P}}$ is K-convergent to $\int_S f d\mu$, that is $(K)-\lim_P \sigma_P(f) = \int_S f d\mu$, where $\sigma_P(f) = \sum_{k=1}^{s} f(c_k)\mu(C_k)$ $\in \mathcal{P}_{kc}(\mathbb{R}_+)$ for every $P = \{C_k\}_{k=\overline{1,s}} \in \mathcal{P}$ and for any choice of $c_k \in C_k$.

Since $\mu(A) = [0, m(A)]$ and $m(A) \leq \overline{m}(A)$, $\forall A \in \mathcal{A}$ then from proposition 2.3 and lemma 2.9, i) we obtain that $\sigma_P(f) = \sum_{k=1}^{s} f(c_k)\mu(C_k) = \sum_{k=1}^{s} f(c_k)[0, m(C_k)] = [0, \sum_{k=1}^{s} f(c_k)m(C_k)] \lesssim [0, \sum_{k=1}^{s} f(c_k)\overline{m}(C_k)]$. Hence, using lemma 2.6, we have that $\int_S f d\mu \lesssim [0, \int_S f d\overline{m}]$. □

From proposition 3.8, if $f : S \longrightarrow \mathbb{R}_+$ is μ-integrable on S then for every $A \in \mathcal{A}$ f is μ-integrable on A and consequently the set function

$$I(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A} \tag{4}$$

is well defined.

Proposition 3.11. *Let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation and $f : S \longrightarrow \mathbb{R}_+$ μ-integrable on S. Then the multifunction $I : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ given by* (4) *is a multisubmeasure.*

Proof. From theorem 3.9 we obtain that $I(A) \lesssim I(B)$ for every $A, B \in \mathcal{A}$ with $A \subseteq B$. We now prove that $I(A \cup B) \lesssim I(A) + I(B)$ for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$.

From proposition 3.8 we obtain that f is μ-integrable on $A\cup B$ and from definition 3.1 we have that there exist $\int_{A\cup B} f d\mu = (K) - \lim_P \sigma_P(f,\mu)$, where $\sigma_P(f,\mu) = \sum_{s=1}^r f(t_s)\mu(F_s) \in \mathcal{P}_{kc}(\mathbb{R}_+)$, for every $P = \{F_s\}_{s=\overline{1,r}} \in \mathcal{P}_{A\cup B}$ and for any choice of $t_s \in F_s$.

Since $F_s = F_s \cap (A\cup B) = (F_s\cap A)\cup(F_s\cap B)$, $\forall s \in \{1,2,\dots,r\}$ we have $\{F_s\cap A\}_{s=\overline{1,r}} \in \mathcal{P}_A$ and $\{F_s\cap B\}_{s=\overline{1,r}} \in \mathcal{P}_B$. Because μ is a multisubmeasure we have $\mu(F_s) \lesssim \mu(F_s\cap A) + \mu(F_s\cap B)$ and since $f(t_s)\geq 0$ we obtain that $\sum_{s=1}^r f(t_s)\mu(F_s) \lesssim \sum_{s=1}^r f(t_s)\mu(F_s\cap A) + \sum_{s=1}^r f(t_s)\mu(F_s\cap B)$. Hence, using lemma 2.9, iii) and passing to the K-limit we obtain that $I(A\cup B) \lesssim I(A) + I(B)$. Evidently $I(\emptyset) = \{0\}$ and hence $I(.)$ is a multisubmeasure. □

Proposition 3.12. *The multifunction $I : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ given by* (4) *is a weakly null-additive multisubmeasure.*

Proof. Let be $A, B \in \mathcal{A}$ such that $I(A) = I(B) = \{0\}$. From previous proposition we have that $I(A\cup B) \lesssim I(A) + I(B)$ and $I(A\cup B) \lesssim \{0\}$. From $I(A\cup B)\in \mathcal{P}_{kc}(\mathbb{R}_+)$ and definition 2.2 we have I(A∪B)={0} hence I(.) is a weakly null-additive multisubmeasure.

In the following we associate to the multisubmeasure $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ the two real set functions

$$\mu_1(A) = \inf \mu(A), \quad \forall A \in \mathcal{A} \text{ and } \mu_2(A) = \sup \mu(A), \tag{5}$$

for every $A \in \mathcal{A}$.

According to [5], proposition 2.1, μ_1, μ_2 are submeasures in Drewnowski sense such that for every A∈ $\mathcal{A}$ we have $\mu_1(A) \in \mu(A), \mu_2(A) \in \mu(A)$ that is μ_1, μ_2 are selectors of μ.

Theorem 3.13. *Let $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ be a multisubmeasure of finite variation and let $f : S \longrightarrow \mathbb{R}_+$ be a μ-integrable function on S. Then $\int_S f d\mu = [\int_S f d\mu_1, \int_S f d\mu_2]$ where $\int_S f d\mu_1, \int_S f d\mu_2$ are the Gould integral* ([3]) *with respect to the real submeasures μ_1, μ_2.*

Proof. Let us observe that $\mu(A) = [\mu_1(A), \mu_2(A)]$, for every $A \in \mathcal{A}$. Firstly we shall prove that f is Gould integrable with respect to μ_1 and μ_2 in the sense of [3], definition 2.3.

Since f is μ-integrable on S then for every $\varepsilon > 0$ there exists $P_\varepsilon = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}$ such that for every $P = \{B_j\}_{j=\overline{1,m}} \in \mathcal{P}$ with $P \geq P_\varepsilon$ we have

$$h(\sigma_P(f,\mu), \int_S f d\mu) < \frac{\varepsilon}{2}, \tag{6}$$

where

$$\begin{aligned} \sigma_P(f,\mu) &= \sum_{j=1}^{m} f(b_j)\mu(B_j) = \sum_{j=1}^{m} f(b_j)[\mu_1(B_j), \mu_2(B_j)] \\ &= [\sum_{j=1}^{m} f(b_j)\mu_1(B_j), \sum_{j=1}^{m} f(b_j)\mu_2(B_j)] \\ &= [\sigma_P(f,\mu_1), \sigma_P(f,\mu_2)], \end{aligned} \tag{7}$$

in which $\sigma_P(f,\mu_i) = \sum_{j=1}^{m} f(b_j)\mu_i(B_j)$, $i = 1, 2$.

Because f is μ integrable the net $\{\sigma_P(f,\mu)\}_{P\in(\mathcal{P},\leq)}$ is convergent; hence $\{\sigma_P(f,\mu)\}_{P\in(\mathcal{P},\leq)}$ is a Cauchy net that is for every $\varepsilon > 0$ there exists $P_\varepsilon = \{A_i\}_{i=\overline{1,n}}$ such that for every $P' = \{C_k\}_{k=\overline{1,l}} \in \mathcal{P}$, $P'' = \{D_s\}_{s=\overline{1,r}} \in \mathcal{P}$ with $P' \geq P_\varepsilon$ and $P'' \geq P_\varepsilon$ we have that

$$h(\sigma_{P'}(f), \sigma_{P''}(f)) < \frac{\varepsilon}{2}. \tag{8}$$

Using lemma 2.1 and (8) we obtain

$$|\sigma_{P'}(f,\mu_1) - \sigma_{P''}(f,\mu_1)| \leq h(\sigma_{P'}(f,\mu), \sigma_{P''}(f,\mu)) < \frac{\varepsilon}{2} \tag{9}$$

which assures that $(\sigma_P(f,\mu_1))$ is a Cauchy net in $[0,\infty)$, hence convergent.

Analogously, we obtain that $(\sigma_P(f,\mu_2))$ is a Cauchy net in $[0,\infty)$, hence convergent.

From [3], definition 2.3, we have that f is Gould integrable with respect to the submeasures μ_1 and μ_2. Since $\int_S f d\mu \in \mathcal{P}_{kc}(\mathbb{R}_+)$ we can write it as $\int_S f d\mu = [a,b]$. We shall prove that $[a,b] = [\int_S f d\mu_1, \int_S f d\mu_2]$.

Using (7) we have that

$$h(\sigma_P(f,\mu), \int_S f d\mu) = h(\sigma_P(f,\mu), [a,b]) = h([\sigma_P(f,\mu_1), \sigma_P(f,\mu_2)], [a,b]).$$

Now,applying lemma 2.1 we have $|\sigma_P(f,\mu_1) - a| \leq h(\sigma_P(f,\mu), [a,b]) < \frac{\varepsilon}{2}$.

Analogously we obtain $|\sigma_P(f,\mu_2) - b| < \frac{\varepsilon}{2}$.

Hence the net $(\sigma_P(f,\mu_1))$ is convergent with limit a and the net $(\sigma_P(f,\mu_2))$ is convergent with limit b. Consequently, $a = \int_S f d\mu_1$, $b = \int_S f d\mu_2$ and $\int_S f d\mu = [\int_S f d\mu_1, \int_S f d\mu_2]$. □

4. A multivalued Gould integral for real functions

In what follows we assume that $f : S \longrightarrow \mathbb{R}$ is a real, bounded function and $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ is a multisubmeasure of finite variation.

We denote by $f^+ = \sup\{f, 0\}$ and $f^- = -\inf\{f, 0\}$ and then $f = f^+ - f^-$.

We observe that, for the case of $\mathcal{P}_{kc}(\mathbb{R})$, we can extend the operations with compact intervals from definition 2.7.

Definition 4.1. If $[a, b], [c, d] \in \mathcal{P}_{kc}(\mathbb{R}_+)$ and $k < 0$ then we can define $k[a, b] = [kb, ka] \in \mathcal{P}_{kc}(\mathbb{R})$ and $[a, b]-[c, d] = [a, b]+[-d, -c] = [a-d, b-c] \in \mathcal{P}_{kc}(\mathbb{R})$.

Definition 4.2. 1) We say that the bounded function $f : S \longrightarrow \mathbb{R}$ is *generalized Gould integrable* (briefly *g-μ* -integrable) with respect to the multisubmeasure μ whenever f^+ and f^- are μ- integrable in sense of definition 3.1 and we define the integral of f with respect to μ on S as

$$\int_S f d\mu = \int_S f^+ d\mu - \int_S f^- d\mu \in \mathcal{P}_{kc}(\mathbb{R}). \tag{10}$$

2) We say that f is μ-integrable on $T \in \mathcal{A}$ if the restriction of f to T is μ-integrable on $(T, \mathcal{A}_T, \mu_T)$.

Remark 4.3. Let us observe that if $f \geq 0$ we have $f^- = 0$ and $f = f^+$ and from proposition 3.3 we have $\int_S f d\mu = \int_S f^+ d\mu \in \mathcal{P}_{kc}(\mathbb{R}_+)$.

For the general case of integral, we have the following properties:

Proposition 4.4. *Let $f : S \longrightarrow \mathbb{R}$ be a g-μ-integrable function on S and $\alpha \in \mathbb{R}$. Then αf is g-μ-integrable on S and $\int_S (\alpha f) d\mu = \alpha \int_S f d\mu$.*

Proof. From proposition 3.4, if $f \geq 0$ and $\alpha \geq 0$ we have $\int_S (\alpha f) d\mu = \alpha \int_S f d\mu$.

Now, if $f : S \longrightarrow \mathbb{R}$ we observe that

1) for $\alpha \geq 0$ we have $(\alpha f)^+ = \sup\{\alpha f, 0\} = \alpha f^+$ and $(\alpha f)^- = -\inf\{\alpha f, 0\} = \alpha f^-$ and

2) for $\alpha < 0$ we have that $(\alpha f)^+ = \sup\{\alpha f, 0\} = -\alpha \sup\{-f, 0\} = -\alpha f^-$ and $(\alpha f)^- = -\alpha f^+$.

Since f is g- μ-integrable whenever f^+ and f^- are μ- integrable, we have from proposition 3.4 that αf^+ and αf^- are μ-integrable on S.

Using definitions 2.7 and 4.2 we obtain for $\alpha \geq 0$ that $\int_S (\alpha f) d\mu = \int_S (\alpha f)^+ d\mu - \int_S (\alpha f)^- d\mu = \int_S \alpha f^+ d\mu - \int_S \alpha f^- d\mu = \alpha \int_S f d\mu$ and for $\alpha < 0$ we have $\int_S (\alpha f) d\mu = \int_S (\alpha f)^+ d\mu - \int_S (\alpha f)^- d\mu = (-\alpha) \int_S f^- d\mu - (-\alpha) \int_S f^+ d\mu = \alpha \int_S f d\mu$. □

Analogously, it is easy to prove the following:

Proposition 4.5. *Let $f : S \longrightarrow \mathbb{R}$ be a g-μ-integrable function on S and $\alpha \in \mathbb{R}_+$. Then f is $\alpha\mu$-integrable on S and $\int_S f d(\alpha\mu) = \alpha \int_S f d\mu$.*

Proposition 4.6. *Let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation and $f : S \longrightarrow \mathbb{R}$ g- μ -integrable on S. Then for every $A \in \mathcal{A}$, f is g- μ-integrable on A.*

Proof. Since f is μ-integrable on S whenever f^+ and f^- are μ-integrable on S then from proposition 3.8 we have that, for every $A \in \mathcal{A}$, f^+ and f^- are μ-integrable on A. Hence f is g-μ-integrable on $A \in \mathcal{A}$. □

If f is g-μ-integrable on S, using proposition 4.7 we can define the set function given by

$$I(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}. \tag{11}$$

Proposition 4.7. *Let be $\mu : \mathcal{A} \longrightarrow \mathcal{P}_{kc}(\mathbb{R}_+)$ a multisubmeasure of finite variation, $f : S \longrightarrow \mathbb{R}$ a g-μ-integrable function on S. Then the set function $I(\cdot)$ defined by (11) is absolutely continuous with respect to $\overline{\mu}$, where $\overline{\mu}$ is the variation of μ.*

Proof. Let be $A \in \mathcal{A}$ such that $\overline{\mu}(A) = 0$, then from the definition of $\overline{\mu}(A)$ we have $\overline{\mu}(A) = \sup\{\sum_{k=1}^s |\mu(E_k)| \ , \ E_k \cap E_{k'} = \emptyset, k \neq k', E_k \in \mathcal{A}, k \in \{1, \dots s\}, \bigcup_{k=1}^s E_k = A\} = 0$ which implies $\sum_{k=1}^s |\mu(E_k)| = 0$, for every $\{E_k\}_{k=\overline{1,s}} \in \mathcal{P}_A$. Hence $|\mu(E_k)| = h(\mu(E_k), \{0\}) = 0$ which implies that $\mu(E_k) = \{0\}$ for every $k \in \{1, \dots, s\}$. Since f is g-μ-integrable on A whenever f^+ and f^- are μ- integrable on A then from definition 3.1 there exists the K-limit of the net $\{\sigma_P(f^+, \mu)\}_{P \in (\mathcal{P}_A, \leq)}$, that is $(K) - \lim_P \sigma_P(f^+, \mu) = \int_A f^+ d\mu$ and respectively, there exists the K-limit of the net $\{\sigma_P(f^-, \mu)\}_{P \in (\mathcal{P}_A, \leq)}$, that is $(K) - \lim_P \sigma_P(f^-, \mu) = \int_A f^- d\mu$. Hence,

for every $P = \{E_k\}_{k=\overline{1,s}} \in \mathcal{P}_A$ we have $\sigma_P(f^+, \mu) = \sum_{k=1}^{s} f^+(t_k)\mu(E_k) = \{0\}$ and $\sigma_P(f^-, \mu) = \sum_{k=1}^{s} f^-(t_k)\mu(E_k) = \{0\}$. Then $\int_A f^+ d\mu = \{0\}$ and $\int_A f^- d\mu = \{0\}$ and from (11) we obtain that $I(A) = \{0\}$, so $I \ll \overline{\mu}$. □

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