ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVII, 2011, f.1

HAMILTON–JACOBI–BELLMAN EQUATIONS ASSOCIATED TO SYMMETRIC STABLE PROCESSES

BY

ADRIAN ZĂLINESCU

Abstract. We are concerned with an optimal stochastic control and stopping problem of a jump diffusion process. The main interest of this paper lies in the case where the dynamics has infinite variance, especially in the case of solutions of SDEs driven by symmetric stable processes. We prove that the value function is a viscosity solution of the integro-differential variational inequality arising from the associated dynamic programming. We also establish comparison principles in the class of semi-continuous functions with polynomial growth of a given order.

Mathematics Subject Classification 2000: 93E20, 49L25, 60J75.

Key words: stochastic control, viscosity solutions, jump-diffusion processes, stable processes.

1. Introduction

We study a joint optimal control and stopping time problem in a finite horizon of a controlled finite-dimensional jump diffusion process. This type of problem was already considered by some authors, see for instance PHAM in [7], in the case where the jump diffusion process has finite variance. However, in the current paper we focus on a disjoint class of jump diffusions, in particular solutions of stochastic differential equations driven by symmetric stable processes.

Throughout the paper we will use $|\ \ |$, "$\cdot$" to denote the euclidean norm, respectively the scalar product in $\mathbb{R}^k$, $k \geq 1$. Also, $B(x,r)$ and $\overline{B(x,r)}$ will denote the open, respectively the closed ball of center $x \in \mathbb{R}^k$ and radius $r > 0$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the *usual assumptions* (in this case we say that

$(\Omega, \mathcal{F}, P, \mathbb{F})$ is a *stochastic basis*), and ν a σ-finite measure on $\mathbb{R}^d$ with the property that

$$\int_{\{|z|\geq 1\}} \nu\left(\mathrm{d}z\right) < +\infty$$

(we will often write $\{|z| \geq 1\}$ instead of $\left\{z \in \mathbb{R}^d;\ |z| \geq 1\right\}$, etc.). In this framework we consider a homogeneous Poisson random measure $N\left(\omega, \mathrm{d}t, \mathrm{d}z\right)$ with characteristic measure ν, which is:

i) for each $\omega \in \Omega$, $N\left(\omega, ., .\right)$ is a measure on $\mathbb{R}_+ \times \mathbb{R}^d$;

ii) for each set $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ with $\left(\lambda \otimes \nu\right)(B) < +\infty$ (λ is the Lebesgue measure on $\mathbb{R}_+$), the random variable $N\left(\cdot, B\right)$ is Poisson with parameter $\left(\lambda \otimes \nu\right)(B)$;

iii) if $B_1, \ldots, B_n$ are disjoint Borel sets of $\mathbb{R}_+ \times \mathbb{R}^d$, then $N(\cdot, B_1)$, $\ldots, N(\cdot, B_n)$ are independent.

We also suppose that N is $\mathbb{F}$-*adapted*, *i.e.* the stochastic process $t \mapsto N\left(\cdot, [0,t] \times A\right)$ is $\mathbb{F}$-adapted.

Then, the *compensated random measure*

$$\tilde{N}\left(\mathrm{d}t, \mathrm{d}z\right) := N\left(\mathrm{d}t, \mathrm{d}z\right) - \mathrm{d}t \otimes \nu\left(\mathrm{d}z\right)$$

has the property that $t \longmapsto \tilde{N}\left([0,t], A\right)$ is a martingale for every $A \in \mathcal{B}(\mathbb{R}^d)$ with $\nu\left(A\right) < +\infty$.

We will need some preparation concerning stochastic integration with respect to random measures. Let $\mathcal{P}$ denote the σ-algebra of $\mathbb{F}$-predictable sets on $[0,T] \times \Omega$. A stochastic process $H : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^n$ is called *predictable* if it is measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$. If the process

$$t \to \int_0^t \int_{\mathbb{R}^d} \left(\left|H\left(s,z\right)\right|^2 \wedge \left|H\left(s,z\right)\right|\right) \nu\left(\mathrm{d}z\right) \mathrm{d}s$$

is locally integrable, one can define the stochastic integral

$$\int_0^t \int_{\mathbb{R}^d} H\left(s,z\right) \tilde{N}\left(\mathrm{d}s, \mathrm{d}z\right),$$

which is a n-dimensional purely discontinuous local martingale. We give a list of properties which are of concern in the sequel.

Let τ be a stopping time, and H a predictable process. Then:

(a) if H is positive or $\mathbb{E}\int_0^{t\wedge\tau}\int_{\mathbb{R}^d}|H(s,z)|\,\nu(\mathrm{d}z)\,\mathrm{d}s<+\infty,\ \forall t\geq 0$, then

$$\mathbb{E}\int_0^{t\wedge\tau}\int_{\mathbb{R}^d}H(s,z)\,N(\mathrm{d}s,\mathrm{d}z)=\mathbb{E}\int_0^{t\wedge\tau}\int_{\mathbb{R}^d}H(s,z)\,\nu(\mathrm{d}z)\,\mathrm{d}s,\ \forall t\geq 0 \tag{1.1}$$

(here the integral with respect to N can be also considered in the usual way, as an integral with respect to a measure);

(b) if $\mathbb{E}\int_0^{t\wedge\tau}\int_{\mathbb{R}^d}|H(s,z)|^2\,\nu(\mathrm{d}z)\,\mathrm{d}s<+\infty,\ \forall t\geq 0$, then the process

$$t\to\int_0^{t\wedge\tau}\int_{\mathbb{R}^d}H(s,z)\,\tilde{N}(\mathrm{d}s,\mathrm{d}z),\ t\geq 0 \tag{1.2}$$

is a square-integrable martingale; moreover,

(c) for every $t\geq 0$,

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^{s\wedge\tau}\int_{\mathbb{R}^d}H(r,z)\tilde{N}(\mathrm{d}r,\mathrm{d}z)\right|^2\leq 4\mathbb{E}\int_0^{t\wedge\tau}\int_{\mathbb{R}^d}|H(r,z)|^2\,\nu(\mathrm{d}z)\mathrm{d}r. \tag{1.3}$$

The properties (a), (b), (c) are proved, for example, in [4], in a more general setting. The last inequality is a consequence of Doob's inequality. For more precise definitions and properties of random measures, the readers are referred to [2], [3], or [4].

Let us now describe the problem. For $t_0\in[0,T]$, we introduce the filtration $\mathbb{F}^{t_0}=\left\{\mathcal{F}_t^{t_0}\right\}_{t\in[t_0,T]}$, where $\mathcal{F}_t^{t_0}$ is the σ-algebra $\sigma(\tilde{N}(A);\ B\in\mathcal{B}([t_0,t]\times\mathbb{R}^d))$, augmented by the P-null sets in $\mathcal{F}$. In order to shorten notation, we define $\bar{N}(\mathrm{d}t,\mathrm{d}z):=N(\mathrm{d}t,\mathrm{d}z)-\mathrm{d}t\otimes\mathbf{1}_{\{|z|<1\}}\nu(\mathrm{d}z)$. The state of the system is described by an n-dimensional jump diffusion process X, which is the solution of the following stochastic differential equation (SDE for short):

$$\begin{cases}\mathrm{d}X_t=b(t,X_t,u_t)\,\mathrm{d}t+\int_{\mathbb{R}^d}\gamma(t,X_{t-},u_t,z)\,\bar{N}(\mathrm{d}t,\mathrm{d}z),t\in[t_0,T];\\ X_{t_0}=x_0.\end{cases} \tag{1.4}$$

Here $T>0$ is the finite horizon, U is a compact metric space, and the control u belongs to $\mathcal{U}_{t_0,T}$, the set of U-valued, $\mathbb{F}^{t_0}$-predictable processes defined on $[t_0,T]\times\Omega$. In order to have a unique solution of this equation, we impose conditions of Lipschitz type on the coefficients; they are explicitly given in the next section. We precise that, since $\int_{\{|z|\geq 1\}}|\gamma(t,x,u,z)|\,\nu(\mathrm{d}z)$

is not necessarily finite, the dynamics X can have infinite expectation. This is also the reason for which we consider $\tilde{N}(\mathrm{d}t, \mathrm{d}z)$ instead of $\tilde{N}(\mathrm{d}t, \mathrm{d}z)$ as an integrator.

We consider the problem of minimizing, with respect to stopping times and controls, a running cost g and a terminal cost h with discount rate c. Therefore, for every $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$, we introduce the expected value:

$$V(t_0,x_0) := \inf_{\substack{u\in\mathcal{U}_{t_0,T}\\ \tau\in\mathcal{T}_{t_0,T}}} \mathbb{E}\left[\int_{t_0}^{\tau} \mathrm{e}^{-\int_{t_0}^{t} c(s,X_s)\mathrm{d}s} g(t,X_t,u_t)\,\mathrm{d}t \right.$$
$$\left. + \mathrm{e}^{-\int_{t_0}^{\tau} c(s,X_s)\mathrm{d}s} h(X_\tau)\right], \tag{1.5}$$

where $\mathcal{T}_{t_0,T}$ is the set of all stopping times between t_0 and T. This stochastic control problem applies in finance theory for the American option valuation and the consumption/investment portfolio choice. The assumptions on c, g and h will imply the continuity and polynomial growth of V, regarded as a function of (t_0, x_0); they are also forwarded to the next section.

The reason for which we allow the initial time $t_0 \in [0,T]$, and initial state $x_0 \in \mathbb{R}^n$ of X to vary is that we will implement the dynamic part of the Bellman's principle of optimality, in order to obtain the Hamilton–Jacobi–Bellman (HJB in short) equation associated to this problem. More precisely, we will prove that the value function V is a solution (in the viscosity sense) of the following integro-differential equation:

$$\begin{cases} \min\{-c(t,x)v + \frac{\partial v}{\partial t} + \inf\limits_{u\in U} \mathcal{H}v(t,x,u); h(x) - v\} = 0 \text{ in } (0,T)\times\mathbb{R}^n; \\ v(T,x) = h(x), \ x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

where the operator $\mathcal{H}$, known as the *Hamiltonian* of the system, is given by

$$\mathcal{H}v(t,x,u) := g(t,x,u) + \mathrm{D}v(t,x)\cdot b(t,x,u) + \int_{\mathbb{R}^d} \left[v(t,x+\gamma(t,x,u,z)) \right.$$
$$\left. - v(t,x) - \mathrm{D}v(t,x)\cdot\gamma(t,x,u,z)\,\mathbf{1}_{\{|z|<1\}}\right]\nu(\mathrm{d}z).$$

Here $\mathrm{D}v$ and D^2v denote the first order, respectively the second order differential of v with respect to the state variable.

We investigate further what happens when the state equation is driven by a symmetric stable process. One can easily show that the solution of the equation$\mathrm{d}X_t = b(t,X_t,u_t)\mathrm{d}t + \sigma(t,X_{t-},u_t)\mathrm{d}M_t$, $t \in [t_0,T]$, where M

is a one-dimensional symmetric stable proces of order $\alpha \in (0,2)$, can be regarded as a jump-diffusion of the form (1.4). The more interesting aspect is that, in the case $\alpha \neq 1$, the resulting Hamiltonian will have this form:

$$\mathcal{H}v(t,x,u) := g(t,x,u) + \mathrm{D}v(t,x) \cdot b(t,x,u) + \tilde{c}_\alpha \mathrm{D}^\alpha_{\sigma(t,x,u)} v(t,x),$$

where $\tilde{c}_\alpha$ is a constant depending on α, and D^α_θ is the fractional derivative operator of order α in the direction θ. This allows us to give a unifying theory for the cases $\alpha \in (0,2)$ and $\alpha = 2$, since it is well known that for $\alpha = 2$, M is a Brownian motion, and the corresponding HJB equation is a PDE of the second order.

In the last section we prove a comparison principle for viscosity solutions. We are especially interested in the case $\int_{\{|z|\geq 1\}} |\gamma(t,x,u,z)|^p \nu(\mathrm{d}z) < +\infty$, for $p \in (0,2)$, although we consider the general situation of arbitrary $p > 0$. For different situations where $p \geq 2$, the uniqueness was established in [5], [7], or [8].

2. Assumptions. Estimates

Let us state the conditions on the coefficients of equation (1.4). Throughout the paper we fix a constant $p > 0$ and a function $\rho : \mathbb{R}^d \to \mathbb{R}_+$, bounded on the open unit ball, with the property that

$$\int_{\mathbb{R}^d} \left(\rho(z)^2 \mathbf{1}_{\{|z|<1\}} + \rho(z)^p \mathbf{1}_{\{|z|\geq 1\}} \right) \nu(\mathrm{d}z) < +\infty. \tag{2.1}$$

We assume that the functions $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\gamma : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^d \to \mathbb{R}^n$ satisfy the following conditions:

(A0) b and γ are continuous with respect to $(t,x,u) \in [0,T] \times \mathbb{R}^n \times U$; γ is measurable;

there exists $L > 0$, such that, for all $t \in [0,T]$, $x, y \in \mathbb{R}^n$, $u \in U$, and $z \in \mathbb{R}^d$, we have:

(A1) $|b(t,x,u) - b(t,y,u)| \leq L|x-y|$;

(A2) $|\gamma(t,x,u,z) - \gamma(t,y,u,z)| \leq \rho(z)|x-y|$;

(A3) $|\gamma(t,x,u,z)| \leq \rho(z)(1+|x|)$.

Of course, since U is compact, we have that$|b(t,x,u)| \leq C(1+|x|)$, $\forall\,(t,x,u) \in [0,T]\times\mathbb{R}^n\times U$, for a constant C depending only on L and $\sup b(\cdot,0,\cdot)$. These conditions ensure that, for each $(t_0,x_0)\in[0,T]$, and each $u\in\mathcal{U}_{t_0,T}$, there exists a unique solution of equation (1.4), which we denote $X^{t_0,x_0,u}$. Moreover, since u is $\mathbb{F}^{t_0}$-predictable, $X^{t_0,x_0,u}$ is $\mathbb{F}^{t_0}$-adapted. But, of course, such a solution exists even if we take u only $\mathbb{F}$-predictable.

We give now the conditions on the coefficients of the cost functional in (1.5). The functions $g:[0,T]\times\mathbb{R}^n\times U\to\mathbb{R}$, $h:\mathbb{R}^n\to\mathbb{R}$, and $c:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ satisfy:

(B0) g, h, and c are continuous;

there exist constants $c_0\in\mathbb{R}$, $0\leq p'<p$, and $K>0$ such that, for all $(t,x,u)\in[0,T]\times\mathbb{R}^n\times U$:

(B1) $|g(t,x,u)|+|h(x)|\leq K(1+|x|^{p'})$;

(B2) $c(t,x)\geq c_0$.

It is clear that $V(t,x)\leq h(x)$, $\forall\,(t,x)\in[0,T]\times\mathbb{R}^n$ (by taking $\tau\equiv t_0$ in (1.5)). Moreover, if $t=T$, this becomes an equality.

In order to study other properties of V, we need some estimates on X.

Proposition 2.1. *There exist a constant $C>0$, such that, for all $(t_0,x_0,y_0)\in[0,T)\times\mathbb{R}^n\times\mathbb{R}^n$, $u\in\mathcal{U}_{t_0,T}$, and $t_1\in(t_0,T]$, we have:*

i) $\mathbb{E}\sup\limits_{t\in[t_0,T]}\left|X_t^{t_0,x_0,u}\right|^p\leq C(1+|x_0|^p)$;

ii) $\mathbb{E}\sup\limits_{t\in[t_0,T]}\left|X_t^{t_0,x_0,u}-X_t^{t_0,y_0,u}\right|^p\leq C|x_0-y_0|^p$;

iii) $\mathbb{E}\sup\limits_{t\in[t_0,t_1]}\left|X_t^{t_0,x_0,u}-x_0\right|^{p\wedge 2}\leq C(1+|x_0|^{p_1})(t_1-t_0)^{p_2}$, *where* $p_1=4+2p$, $p_2=p/4$ *if* $p<2$, *and* $p_1=2$, $p_2=1$, *if* $p\geq 2$.

Since $x\longmapsto|x|^p$ is not second-order differentiable in 0, for $p<2$, one cannot use Itô's formula with this function, so we will use a C^2-approximation. We state, without proof, an elementary result which proves useful in the next sections, too.

Lemma 2.2. *There exists a function $\beta \in C^2(\mathbb{R}^n)$ satisfying: $\beta(x) = |x|^2$ if $|x| \leq 1$, $\beta(x) = |x|^p$ if $|x| \geq 2$, and $|x| \leq |y| \Rightarrow \beta(x) \leq \beta(y)$, if $x, y \in \mathbb{R}^n$. For such a function, there exists a constant $C = C_\beta > 0$, depending only on β, such that*

1. *if x, $a \in \mathbb{R}^n$, $\rho > 0$, and $|a| \leq \rho(1 + |x|)$, then*

$$|\mathrm{D}\beta(x)|(1 + |x|) \leq C(1 + \beta(x)); \tag{2.2}$$

$$|\beta(x + a) - \beta(x)| \leq C(1 + \rho^p)(1 + \beta(x)); \tag{2.3}$$

$$|\beta(x + a) - \beta(x)| \leq C\rho(1 + \rho_p)(1 + \beta(x)); \tag{2.4}$$

$$|\beta(x + a) - \beta(x) - \mathrm{D}\beta(x) \cdot a| \leq C\rho^2(1 + \rho_p)(1 + \beta(x)); \tag{2.5}$$

2. *if x, $a \in \mathbb{R}^n$, $\rho > 0$, and $|a| \leq \rho|x|$, then*

$$|\mathrm{D}\beta(x)||x| \leq C\beta(x); \tag{2.6}$$

$$|\beta(x + a) - \beta(x)| \leq C(1 + \rho^p)(\beta(x) + |x|^p); \tag{2.7}$$

$$|\beta(x + a) - \beta(x)| \leq C\rho(1 + \rho_p)(\beta(x) + |x|^p); \tag{2.8}$$

$$|\beta(x + a) - \beta(x) - \mathrm{D}\beta(x) \cdot a| \leq C\rho^2(1 + \rho_p)(\beta(x) + |x|^p). \tag{2.9}$$

Here $\rho_p := 0$ if $p \leq 1$ and $\rho_p := \rho^{p-1}$ if $p > 1$.

Proof of Proposition 2.1. For the sake of simplicity, by C we will denote different positive constants appearing in the proof, not depending on t_0, t_1, x_0, y_0, $u \in \mathcal{U}_{t_0,T}$, and which may change from line to line.

i) For brevity, we shall write X instead of $X^{t_0,x_0,u}$. Let us apply Itô's formula to $\beta(X_t)$. We have a.s., for every $t \in [t_0, T]$:

$$\begin{aligned}\beta(X_t) =& \beta(X_{t_0}) + \int_{t_0}^t \mathrm{D}\beta(X_s) \cdot b(s, X_s, u_s)\,\mathrm{d}s \\ &+ \int_{t_0}^t \int_{\{|z|<1\}} \mathcal{D}_1\beta(X_s, \gamma(s, X_s, u_s, z))\,\nu(\mathrm{d}z)\,\mathrm{d}s \\ &+ \int_{t_0}^t \int_{\mathbb{R}^d} \mathcal{D}_2\beta(X_{s-}, \gamma(s, X_{s-}, u_s, z))\,\bar{N}(\mathrm{d}s, \mathrm{d}z),\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}&\mathcal{D}_1\beta(x, a) := \beta(x + a) - \beta(x) - \mathrm{D}\beta(x) \cdot a; \\ &\mathcal{D}_2\beta(x, a) := \beta(x + a) - \beta(x),\end{aligned}$$

for $x,\ a \in \mathbb{R}^n$.

By (2.2) and (2.5) with $x = X_s$, $a = b(s, X_s, u_s)$ or $a = \gamma(s, X_s, u_s, z)$, $\rho = \rho(z)$, the linear growth of b and condition (A3), we have that

$$\int_{t_0}^{t} \mathrm{D}\beta(X_s)\cdot b(s, X_s, u_s)\,\mathrm{d}s + \int_{t_0}^{t}\int_{\{|z|<1\}} \mathcal{D}_1\beta(X_s, \gamma(s, X_s, u_s, z))\,\nu(\mathrm{d}z)\,\mathrm{d}s \le C\left(1 + \int_{\{|z|<1\}} \rho(z)^2\,\nu(\mathrm{d}z)\right)\int_{t_0}^{t}[1+\beta(X_s)]\,\mathrm{d}s.$$

We consider, for every $k \ge 1$, the stopping time:

$$\tau_k := \inf\{t \ge t_0;\ |X_t| \ge k\} \wedge T.$$

Clearly, τ_k converges to T as $k \to \infty$. From (1.1) and (2.3),

$$\begin{aligned}
&\mathbb{E}\int_{t_0}^{t\wedge\tau_k}\int_{\{|z|\ge 1\}} |\mathcal{D}_2\beta(X_{s-}, \gamma(s, X_{s-}, u_s, z))|\,N(\mathrm{d}s, \mathrm{d}z)\\
&= \mathbb{E}\int_{t_0}^{t\wedge\tau_k}\int_{\{|z|\ge 1\}} |\mathcal{D}_2\beta(X_s, \gamma(s, X_s, u_s, z))|\,\nu(\mathrm{d}z)\,\mathrm{d}s\\
&\le C\int_{\{|z|\ge 1\}} (1+\rho(z)^p)\,\nu(\mathrm{d}z)\,\mathbb{E}\int_{t_0}^{t\wedge\tau_k}[1+\beta(X_s)]\,\mathrm{d}s.
\end{aligned}$$

Also, by (2.4),

$$\begin{aligned}
&\mathbb{E}\int_{t_0}^{\tau_k}\int_{\{|z|<1\}} |\mathcal{D}_2\beta(X_s, \gamma(s, X_s, u_s, z))|^2\,\nu(\mathrm{d}z)\,\mathrm{d}s\\
&\le C\left(1+|k|^{2p}\right)\int_{\{|z|<1\}} \rho(z)^2\,\nu(\mathrm{d}z) < +\infty.
\end{aligned}$$

Hence, property (1.2) implies that

$$t \to \int_{t_0}^{t\wedge\tau_k}\int_{\{|z|<1\}} \mathcal{D}_2\beta(X_{s-}, \gamma(s, X_{s-}, u_s, z))\,\tilde{N}(\mathrm{d}s, \mathrm{d}z),\ t \in [t_0, T]$$

is a martingale; thus

$$\mathbb{E}\int_{t_0}^{t\wedge\tau_k}\int_{\{|z|<1\}} \mathcal{D}_2\beta(X_{s-}, \gamma(s, X_{s-}, u_s, z))\,\tilde{N}(\mathrm{d}s, \mathrm{d}z) = 0,\ t \in [t_0, T].$$

Using these formulae in relation (2.10), we obtain, for every $t \in [t_0, T]$:

$$\mathbb{E}\beta\left(X_{t\wedge\tau_k}\right) \leq C + \beta\left(x_0\right) + C\int_{t_0}^{t} \mathbb{E}\beta\left(X_{s\wedge\tau_k}\right) \mathrm{d}s.$$

Gronwall's inequality yields $\mathbb{E}\beta\left(X_{t\wedge\tau_k}\right) \leq C\left(1+\beta\left(x_0\right)\right),\ t \in [t_0, T]$; letting $k \to \infty$,

$$\mathbb{E}\beta\left(X_t\right) \leq C\left(1+\beta\left(x_0\right)\right),\ t \in [t_0, T]. \tag{2.11}$$

The following step is to obtain an estimate for $\mathbb{E}\sup_{t\in[t_0,T]} \beta\left(X_t\right)$. For that, let $\tilde{\beta}$ be defined in the same way as β, but replacing p by $p/2$.

We apply once again the Itô's formula, but for $\tilde{\beta}\left(X_t\right)$:

$$\begin{aligned}\tilde{\beta}\left(X_t\right) \leq & \tilde{\beta}\left(x_0\right) + \int_{t_0}^{T} \left|\mathrm{D}\tilde{\beta}\left(X_s\right) \cdot b\left(s, X_s, u_s\right)\right| \mathrm{d}s \\ & + \int_{t_0}^{T}\int_{\{|z|<1\}} \left|\mathcal{D}_1\tilde{\beta}\left(X_s, \gamma\left(s, X_s, u_s, z\right)\right)\right| \nu\left(\mathrm{d}z\right)\mathrm{d}s \\ & + \int_{t_0}^{T}\int_{\{|z|\geq 1\}} \left|\mathcal{D}_2\tilde{\beta}\left(X_{s-}, \gamma\left(s, X_{s-}, u_s, z\right)\right)\right| \nu\left(\mathrm{d}z\right)\mathrm{d}s + Z_t,\end{aligned}$$

where

$$Z_t := \int_{t_0}^{t}\int_{\mathbb{R}^d} \mathcal{D}_2\tilde{\beta}\left(X_{s-}, \gamma\left(s, X_{s-}, u_s, z\right)\right) \tilde{N}\left(\mathrm{d}s, \mathrm{d}z\right),\ t \in [t_0, T].$$

Using, as in the previous argument, (2.2), (2.3) and (2.5) with $\tilde{\beta}$ instead of β, we obtain:

$$\begin{aligned}&\sup_{t\in[t_0,T]} \tilde{\beta}\left(X_t\right)^2 \leq C\tilde{\beta}\left(x_0\right)^2 + C\sup_{t\in[t_0,T]} |Z_t|^2 \\ &+C\left(\int_{\mathbb{R}^d}\left(\rho\left(z\right)^2 \mathbf{1}_{\{|z|<1\}} + (1+\rho\left(z\right)^{\frac{p}{2}})\mathbf{1}_{\{|z|\geq 1\}}\right)\nu\left(\mathrm{d}z\right)\right)^2 \int_{t_0}^{T} [1+\tilde{\beta}(X_s)^2]\mathrm{d}s.\end{aligned}$$

On the other hand, from (1.3), (2.3) and (2.4),

$$\begin{aligned}&\mathbb{E}\sup_{t\in[t_0,T]} |Z_t|^2 \leq 4\mathbb{E}\int_{t_0}^{T}\int_{\mathbb{R}^d} \left|\mathcal{D}_2\tilde{\beta}\left(X_{s-}, \gamma\left(s, X_{s-}, u_s, z\right)\right)\right|^2 \nu\left(\mathrm{d}z\right)\mathrm{d}s \\ &\leq C\int_{\mathbb{R}^d}\left(\rho\left(z\right)^2 \mathbf{1}_{\{|z|<1\}} + \left(1+\rho\left(z\right)^p\right)\mathbf{1}_{\{|z|\geq 1\}}\right)\nu\left(\mathrm{d}z\right)\mathbb{E}\int_{t_0}^{T}\left[1+\tilde{\beta}\left(X_s\right)^2\right]\mathrm{d}s.\end{aligned}$$

Since $\tilde{\beta}^2(x) \leq C\beta(x)$, $\forall x \in \mathbb{R}^n$, we conclude that

$$\mathbb{E}\sup_{t\in[t_0,T]}\tilde{\beta}(X_t)^2 \leq C(1+\beta(x_0)) + C\int_{t_0}^{T}[1+\mathbb{E}\beta(X_s)]\,\mathrm{d}s \leq C(1+\beta(x_0)),$$

by (2.11). We then easily obtain that $\mathbb{E}\sup_{t\in[t_0,T]}|X_t|^p \leq C(1+|x_0|^p)$, because $\tilde{\beta}(x)^2 = \beta(x) = |x|^p$ if $|x| \geq 2$.

ii) We will write X, Y, $\hat{X}$, instead of $X^{t_0,x_0,u}$, $X^{t_0,y_0,u}$, $X^{t_0,x_0,u} - X^{t_0,y_0,u}$, respectively. Obviously, we can suppose that $x_0 \neq y_0$. Let $\varepsilon := |x_0 - y_0|$. In order to obtain the second estimate of the proposition, we apply Itô's formula for $\tilde{\beta}(\frac{1}{\varepsilon}\hat{X}_t)$:

$$\begin{aligned}\tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_t) &= 1 + \int_{t_0}^{t}\tfrac{1}{\varepsilon}\mathrm{D}\tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_s)\cdot[b(s,X_s,u_s)-b(s,Y_s,u_s)]\,\mathrm{d}s + \tilde{Z}_t\\ &+ \int_{t_0}^{t}\int_{\{|z|<1\}}\mathcal{D}_1\tilde{\beta}\left(\tfrac{1}{\varepsilon}\hat{X}_s, \tfrac{1}{\varepsilon}(\gamma(s,X_s,u_s,z)-\gamma(s,Y_s,u_s,z))\right)\nu(\mathrm{d}z)\,\mathrm{d}s\\ &+ \int_{t_0}^{t}\int_{\{|z|\geq 1\}}\mathcal{D}_2\tilde{\beta}\left(\tfrac{1}{\varepsilon}\hat{X}_s, \tfrac{1}{\varepsilon}(\gamma(s,X_{s-},u_s,z)-\gamma(s,Y_{s-},u_s,z))\right)\nu(\mathrm{d}z)\,\mathrm{d}s,\end{aligned}$$

where

$$\tilde{Z}_t := \int_{t_0}^{t}\int_{\mathbb{R}^d}\mathcal{D}_2\tilde{\beta}\left(\tfrac{1}{\varepsilon}\hat{X}_s, \tfrac{1}{\varepsilon}(\gamma(s,X_{s-},u_s,z)-\gamma(s,Y_{s-},u_s,z))\right)\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

Using inequalities (2.6), (2.7), (2.9) with $a = \frac{1}{\varepsilon}(b(s,X_s,u_s)-b(s,Y_s,u_s))$ or $a = \frac{1}{\varepsilon}(\gamma(s,X_s,u_s,z)-\gamma(s,Y_s,u_s,z))$, $x = \frac{1}{\varepsilon}\hat{X}_s$, $\rho = \rho(z)$, and conditions (A1), (A2), we obtain

$$\sup_{s\in[t_0,t]}\tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_s)^2 \leq C + C\int_{t_0}^{t}\left(\tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_s) + |\tfrac{1}{\varepsilon}\hat{X}_s|^{p/2}\right)^2\mathrm{d}s + C\sup_{s\in[t_0,t]}|\tilde{Z}_s|^2.$$

Then, from (2.7), (2.8), taking into account (1.3),

$$\begin{aligned}&\mathbb{E}\sup_{s\in[t_0,t]}\tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_s)^2 \leq C + C\mathbb{E}\int_{t_0}^{t}\left(\tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_s)^2 + |\tfrac{1}{\varepsilon}\hat{X}_s|^p\right)\mathrm{d}s\\ &+ C\mathbb{E}\int_{t_0}^{t}\int_{\mathbb{R}^d}\left|\mathcal{D}_2\tilde{\beta}\left(\tfrac{1}{\varepsilon}\hat{X}_s, \tfrac{1}{\varepsilon}(\gamma(s,X_{s-},u_s,z)-\gamma(s,Y_{s-},u_s,z))\right)\right|^2\nu(\mathrm{d}z)\,\mathrm{d}s\\ &\leq C + C\mathbb{E}\int_{t_0}^{t}\tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_s)^2\mathrm{d}s + \mathbb{E}\int_{t_0}^{t}\left|\tfrac{1}{\varepsilon}\hat{X}_s\right|^p\mathrm{d}s.\end{aligned}$$

Since

$$|x|^p \leq 1 + C\tilde{\beta}(x)^2, \ \forall x \in \mathbb{R}^n, \tag{2.12}$$

we get

$$\mathbb{E} \sup_{s\in[t_0,t]} \tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_s)^2 \leq C + C\mathbb{E}\int_{t_0}^{t} \sup_{r\in[t_0,s]} \tilde{\beta}(\tfrac{1}{\varepsilon}\hat{X}_r)^2 \mathrm{d}s.$$

By Gronwall's inequality, $\mathbb{E}\sup_{t\in[t_0,T]} \tilde{\beta}(\frac{1}{\varepsilon}\hat{X}_t)^2 \leq C$. Applying once again (2.12), this yields $\mathbb{E}\sup_{t\in[t_0,T]} |\frac{1}{\varepsilon}\hat{X}_t|^p \leq 1 + C\mathbb{E}\sup_{t\in[t_0,T]} \tilde{\beta}(\frac{1}{\varepsilon}\hat{X}_t)^2 \leq C$. Hence, from the choice $\varepsilon = |x_0 - y_0|$, $\mathbb{E}\sup_{t\in[t_0,T]} |\hat{X}_t|^p \leq C\,|x_0 - y_0|^p$.

iii) We make first the estimate in the case $p < 2$. We apply Itô's formula for $\tilde{\beta}(X_t - x_0)$:

$$\begin{aligned}
\tilde{\beta}(X_t - x_0) &= \int_{t_0}^{t} \mathrm{D}\tilde{\beta}(X_s - x_0) \cdot b(s, X_s, u_s)\,\mathrm{d}s \\
&+ \int_{t_0}^{t}\int_{\{|z|<1\}} \mathcal{D}_1\tilde{\beta}(X_s - x_0, \gamma(s, X_{s-}, u_s, z))\,\nu(\mathrm{d}z)\,\mathrm{d}s \\
&+ \int_{t_0}^{t}\int_{\{|z|\geq 1\}} \mathcal{D}_2\tilde{\beta}(X_{s-} - x_0, \gamma(s, X_{s-}, u_s, z))\,\nu(\mathrm{d}z)\,\mathrm{d}s \\
&+ \int_{t_0}^{t}\int_{\mathbb{R}^d} \mathcal{D}_2\tilde{\beta}(X_{s-} - x_0, \gamma(s, X_{s-}, u_s, z))\,\tilde{N}(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

Using inequalities (2.2), (2.3), (2.5) with $x = X_s - x_0$, $a = b(s, X_s, u_s)$ or $a = \gamma(s, X_s, u_s, z)$, $\rho = \rho(z)$, and the growth estimates:

$$\begin{aligned}
|b(s, X_s, u_s)| &\leq C(1 + |x_0|)(1 + |X_s - x_0|), \\
|\gamma(s, X_{s-}, u_s, z)| &\leq \rho(z)(1 + |x_0|)(1 + |X_s - x_0|),
\end{aligned}$$

one can show that

$$\begin{aligned}
\sup_{t\in[t_0,t_1]} \tilde{\beta}(X_t - x_0)^2 &\leq C(1 + |x_0| + |x_0|^4 + |x_0|^p)\left(\int_{t_0}^{t_1} [1 + \tilde{\beta}(X_s - x_0)]\mathrm{d}s\right)^2 \\
&+ \sup_{t\in[t_0,t_1]} \left(\int_{t_0}^{t}\int_{\mathbb{R}^d} \mathcal{D}_2\tilde{\beta}(X_{s-} - x_0, \gamma(s, X_{s-}, u_s, z))\,\tilde{N}(\mathrm{d}s, \mathrm{d}z)\right)^2.
\end{aligned}$$

From (1.3), together with (2.3), (2.4), we obtain

$$\mathbb{E} \sup_{t\in[t_0,t_1]} \tilde{\beta}(X_t - x_0)^2 \leq C(1 + |x_0|^4)(t_1 - t_0)^2(1 + \mathbb{E} \sup_{t\in[t_0,T]} \tilde{\beta}(X_t - x_0)^2)$$

$$+\int_{t_0}^{t}\int_{\mathbb{R}^d}|\mathcal{D}_2\tilde{\beta}(X_{s-}-x_0,\gamma(s,X_{s-},u_s,z))|^2\nu(\mathrm{d}z)\mathrm{d}s$$
$$\leq C(1+|x_0|^4)(t_1-t_0)(1+\mathbb{E}\sup_{t\in[t_0,T]}\tilde{\beta}(X_t-x_0)^2).$$

But $\tilde{\beta}(x-x_0)^2\leq C(1+|x_0|^p)(1+\tilde{\beta}(x)^2)$, $\forall x\in\mathbb{R}^n$; hence, from i),

$$\mathbb{E}\sup_{t\in[t_0,T]}\tilde{\beta}(X_t-x_0)^2\leq C(1+|x_0|^p)(1+\mathbb{E}\sup_{t\in[t_0,T]}\tilde{\beta}(X_t)^2)\leq C(1+|x_0|^{2p}).$$

Therefore $\mathbb{E}\sup_{t\in[t_0,t_1]}\tilde{\beta}(X_t-x_0)^2\leq C(1+|x_0|^{4+2p})(t_1-t_0)$. Then

$$\mathbb{E}\sup_{t\in[t_0,t_1]}|X_t-x_0|^p\leq\left[\mathbb{E}\sup_{t\in[t_0,t_1]}\tilde{\beta}\left(X_t-x_0\right)^2\right]^{p/4}+C\mathbb{E}\sup_{t\in[t_0,t_1]}\tilde{\beta}\left(X_t-x_0\right)^2$$
$$\leq C(1+|x_0|^{4+2p})(t_1-t_0)^{p/4}.$$

If $p\geq 2$, one can obtain better estimates. Indeed, in this case $\int\rho\left(z\right)^2\nu\left(\mathrm{d}z\right)$ is finite; consequently $\gamma\left(s,X_{s-},u_s,z\right)$ is integrable with respect to $\tilde{N}$ on $[t_0,T]\times\mathbb{R}^d$, and we have

$$\mathbb{E}\sup_{t\in[t_0,t_1]}|X_t-x_0|^2\leq\mathbb{E}\left(\int_{t_0}^{t_1}\left|b\left(s,X_s,u_s\right)\right|\mathrm{d}s\right)^2$$
$$+\mathbb{E}\left(\int_{t_0}^{t_1}\int_{\{|z|\geq 1\}}\left|\gamma\left(s,X_{s-},u_s,z\right)\right|\nu\left(\mathrm{d}z\right)\mathrm{d}s\right)^2$$
$$+\mathbb{E}\sup_{t\in[t_0,t_1]}\left|\int_{t_0}^{t}\int_{\mathbb{R}^d}\gamma\left(s,X_{s-},u_s,z\right)\tilde{N}\left(\mathrm{d}s,\mathrm{d}z\right)\right|^2.$$

By (1.3) we get

$$\mathbb{E}\sup_{t\in[t_0,t_1]}|X_t-x_0|^2\leq C(t_1-t_0)^2\mathbb{E}(1+\sup_{t\in[t_0,T]}|X_s|^2)$$
$$+\mathbb{E}\int_{t_0}^{t_1}\int_{\mathbb{R}^d}|\gamma(s,X_{s-},u_s,z)|^2\nu(\mathrm{d}z)\mathrm{d}s\leq C(1+|x_0|^2)(t_1-t_0).$$

□

Remark. In fact, one can generalize the estimate ii) for random initial conditions, by the same method of proof. Then, the estimate iii) enables us

to conclude that $\mathbb{E}\sup_{t\in[t_0,T]}|X_t^{t_1,x_0,u}-X_t^{t_0,x_0,u}|^{p\wedge 2}\leq C(1+|x_0|^{p_1})(t_1-t_0)^{p_2}$, where $t_1>t_0$ and $X_t^{t_1,x_0,u}$ is defined as x_0 on $[t_0,t_1)$.

This proposition allows us to retrieve some properties of V, which we can find in the continuous-diffusion case.

First of all, we can easily derive from i) that $|V(t,x)|\leq C(1+|x|^{p'})$, $\forall x\in\mathbb{R}^d$, for a constant $C>0$.

From i), ii) and the remark above, one can prove the following technical result:

Lemma 2.3. *For $a,\delta>0$, let*

$$\omega(\delta,a):=\sup_{\substack{t\in[0,T],\ u\in U,\\ x,y\in\overline{B(0,a)},\ |x-y|\leq\delta}}\left(|c(t,x)-c(t,y)|+|g(t,x,u)-g(t,y,u)|\right.$$
$$\left.+|h(x)-h(y)|\right)$$

be the joint modulus of continuity of c, g, and h.

Then, for all δ, $a>0$, $(t,x),(s,y)\in[0,T]\times\mathbb{R}^n$,

$$|V(t,x)-V(s,y)|\leq C\omega(\delta,a)\left(1+|x|^{p'}\right)$$
$$+C(1+|x|^{p'}+|y|^{p'})\left(\frac{|x-y|^{p-p'}+\left[\left(1+|x|^{p_1/p}\right)|t-s|^{p_2/p}\right]^{p-p'}}{\delta^{\frac{p\wedge 2}{p}(p-p')}}+\frac{1+|x|^{p-p'}+|y|^{p-p'}}{a^{p-p'}}\right),$$

where C is a constant not depending on t,s,x,y,δ, or a.

As an important consequence, we have:

Proposition 2.4. *Under assumptions $(A0)$–$(A3)$, $(B0)$–$(B2)$, V is continuous.*

Proof. Let $(t_0,x_0)\in[0,T]\times\mathbb{R}^n$ and a sequence (t_n,x_n) in $[0,T]\times\mathbb{R}^n$ converging to (t_0,x_0). Then, by the previous lemma,

$$\limsup_{n\to\infty}|V(t_n,x_n)-V(t_0,x_0)|\leq C\left(1+2|x_0|^{p'}\right)\left(\omega(\delta,a)+\frac{1}{a^{p-p'}}\right),$$

for every a, $\delta>0$. Letting now $\delta\to 0$ and afterwards $a\to\infty$, we obtain $\lim_{n\to\infty}V(t_n,x_n)=V(t_0,x_0)$. This proves that V is continuous. □

3. HJB equations and dynamic programming

Let us introduce some notations: if $q \geq 0$, I is an interval in $[0,T]$, we say that a function $f : I \times \mathbb{R}^n \to \mathbb{R}$ has *q-growth*, if

$$\sup_{(t,x)\in I\times\mathbb{R}^n} \frac{|f(t,x)|}{1+|x|^q} < +\infty.$$

Then $C_q(I\times\mathbb{R}^n)$, $C_q^{1,2}(I\times\mathbb{R}^n)$, $USC_q(I\times\mathbb{R}^n)$, $LSC_q(I\times\mathbb{R}^n)$ will stand for the sets of functions in $C(I\times\mathbb{R}^n)$, $C^{1,2}(I\times\mathbb{R}^n)$[1], upper semi-continuous on $I\times\mathbb{R}^n$, and lower semi-continuous on $I\times\mathbb{R}^n$ respectively, which have q-growth.

For a function $\phi \in C_p^{1,2}((0,T)\times\mathbb{R}^n)$, we define the operators:

$$\begin{aligned}\mathcal{I}_1\phi(t,x,u) &:= \int_{\{|z|<1\}} [\phi(t,x+\gamma(t,x,u,z)) - \phi(t,x) \\ &\quad - \mathrm{D}\phi(t,x)\cdot\gamma(t,x,u,z)]\nu(\mathrm{d}z); \\ \mathcal{I}_2\phi(t,x,u) &:= \int_{\{|z|\geq 1\}} [\phi(t,x+\gamma(t,x,u,z)) - \phi(t,x)]\,\nu(\mathrm{d}z),\end{aligned}$$

with $(t,x,u)\in(0,T)\times\mathbb{R}^n\times U$.

In view of (A3) and the boundednes of ρ on the open unit ball, we have,

$$\begin{aligned}&|\phi(t,x+\gamma(t,x,u,z)) - \phi(t,x) - \mathrm{D}\phi(t,x)\cdot\gamma(t,x,u,z)|\,\mathbf{1}_{\{|z|<1\}} \\ &\qquad\leq 2 \sup_{|y|\leq C(1+|x|)} \left|\mathrm{D}^2\phi(t,y)\right| \rho(z)^2 \left(1+|x|^2\right);\end{aligned}$$

$$|\phi(t,x+\gamma(t,x,u,z)) - \phi(t,x)| \leq C(1+\rho(z)^p)(1+|x|^p), \tag{3.1}$$

where $C>0$ does not depend on $(t,x,u,z)\in(0,T)\times\mathbb{R}^n\times U\times\mathbb{R}^d$. Consequently we have the continuity of $\mathcal{I}_1\phi$ and $\mathcal{I}_2\phi$.

The Hamiltonian operator can be more rigorously defined as

$$\mathcal{H}\phi(t,x,u) := g(t,x,u) + \mathrm{D}\phi(t,x)\cdot b(t,x,u) + (\mathcal{I}_1+\mathcal{I}_2)\phi(t,x,u),$$

for $\phi\in C_p^{1,2}((0,T)\times\mathbb{R}^n)$, $(t,x,u)\in(0,T)\times\mathbb{R}^n\times U$.

[1] $C^{1,2}(I\times\mathbb{R}^n)$ stands for the class of functions on $I\times\mathbb{R}^n$ which have continuous first-order derivatives with respect to $t\in I$ and second-order differentials with respect to $x\in\mathbb{R}^n$.

Our aim is to prove that V is a solution of the integro-differential variational inequality (1.6). Of course, as it is the case when the state equation is driven by a Brownian motion (see [1], for example), one cannot expect that the value function V to be differentiable, and equation (1.6) should be interpreted in a weaker sense.

Definition 1. We say that the function $v : (0,T]\times\mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution (supersolution) of equation (1.6) if

1. v has p-growth and is upper (lower) semi-continuous on $(0,T]\times\mathbb{R}^n$;
2. $v(T,x) \le h(x)$ $\qquad$ ($v(T,x) \ge h(x)$), $\forall x \in \mathbb{R}^n$;
3. whenever $\phi \in C_p^{1,2}((0,T)\times\mathbb{R}^n)$ and $v-\phi$ has a global maximum (minimum) at $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$, we have:

$$\min\left\{-cv+\frac{\partial\phi}{\partial t}+\inf_{u\in U}\mathcal{H}\phi(\cdot,\cdot,u);h-v\right\}(t_0,x_0)\ge 0 \qquad (\le 0).$$

The function v is called a viscosity solution of equation (1.6) if it is both a subsolution, and a supersolution in the viscosity sense.

The usual way for proving that the value function V is a solution of (1.6) goes through the Bellman's principle of optimality, which we will state here in two forms.

Theorem 3.1. *Let $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$.*

1. For all admissible controls $u\in\mathcal{U}_{t_0,T}$, and all stopping times $\tau\in\mathcal{T}_{t_0,T}$,

$$V(t_0,x_0)\le\mathbb{E}\left[\int_{t_0}^{\tau}\mathrm{e}^{-\int_{t_0}^{t}c(s,X_s)\mathrm{d}s}g(t,X_t,u_t)\,\mathrm{d}t+\mathrm{e}^{-\int_{t_0}^{\tau}c(s,X_s)\mathrm{d}s}V(\tau,X_\tau)\right]$$

(here and below X stands for $X^{t_0,x_0,u}$).

2. Let $\varepsilon>0$. For all admissible controls $u\in\mathcal{U}_{t_0,T}$, define the stopping time

$$\tau^{\varepsilon}_{t_0,x_0,u}:=\inf\{t\in[t_0,T];\ V(t,X_t)\ge h(X_t)-\varepsilon\}\wedge T.$$

If $\{\tau_u\}_{u\in\mathcal{U}_{t_0,T}}$ are stopping times such that $\tau_u\le\tau^{\varepsilon}_{t_0,x_0,u}$ for all $u\in\mathcal{U}_{t_0,T}$, then we have:

$$\begin{aligned}V(t_0,x_0)=\inf_{u\in\mathcal{U}_{t_0,T}}\mathbb{E}[&\int_{t_0}^{\tau_u}\mathrm{e}^{-\int_{t_0}^{t}c(s,X_s)\mathrm{d}s}g(t,X_t,u_t)\mathrm{d}t\\&+\mathrm{e}^{-\int_{t_0}^{\tau_u}c(s,X_s)\mathrm{d}s}V(\tau_u,X_{\tau_u})].\end{aligned}$$

Proof. In order to simplify the calculus, we will assume that $c \equiv 0$; but one can easily reconstruct the arguments for the general case. Let us introduce the following notation, for $(t,x) \in [0,T] \times \mathbb{R}^n$, $\tau \in \mathcal{T}_{t,T}$, and an $\mathbb{F}$-predictable process $u : [t,T] \times \Omega \to U$:

$$J(t,x,u,\tau) := \mathbb{E}\left[\int_t^\tau g\left(s, X_s^{t,x,u}, u_s\right) \mathrm{d}s + h\left(X_\tau\right)\right].$$

We first give, without proof, the following lemma, which asserts the Markovian character of the solution X:

Lemma 3.2. *Let $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$, $u \in \mathcal{U}_{t_0,T}$, and $\tau_1,\ \tau_2 \in \mathcal{T}_{t_0,T}$ with $\tau_1 \leq \tau_2$ a.s. Then, $P(\mathrm{d}\omega)$-a.s.,*

$$J(\tau_1(\omega), X_{\tau_1}^{t_0,x_0,u}(\omega), u, \tau_2) = \mathbb{E}\left[\int_{\tau_1}^{\tau_2} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + h(X_{\tau_2}^{t_0,x_0,u})\Big|\mathcal{F}_{\tau_1}^{t_0}\right](\omega).$$

For the continuous-diffusion case, the interested reader can find a proof of this result in [9]; it can be easily adapted to ours.

1. We suppose first that $\tau \equiv t_1$ is deterministic. Let us fix $\varepsilon \in (0,1)$. We can choose $a > 0$ such that $a^{p-p'} > \frac{1}{\varepsilon}$ and $\delta > 0$ such that $\delta^{(2-\frac{p\wedge 2}{p})(p-p')} < \varepsilon$ and $\omega(\delta,a) < \varepsilon$ (see Lemma 2.3 for the definition of $\omega(\delta,a)$).

Let $\{D_j\}_{j\geq 1}$ be a Borelian partition of $\mathbb{R}^n$, with fixed points $x_j \in D_j$, and $\operatorname{diam} D_j \leq \delta^2$, for every $j \geq 1$.

For all $j \geq 1$, there exist $u_j \in \mathcal{U}_{t_1,T}$ and $\tau_j \in \mathcal{T}_{t_1,T}$ such that $V(t_1,x_j) + \varepsilon \geq J(t_1,x_j,u_j,\tau_j)$. Since J satisfies an analoguos relation as V does in Lemma 2.3, it is immediately obtained that, for $x \in D_j$,

$$\begin{aligned} J(t_1,x,u_j,\tau_j) \leq & \varepsilon + V(t_1,x) \\ & + C_1\left(1+|x|^{p'}\right)\left(\omega(\delta,a) + \frac{1+|x|^{p-p'}}{a^{p-p'}} + \frac{|x-x_j|^{p-p'}}{\delta^{\frac{p\wedge 2}{p}(p-p')}}\right) \\ & \leq V(t_1,x) + C_2\varepsilon\left(1+|x|^p\right), \end{aligned} \tag{3.2}$$

where C_1 and C_2 are positive constants which do not depend on ε, j and x.

We consider the following controls and the following stopping time:

$$\tilde{u}_j(t,\omega) := \begin{cases} u(t,\omega), & t \in [t_0, t_1]; \\ u_j(t,\omega), & t \in (t_1, T]; \end{cases}$$
$$\tilde{u}(t,\omega) := \tilde{u}_j(t,\omega), \text{ if } X_{t_1}^{t_0,x_0,u}(\omega) \in D_j;$$
$$\tau(\omega) := \tau_j(t,\omega), \text{ if } X_{t_1}^{t_0,x_0,u}(\omega) \in D_j.$$

The fact that $X_{t_1}^{t_0,x_0,u}$ is $\mathcal{F}_{t_1}^{t_0}$-measurable ensures us that $\tilde{u}_j,\ \tilde{u} \in \mathcal{U}_{t_0,T}$ and $\tau \in \mathcal{T}_{t_1,T}$. We have:

$$V(t_0,x_0) \le J(t_0,x_0,\tilde{u},\tilde{\tau}) = \mathbb{E}\left[\int_{t_0}^{t_1} g(t, X_t^{t_0,x_0,\tilde{u}}, \tilde{u}_t)\mathrm{d}t\right]$$
$$+ \mathbb{E}\left[\int_{t_1}^{\tilde{\tau}} g(t, X_t^{t_0,x_0,\tilde{u}}, \tilde{u}_t)\mathrm{d}t + h(X_{\tilde{\tau}}^{t_0,x_0,\tilde{u}})\right] = \mathbb{E}\left[\int_{t_0}^{t_1} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t\right]$$
$$+ \sum_{j\ge 1} \mathbb{E}\left[\left(\int_{t_1}^{\tau_j} g(t, X_t^{t_0,x_0,\tilde{u}_j}, (\tilde{u}_j)_t)\mathrm{d}t + h(X_{\tau_j}^{t_0,x_0,\tilde{u}_j})\right)\mathbf{1}_{D_j}(X_{t_1}^{t_0,x_0,u})\right].$$

By Lemma 3.2,

$$\mathbb{E}\left[\left(\int_{t_1}^{\tau_j} g(t, X_t^{t_0,x_0,\tilde{u}_j}, (\tilde{u}_j)_t)\mathrm{d}t + h(X_{\tau_j}^{t_0,x_0,\tilde{u}_j})\right)\mathbf{1}_{D_j}(X_{t_1}^{t_0,x_0,u})\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left(\int_{t_1}^{\tau_j} g(t, X_t^{t_0,x_0,\tilde{u}_j}, (\tilde{u}_j)_t)\mathrm{d}t + h(X_{\tau_j}^{t_0,x_0,\tilde{u}_j})\middle|\mathcal{F}_{t_1}^{t_0}\right)\mathbf{1}_{D_j}(X_{t_1}^{t_0,x_0,u})\right]$$
$$= \mathbb{E}[J(t_1, X_{t_1}^{t_0,x_0,\tilde{u}_j}, \tilde{u}_j, \tau_j)\mathbf{1}_{D_j}(X_{t_1}^{t_0,x_0,u})]$$
$$= \mathbb{E}[J(t_1, X_{t_1}^{t_0,x_0,u}, u_j, \tau_j)\mathbf{1}_{D_j}(X_{t_1}^{t_0,x_0,u})].$$

Hence, by (3.2),

$$V(t_0,x_0) \le \mathbb{E}\left[\int_{t_0}^{t_1} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t\right]$$
$$+ \sum_{j\ge 1} \mathbb{E}\left[J(t, X_t^{t_0,x_0,u}, u_j, \tau_j)\mathbf{1}_{D_j}(X_{t_1}^{t_0,x_0,u})\right] \le \mathbb{E}\left[\int_{t_0}^{t_1} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t\right]$$
$$+ \sum_{j\ge 1} \mathbb{E}\left[\left(V(t_1, X_{t_1}^{t_0,x_0,u}) + C_2\varepsilon\left(1 + |X_{t_1}^{t_0,x_0,u}|^p\right)\right)\mathbf{1}_{D_j}(X_{t_1}^{t_0,x_0,u})\right]$$
$$= \mathbb{E}\left[\int_{t_0}^{t_1} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + V(t_1, X_{t_1}^{t_0,x_0,u})\right] + C_2\varepsilon\left(1 + \mathbb{E}|X_{t_1}^{t_0,x_0,u}|^p\right)$$

$$\leq \mathbb{E}\left[\int_{t_0}^{t_1} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + V(t_1, X_{t_1}^{t_0,x_0,u})\right] + C_2\varepsilon\left(1 + C\left(1 + |x_0|^p\right)\right),$$

where C is the constant which appears in Proposition 2.1. Since ε was arbitrarily taken, we can conclude that

$$V(t_0, x_0) \leq \mathbb{E}\left[\int_{t_0}^{t_1} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + V(t_1, X_{t_1}^{t_0,x_0,u})\right].$$

Let us prove that the inequality holds also for general stopping times. Applying a variant of Lemma 3.2 (instead of $h(x)$, one is allowed to consider $V(t,x)$), we get

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^{t_1} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + V(t_1, X_{t_1}^{t_0,x_0,u})\middle| F_s^{t_0}\right] = \int_{t_0}^{s} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t \\
&+ \left(\mathbb{E}\left[\int_s^{t_1} g(t, X_t^{s,x,u}, u_t)\mathrm{d}t + V\left(t_1, X_{t_1}^{s,x,u}\right)\right]\right)_{x=X_s^{t_0,x_0,u}} \\
&\geq \int_{t_0}^{s} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + V\left(s, X_s^{t_0,x_0,u}\right).
\end{aligned}$$

Hence $s \mapsto \int_{t_0}^{s} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + V(s, X_s^{t_0,x_0,u})$, $s \in [t_0, T]$, is a submartingale. By the version for submartingales of Doob's Optional Sampling Theorem we obtain

$$V(t_0, x_0) \leq \mathbb{E}\left[\int_{t_0}^{\tau} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + V\left(\tau, X_\tau^{t_0,x_0,u}\right)\right],$$

for every stopping time $\tau \in \mathcal{T}_{t_0,T}$.

2. We only have to prove that

$$V(t_0, x_0) \geq \inf_{u \in \mathcal{U}_{t_0,T}} \mathbb{E}\left[\int_{t_0}^{\tau_u} g(t, X_t^{t_0,x_0,u}, u_t)\mathrm{d}t + V\left(\tau_u, X_{\tau_u}^{t_0,x_0,u}\right)\right], \tag{3.3}$$

because the reverse inequality follows immediately from the previous result.

For every $j \geq 1$, there exist $u_j \in \mathcal{U}_{t_0,T}$ and $\tau_j \in \mathcal{T}_{t_0,T}$ such that $V(t_0, x_0) + \frac{\varepsilon}{j} \geq J(t_0, x_0, u_j, \tau_j)$. Let X^j denote X^{t_0,x_0,u_j}. By Lemma 3.2

we obtain:

$$
\begin{aligned}
&J(t_0,x_0,u_j,\tau_j)=\mathbb{E}\left[\int_{t_0}^{\tau_j} g(t,X_t^j,(u_j)_t)\mathrm{d}t+h(X_{\tau_j}^j)\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[\int_{t_0}^{\tau_j\vee\tau_{u_j}} g(t,X_t^j,(u_j)_t)\mathrm{d}t+h(X_{\tau_j}^j)\middle|\mathcal{F}_{\tau_{u_j}}^{t_0}\right]\mathbf{1}_{\{\tau_{u_j}\le\tau_j\}}\right]\\
&+\mathbb{E}\left(\left[\int_{t_0}^{\tau_j} g(t,X_t^j,(u_j)_t)\mathrm{d}t+h(X_{\tau_j}^j)\right]\mathbf{1}_{\left\{\tau_{u_j}>\tau_j\right\}}\right)\\
&=\mathbb{E}\left(\left[\int_{t_0}^{\tau_{u_j}} g(t,X_t^j,(u_j)_t)\mathrm{d}t+J(\tau_{u_j},X_{\tau_{u_j}}^j,u_j,\tau_j\vee\tau_{u_j})\right]\mathbf{1}_{\{\tau_{u_j}\le\tau_j\}}\right)\\
&+\mathbb{E}\left(\left[\int_{t_0}^{\tau_j} g(t,X_t^j,(u_j)_t)\mathrm{d}t+h(X_{\tau_j}^j)\right]\mathbf{1}_{\left\{\tau_{u_j}>\tau_j\right\}}\right)\\
&\ge\mathbb{E}\left(\left[\int_{t_0}^{\tau_{u_j}} g(t,X_t^j,(u_j)_t)\mathrm{d}t+V(\tau_{u_j},X_{\tau_{u_j}}^j)\right]\mathbf{1}_{\left\{\tau_{u_j}\le\tau_j\right\}}\right)\\
&+\mathbb{E}\left(\left[\int_{t_0}^{\tau_j} g(t,X_t^j,(u_j)_t)\mathrm{d}t+h(X_{\tau_j}^j)\right]\mathbf{1}_{\left\{\tau_{u_j}>\tau_j\right\}}\right).
\end{aligned}
\tag{3.4}
$$

By the previous result and assumption $\tau_{u_j}\le\tau_{t_0,x_0,u}^{\varepsilon}$,

$$
\begin{aligned}
&J\left(t_0,x_0,u_j,\tau_j\right)\ge\mathbb{E}\left[\int_{t_0}^{\tau_{u_j}} g(t,X_t^j,(u_j)_t)\mathrm{d}t+V(\tau_{u_j}\wedge\tau_j,X_{\tau_{u_j}\wedge\tau_j}^j)\right]\\
&+\mathbb{E}\left(\left[h(X_{\tau_j}^j)-V(\tau_{\tau_j},X_{\tau_{u_j}}^j)\right]\mathbf{1}_{\left\{\tau_{u_j}>\tau_j\right\}}\right)\ge V\left(t_0,x_0\right)+\varepsilon P\left(\tau_{u_j}>\tau_j\right).
\end{aligned}
$$

Hence $P\left(\tau_{u_j}>\tau_j\right)<\frac{1}{j},\ \forall j\ge 1$. On the other hand, (3.4) gives

$$
\begin{aligned}
&J\left(t_0,x_0,u_j,\tau_j\right)\ge\mathbb{E}\left[\int_{t_0}^{\tau_{u_j}} g(t,X_t^j,(u_j)_t)\mathrm{d}t+V(\tau_{u_j},X_{\tau_{u_j}}^j)\right]\\
&-\mathbb{E}\left(\left[\int_{\tau_j}^{\tau_{u_j}} g(t,X_t^j,(u_j)_t)\mathrm{d}t+V(\tau_{u_j},X_{\tau_{u_j}}^j)-h(X_{\tau_j}^j)\right]\mathbf{1}_{\left\{\tau_{u_j}>\tau_j\right\}}\right)\\
&\ge\mathbb{E}\left[\int_{t_0}^{\tau_{u_j}} g(t,X_t^j,(u_j)_t)\mathrm{d}t+V(\tau_{u_j},X_{\tau_{u_j}}^j)\right]\\
&-\mathbb{E}\left(\left[\int_{t_0}^{T}\left|g(t,X_t^j,(u_j)_t)\right|\mathrm{d}t+|h(X_{\tau_{u_j}}^j)|+|h(X_{\tau_j}^j)|\right]\mathbf{1}_{\left\{\tau_{u_j}>\tau_j\right\}}\right).
\end{aligned}
$$

By the assumptions on the growths of g and h, we obtain that

$$\begin{aligned}
&\mathbb{E}\left(\left[\int_{t_0}^{T}\left|g(t,X_t^j,(u_j)_t)\right|\mathrm{d}t+|h(X^j_{\tau_{u_j}})|+|h(X^j_{\tau_j})|\right]\mathbf{1}_{\{\tau_{u_j}>\tau_j\}}\right)\\
&\quad\leq K\left(T+2\right)\mathbb{E}\left(\sup_{t\in[t_0,T]}|X_t^j|^{p'}\mathbf{1}_{\{\tau_{u_j}>\tau_j\}}\right)\\
&\quad\leq K\left(T+2\right)\left[C\left(1+|x_0|^p\right)\right]^{\frac{p'}{p}}P\left(\tau_{u_j}>\tau_j\right)^{1-\frac{p'}{p}}\\
&\quad\leq K\left(T+2\right)\left[C\left(1+|x_0|^p\right)\right]^{\frac{p'}{p}}j^{\frac{p'}{p}-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
V\left(t_0,x_0\right)+\frac{\varepsilon}{j}\geq\mathbb{E}\left[\int_{t_0}^{\tau_{u_j}}g(t,X_t^j,(u_j)_t)\mathrm{d}t+V(\tau_{u_j},X^j_{\tau_{u_j}})\right]\\
-K\left(T+2\right)\left[C\left(1+|x_0|^p\right)\right]^{\frac{p'}{p}}j^{\frac{p'}{p}-1}.
\end{aligned}$$

Letting $j\to\infty$, (3.3) follows. □

This result leads to the more usual form of the Dynamic Programming Principle:

Proposition 3.3. *For every $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$, and every $t_1\in[t_0,T]$,*

$$\begin{aligned}
V\left(t_0,x_0\right)=\inf_{\substack{u\in\mathcal{U}_{t_0,T}\\ \tau\in\mathcal{T}_{t_0,T}}}\mathbb{E}\left[\int_{t_0}^{\tau\wedge t_1}\mathrm{e}^{-\int_{t_0}^{t}c(s,X_s)\mathrm{d}s}g\left(t,X_t,u_t\right)\mathrm{d}t\right.\\
\left.+\mathrm{e}^{-\int_{t_0}^{\tau}c(s,X_s)\mathrm{d}s}h\left(X_\tau\right)\mathbf{1}_{\{\tau<t_1\}}+\mathrm{e}^{-\int_{t_0}^{t_1}c(s,X_s)\mathrm{d}s}V\left(t_1,X_{t_1}\right)\mathbf{1}_{\{\tau\geq t_1\}}\right],
\end{aligned}$$

where X stands for $X^{t_0,x_0,u}$.

4. HJB equations related to stable processes

In this section, we will take a look at how an optimal control/stopping problem for solutions of SDEs driven by symmetric stable processes fits in our jump-diffusion framework. For the convenience of the reader, we give here the definition of symmetric stable processes of order $\alpha\in(0,2]$.

Definition 2. Let $\alpha\in(0,2]$. An $\mathbb{F}$-adapted process $(M_t)_{t\geq0}$ is called a symmetric α-stable process if it has homogeneous increments, M_t-M_s is

independent of $\mathcal{F}_s$ for all $0 \leq s < t$, and the characteristic function of M_t has the form

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\lambda M_t} = \mathrm{e}^{-t|\lambda|^\alpha},\ \lambda \in \mathbb{R},$$

for every $t \geq 0$.

For $\alpha = 2$, M is a Brownian motion, and we exclude this well-studied case. If $\alpha \in (0,2)$, M is a *purely discontinuous* semimartingale, with infinite variance. In fact, the Brownian motion is the only symmetric stable process with continuous paths.

We consider the following SDE:

$$\mathrm{d}X_t = b\left(t, X_t, u_t\right)\mathrm{d}t + \sigma\left(t, X_{t-}, u_t\right)\mathrm{d}M_t,\ t \in [t_0, T], \tag{4.1}$$

where the continuous coefficients b, $\sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ are uniformly Lipschitz in $x \in \mathbb{R}^n$. Under these assumptions, equation (4.1) has a unique solution X with initial condition $X_{t_0} = x_0$.

Let $N_t(\mathrm{d}z)$ denote the *counting measure* of the jumps of M, *i.e.* $N_t(B) := \sum_{0<s\leq t} \mathbf{1}_{\{M_t - M_{t-} \in B\}}$, $t \geq 0$, $B \in \mathcal{B}(\mathbb{R}\backslash\{0\})$. Then, for $B \in \mathcal{B}(\mathbb{R}\backslash\{0\})$, $\mathbb{E}N_t(B) = t\nu_\alpha(B)$, where $\nu_\alpha(\mathrm{d}z) := c_\alpha |z|^{-\alpha-1}\mathrm{d}z$, for a constant $c_\alpha > 0$. The measure ν_α is known as the *Lévy measure* of M. The process N, when viewed as a random measure on $\mathbb{R}_+ \times \mathbb{R}$, is a homogeneous Poisson random measure. It is well known that M_t can be written as

$$\begin{aligned} M_t &= \int_0^t \int_{\{|z|<1\}} z\tilde{N}\left(\mathrm{d}s, \mathrm{d}z\right) + \int_0^t \int_{\{|z|\geq 1\}} zN\left(\mathrm{d}s, \mathrm{d}z\right) \\ &= \int_0^t \int_{\mathbb{R}^d} z\bar{N}\left(\mathrm{d}s, \mathrm{d}z\right),\ t \geq 0. \end{aligned}$$

Hence equation (4.1) becomes

$$\mathrm{d}X_t = b\left(t, X_t, u_t\right)\mathrm{d}t + \int_{\mathbb{R}^d} z\sigma\left(t, X_{t-}, u_t\right) d\bar{N}\left(\mathrm{d}t, \mathrm{d}z\right),\ t \in [t_0, T].$$

Of course, for the choice $\gamma\left(t,x,u,z\right) = z\sigma\left(t,x,u\right)$, the conditions (A2) and (A3) are satisfied with $\rho\left(z\right) := |z|$ and $p < \alpha$. In fact, this explains why the case $p < 2$ *must* be considered if we want to study this problem via jump-diffusions.

Now, the corresponding HJB equation (1.6) has the particular Hamiltonian

$$\begin{aligned}\mathcal{H}_\alpha v(t,x,u) := g(t,x,u) &+ \mathrm{D}v(t,x)\cdot b(t,x,u)\\ &+ c_\alpha \int_{-\infty}^{+\infty} \left[v(t,x+z\sigma(t,x,u)) - v(t,x)\right.\\ &\left. -\mathrm{D}v(t,x)\cdot z\sigma(t,x,u)\,\mathbf{1}_{\{|z|<1\}}\right] |z|^{-\alpha-1}\,\mathrm{d}z.\end{aligned}$$

If $\alpha<1$, then $\int_{-1}^{1} z\nu_\alpha(\mathrm{d}z)=0$; if $\alpha>1$, then $\int_{\mathbb{R}\setminus[-1,1]} z\nu_\alpha(\mathrm{d}z)=0$. Hence, for $\alpha\in(0,1)\cup(1,2)$,

$$\begin{aligned}\mathcal{H}_\alpha v(t,x,u) &:= g(t,x,u) + \mathrm{D}v(t,x)\,b(t,x,u)\\ &\quad + \tilde{c}_\alpha\left(\mathrm{D}^\alpha_{\sigma(t,x,u)} + \mathrm{D}^\alpha_{-\sigma(t,x,u)}\right) v(t,x),\end{aligned} \tag{4.2}$$

where, for $\phi\in C^2_p(\mathbb{R}^n)$ and $\theta\in\mathbb{R}^n$, $\mathrm{D}^\alpha_\theta\phi$ is denoting the fractional derivative of order α in the direction θ (see [6]), defined by

$$\mathrm{D}^\alpha_\theta\phi(x) := \begin{cases} -\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(\phi(x-r\theta)-\phi(x)\right) r^{-\alpha-1}\mathrm{d}r, \\ \qquad\qquad\qquad\qquad\qquad \text{if } 0<\alpha<1; \\ \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_0^\infty \left(\phi(x-r\theta)-\phi(x)+\mathrm{D}\phi(x)\cdot r\theta\right) r^{-\alpha-1}\mathrm{d}r, \\ \qquad\qquad\qquad\qquad\qquad \text{if } 1<\alpha<2. \end{cases}$$

This formulation has the great advantage that one can unify the results obtained in the case $\alpha\in(0,1)\cup(1,2)$, on one hand, and the case $\alpha=2$, on the other. In fact, if $\alpha\in(0,1)\cup(1,2)$, $\mathrm{D}^\alpha_\theta\phi(x)$ is the positive fractional derivative of order α (*i.e.* D^α_1) of the function $s\longmapsto\phi(x+s\theta)$ evaluated at $s=0$. With this definition, $\mathrm{D}^2_\theta\phi(x)=\mathrm{D}^2\phi(x)\,\theta\cdot\theta$, and the operator given by (4.2) with $\alpha=2$ coincides with the Hamiltonian obtained in the continuous-diffusion case.

5. Existence

Theorem 5.1. *Under conditions* $(A0)$–$(A3)$, $(B0)$–$(B2)$, *the value function* V *is a viscosity solution of equation* (1.6).

We will prove this theorem with the assumption that $c\equiv 0$, in order to avoid complicated formulae; however, the proof in the general case follows the same direction.

Proof. We have already seen that $V \in C_p([0,T]\times\mathbb{R}^n)$ and $V(T,x) = h(x)$, $\forall x \in \mathbb{R}^n$. It remains to show property 3. of the definition of viscosity solutions.

Let us take arbitrary $(t_0,x_0) \in (0,T)\times\mathbb{R}^n$ and $\phi \in C_p^{1,2}((0,T)\times\mathbb{R}^n)$. To simplify notation, for each control $u \in \mathcal{U}_{t_0,T}$, we write X^u instead $X^{t_0,x_0,u}$. We fix $T' \in (t_0,T)$, for instance $T' := \frac{1}{2}(t_0+T)$, and consider the stopping time

$$\tau_u := \inf\left\{t \in \left[t_0,T'\right];\ |X_t^u - x_0| \geq 1\right\} \wedge T'. \tag{5.1}$$

Obviously, $\tau_u > t_0$ a.s.

Let us apply Itô's formula to $\phi(t,X_t^u)$:

$$\begin{aligned}\phi(t,X_t^u) &= \phi(t_0,x_0) + \int_{t_0}^t [\tfrac{\partial\phi}{\partial t}(s,X_s^u) + \mathrm{D}\phi(s,X_s^u)\cdot b(s,X_s^u,u_s)]\mathrm{d}s \\ &+ \int_{t_0}^t \mathcal{I}_1\phi(s,X_s^u,u_s)\mathrm{d}s \\ &+ \int_{t_0}^t\int_{\mathbb{R}^d}[\phi(s,X_{s-}^u+\gamma(s,X_{s-}^u,u_s,z)) - \phi(s,X_{s-}^u)]\bar{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{5.2}$$

Now, if $|z|<1$, $|x-x_0|\leq 1$, and $t\in[t_0,T']$, then $|\phi(t,x+\gamma(t,x,u,z)) - \phi(t,x)| \leq C\rho(z)$, where the constant C depends only on ϕ, t_0, T', x_0, and $\sup_{|z|<1}\rho(z)$. Hence

$$\mathbb{E}\int_{t_0}^{\tau_u}\int_{\{|z|<1\}} |\phi(s,X_s^u+\gamma(s,X_s^u,u_s,z)) - \phi(s,X_s^u)|^2\,\nu(\mathrm{d}z)\,\mathrm{d}s < +\infty;$$

therefore, by (1.2), for every $\tau \in \mathcal{T}_{t_0,T'}$,

$$\mathbb{E}\int_{t_0}^{\tau\wedge\tau_u}\int_{\{|z|<1\}} \left[\phi\left(s,X_{s-}^u+\gamma\left(s,X_{s-}^u,u_s,z\right)\right) - \phi\left(s,X_{s-}^u\right)\right]\tilde{N}(\mathrm{d}s,\mathrm{d}z) = 0.$$

On the other hand, from (3.1) and Proposition 2.1,

$$\mathbb{E}\int_{t_0}^{\tau_u}\int_{\{|z|\geq 1\}} |\phi(s,X_s^u+\gamma(s,X_s^u,u_s,z)) - \phi(s,X_s^u)|\,\nu(\mathrm{d}z)\,\mathrm{d}s < +\infty,$$

which implies, by (1.1), that for every $\tau \in \mathcal{T}_{t_0,T'}$,

$$\begin{aligned}\mathbb{E}\int_{t_0}^{\tau\wedge\tau_u}\int_{\{|z|\geq 1\}} \left[\phi\left(s,X_{s-}^u+\gamma\left(s,X_{s-}^u,u_s,z\right)\right) - \phi\left(s,X_{s-}^u\right)\right]N(\mathrm{d}s,\mathrm{d}z) \\ = \mathbb{E}\int_{t_0}^{\tau\wedge\tau_u}\mathcal{I}_2\phi(s,X_s^u,u_s)\,\mathrm{d}s.\end{aligned}$$

From (5.2) it follows that

$$\begin{aligned}(5.3)\quad &\mathbb{E}\phi\left(\tau\wedge\tau_u, X^u_{\tau\wedge\tau_u}\right)+\mathbb{E}\int_{t_0}^{\tau\wedge\tau_u} g\left(s, X^u_s, u_s\right)\mathrm{d}s\\ &\qquad=\phi\left(t_0,x_0\right)+\mathbb{E}\int_{t_0}^{\tau\wedge\tau_u}\left[\frac{\partial\phi}{\partial t}\left(s, X^u_s\right)+\mathcal{H}\phi\left(s, X^u_s, u_s\right)\right]\mathrm{d}s,\end{aligned}$$

for every $\tau\in\mathcal{T}_{t_0,T'}$.

(i) We prove now that V is a viscosity subsolution of equation (1.6). Let $\phi\in C_p^{1,2}\left((0,T)\times\mathbb{R}^n\right)$ such that $V-\phi$ has a global maximum at $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$. Without loss of generality we can suppose that $\phi\left(t_0,x_0\right)=V\left(t_0,x_0\right)$; we then have $\phi\left(t,x\right)\geq V\left(t,x\right),\ \forall\left(t,x\right)\in(0,T)\times\mathbb{R}^n$.

Let us take the control u to be constant, $u\equiv\bar{u}\in U$, where $\bar{u}$ is arbitrary. Then, by Theorem 3.1 we get

$$V\left(t_0,x_0\right)\leq\mathbb{E}\left[\int_{t_0}^{t\wedge\tau_{\bar{u}}} g\left(s, X^{\bar{u}}_s, \bar{u}\right)\mathrm{d}s+V\left(t\wedge\tau_{\bar{u}}, X^{\bar{u}}_{t\wedge\tau_{\bar{u}}}\right)\right].$$

On the other hand, equality (5.3) gives, taking $\tau\equiv t\in(t_0,T']$,

$$\begin{aligned}&\mathbb{E}V\left(t\wedge\tau_{\bar{u}}, X^{\bar{u}}_{t\wedge\tau_{\bar{u}}}\right)+\mathbb{E}\int_{t_0}^{t\wedge\tau_{\bar{u}}} g\left(s, X^{\bar{u}}_s, \bar{u}\right)\mathrm{d}s\\ &\qquad\leq V\left(t_0,x_0\right)+\mathbb{E}\int_{t_0}^{t\wedge\tau_{\bar{u}}}\left[\frac{\partial\phi}{\partial t}\left(s, X^{\bar{u}}_s\right)+\mathcal{H}\phi\left(s, X^{\bar{u}}_s, \bar{u}\right)\right]\mathrm{d}s.\end{aligned}$$

Combining the two inequalities we get

$$0\leq\mathbb{E}\int_{t_0}^{t\wedge\tau_{\bar{u}}}\left[\frac{\partial\phi}{\partial t}\left(s, X^{\bar{u}}_s\right)+\mathcal{H}\phi\left(s, X^{\bar{u}}_s, \bar{u}\right)\right]\mathrm{d}s.$$

Since $\mathcal{H}\phi$ is continuous, we have that

$$s\to\frac{\partial\phi}{\partial t}\left(s, X^{\bar{u}}_s\right)+\mathcal{H}\phi\left(s, X^{\bar{u}}_s, \bar{u}\right)\ \text{ is right continuous, a.s.}$$

Consequently, since $\tau_{\bar{u}}>t_0$ a.s.,

$$\lim_{t\searrow t_0}\frac{1}{t-t_0}\int_{t_0}^{t\wedge\tau_{\bar{u}}}\left[\frac{\partial\phi}{\partial t}(s, X^{\bar{u}}_s)+\mathcal{H}\phi(s, X^{\bar{u}}_s, \bar{u})\right]\mathrm{d}s=\frac{\partial\phi}{\partial t}(t_0,x_0)+\mathcal{H}\phi(t_0,x_0,\bar{u}),\ \text{a.s.}$$

By Lebesgue's dominated convergence theorem ($\sup_{s\in[t_0,\tau_{\bar u})}|X_s^{\bar u}|\le 1+|x_0|$), we obtain

$$\begin{aligned}0 &\le \lim_{t\searrow t_0}\mathbb{E}\frac{1}{t-t_0}\int_{t_0}^{t\wedge\tau_{\bar u}}\left[\frac{\partial\phi}{\partial t}\left(s,X_s^{\bar u}\right)+\mathcal{H}\phi\left(s,X_s^{\bar u},\bar u\right)\right]\mathrm{d}s\\ &= \frac{\partial\phi}{\partial t}\left(t_0,x_0\right)+\mathcal{H}\phi\left(t_0,x_0,\bar u\right).\end{aligned}$$

On the other hand, we have that $\phi\left(t_0,x_0\right)=V\left(t_0,x_0\right)\le h\left(x_0\right)$; the variational inequality in the definition of viscosity subsolutions is therefore satisfied.

(ii) We show now that V is a viscosity supersolution of equation (1.6). Let $\phi\in C_p^{1,2}\left((0,T)\times\mathbb{R}^n\right)$ such that $V-\phi$ has a global minimum at $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$. Once again we can suppose that $\phi\left(t_0,x_0\right)=V\left(t_0,x_0\right)$; we then have $\phi\left(t,x\right)\le V\left(t,x\right),\forall\left(t,x\right)\in(0,T)\times\mathbb{R}^n$. Also, since $V\left(t_0,x_0\right)\le h\left(x_0\right)$, we can assume that $V\left(t_0,x_0\right)<h\left(x_0\right)$; otherwise the variational inequality in the definition of viscosity supersolutions would be automatically satisfied. We define $\varepsilon:=\frac{1}{2}\left(h\left(x_0\right)-V\left(t_0,x_0\right)\right)$; let $\tau_u^\varepsilon:=\inf\{t\in[t_0,T];\ V(t,X_t^u)\ge h(X_t^u)-\varepsilon\}\wedge T$ and $\tilde\tau_u^\varepsilon:=\tau_u^\varepsilon\wedge\tau_u$, with τ_u defined by (5.1). We apply Theorem 3.1 with $t\wedge\tilde\tau_u^\varepsilon$, where $t\in(t_0,T']$:

$$V\left(t_0,x_0\right)=\inf_{u\in\mathcal{U}_{t_0,T}}\mathbb{E}\left[\int_{t_0}^{t\wedge\tilde\tau_u^\varepsilon}g\left(s,X_s^u,u_s\right)\mathrm{d}s+V\left(t\wedge\tilde\tau_u^\varepsilon,X_{t\wedge\tilde\tau_u^\varepsilon}^u\right)\right].$$

On the other hand, by (5.3),

$$\begin{aligned}\mathbb{E}V\left(t\wedge\tilde\tau_u^\varepsilon,X_{t\wedge\tilde\tau_u^\varepsilon}^u\right)+\mathbb{E}\int_{t_0}^{t\wedge\tilde\tau_u^\varepsilon}g\left(s,X_s^u,u_s\right)\mathrm{d}s &\ge V\left(t_0,x_0\right)\\ &+\mathbb{E}\int_{t_0}^{t\wedge\tilde\tau_u^\varepsilon}\left[\tfrac{\partial\phi}{\partial t}\left(s,X_s^u\right)+\mathcal{H}\phi\left(s,X_s^u,u_s\right)\right]\mathrm{d}s,\end{aligned}$$

for every $u\in\mathcal{U}_{t_0,T}$. It follows that

$$0\ge\inf_{u\in\mathcal{U}_{t_0,T}}\mathbb{E}\int_{t_0}^{t\wedge\tilde\tau_u^\varepsilon}\left[\tfrac{\partial\phi}{\partial t}\left(s,X_s^u\right)+\mathcal{H}\phi\left(s,X_s^u,u_s\right)\right]\mathrm{d}s. \tag{5.4}$$

Let ω_δ, $0<\delta<1\wedge(T'-t_0)$, be the modulus of continuity of $\frac{\partial\phi}{\partial t}+\mathcal{H}\phi$ in (t_0,x_0), *i.e.*

$$\omega_\delta:=\sup_{\substack{t\in[t_0,t_0+\delta],\\ |x-x_0|\le\delta,\\ u\in U}}\left|\tfrac{\partial\phi}{\partial t}\left(t,x\right)-\tfrac{\partial\phi}{\partial t}\left(t_0,x_0\right)+\mathcal{H}\phi\left(t,x,u\right)-\mathcal{H}\phi\left(t_0,x_0,u\right)\right|.$$

We fix for the moment $t \in [t_0, t_0 + \delta]$. Then, for every $u \in \mathcal{U}[t_0, T]$,

$$\mathbb{E}\int_{t_0}^{t\wedge\tilde{\tau}_u^\varepsilon} \left|\left(\frac{\partial\phi}{\partial t} + \mathcal{H}\phi\right)(s, X_s^u, u_s) - \left(\frac{\partial\phi}{\partial t} + \mathcal{H}\phi\right)(t_0, x_0, u_s)\right| \mathrm{d}s$$
$$\leq \omega_\delta(t - t_0) + 2(t - t_0) \sup_{\substack{(s,u)\in[t_0,T']\times U,\\ |x-x_0|\leq 1}} \left(\frac{\partial\phi}{\partial t} + \mathcal{H}\phi\right)(s, x, u) \cdot$$
$$\cdot P\left(\sup_{s\in[t_0,t]} |X_s^u - x_0| > \delta\right).$$

By (5.4) and Proposition 2.1,

$$(t - t_0)\left[\omega_\delta + C'\frac{(t - t_0)^{p_2}}{\delta^p}\right] \geq \mathbb{E}(t \wedge \tilde{\tau}_u^\varepsilon - t_0) \cdot$$
$$\cdot \left(\frac{\partial\phi}{\partial t}(t_0, x_0) + \inf_{\bar{u}\in U} \mathcal{H}\phi(t_0, x_0, \bar{u})\right), \tag{5.5}$$

where the constant $C' > 0$ does not depend on δ, t, and u.

Let us estimate $\mathbb{E}(t \wedge \tilde{\tau}_u^\varepsilon - t_0)$. We have

$$P(\tilde{\tau}_u^\varepsilon < t) \leq P(\tau_u^\varepsilon < t)$$
$$+ P(\tau_u < t) \leq P\left(\sup_{s\in[t_0,t\wedge\tau_u]} |h(X_s^u) - h(x_0) - V(s, X_s^u) + V(t_0, x_0)| \geq \varepsilon\right)$$
$$+ P\left(\sup_{s\in[t_0,t]} |X_s^u - x_0| \geq 1\right).$$

Let $\tilde{\omega}_\delta$ be the modulus of continuity of $h - V$ in (t_0, x_0), *i.e.*,

$$\tilde{\omega}_\delta := \sup_{\substack{s\in[t_0,t_0+\delta],\\ |x-x_0|\leq\delta}} |h(x) - h(x_0) - V(s, x) + V(t_0, x_0)|.$$

For δ sufficiently small we have $\tilde{\omega}_\delta < \varepsilon$; therefore,

$$\left\{\sup_{s\in[t_0,t\wedge\tau_u]} |h(X_s^u) - h(x_0) - V(s, X_s^u) + V(t_0, x_0)| \geq \varepsilon\right\}$$
$$\subseteq \left\{\sup_{s\in[t_0,t]} |X_s^u - x_0| > \delta\right\}.$$

Using again Proposition 2.1 we obtain $P(\tilde{\tau}_u^\varepsilon < t) \leq C'' \frac{(t-t_0)^{p_2}}{\delta^p}$, where the constant $C'' > 0$ does not depend on the choice of δ, t, and u. This implies

$$\mathbb{E}\left(t \wedge \tilde{\tau}_u^\varepsilon - t_0\right) \geq (t-t_0)\, P\left(\tilde{\tau}_u^\varepsilon \geq t\right) \geq (t-t_0)\left(1 - C'' \tfrac{(t-t_0)^{p_2}}{\delta^p}\right)^+.$$

From (5.5) it follows that

$$\tfrac{\partial \phi}{\partial t}(t_0, x_0) + \inf_{\bar{u} \in U} \mathcal{H}\phi(t_0, x_0, \bar{u}) \leq \frac{\delta^p \omega_\delta + C'(t-t_0)^{p_2}}{(\delta^p - C''(t-t_0)^{p_2})^+}.$$

Hence, letting $t \to t_0$, we get $\frac{\partial \phi}{\partial t}(t_0, x_0) + \inf_{\bar{u} \in U} \mathcal{H}\phi(t_0, x_0, \bar{u}) \leq \omega_\delta$. The conclusion is derived letting $\delta \to 0$. □

6. Uniqueness

The goal of this section is to prove a comparison principle between solutions of equation (1.6).

As we have seen, the given definition of the viscosity solution is convenient for establishing that the value function V is a solution of equation (1.6). However, for proving the comparison principle, the p-growth of the test functions ϕ may be inappropriate. We will give an alternative definition which does not involve a growth condition on the test functions.

For $\eta \in (0,1)$, $\phi \in C^{1,2}((0,T) \times \mathbb{R}^n)$, and $\psi : (0,T) \times \mathbb{R}^n \to \mathbb{R}$ a measurable function with p-growth, we introduce the operators

$$\begin{aligned}
\mathcal{J}^\eta \phi(t,x,u) := \int_{\{|z|<\eta\}} & \left[\phi(t, x+\gamma(t,x,u,z)) - \phi(t,x)\right. \\
& \left. - \mathrm{D}\phi(t,x) \cdot \gamma(t,x,u,z)\right] \nu(\mathrm{d}z); \\
\mathcal{J}_\eta \psi(t,x,u,a) := \int_{\{|z|\geq\eta\}} & \left[\psi(t, x+\gamma(t,x,u,z)) - \psi(t,x)\right. \\
& \left. - a \cdot \gamma(t,x,u,z)\, \mathbf{1}_{\{|z|<1\}}\right] \nu(\mathrm{d}z),
\end{aligned}$$

where $(t,x,u,a) \in (0,T) \times \mathbb{R}^n \times U \times \mathbb{R}^n$. As we noticed in Section 2, these integrals are well defined, thanks to the C^2-differentiability of ϕ and the p-growth of ψ. Moreover, $\mathcal{J}^\eta \phi$ is continuous, and if $\psi \in LSC_p((0,T) \times \mathbb{R}^n)$ ($\psi \in LSC_p((0,T) \times \mathbb{R}^n)$), then $\mathcal{J}_\eta \psi(t,x,u,a)$ is lower (upper) semi-continuous.

We define the operator

$$\mathcal{H}_\eta(\phi,\psi)(t,x,u) := g(t,x,u) + \mathrm{D}\phi(t,x)\cdot b(t,x,u) + \mathcal{J}^\eta\phi(t,x,u) + \mathcal{J}_\eta\psi(t,x,u,\mathrm{D}\phi(t,x)),$$

for $(t,x,u) \in (0,T)\times\mathbb{R}^n\times U$. We observe that $\mathcal{H}_\eta(\phi,\phi) = \mathcal{H}\phi$ if $\phi \in C_p^{1,2}((0,T)\times\mathbb{R}^n)$; this property explains, at least heuristically, why we can use this operator as a Hamiltonian in the definition of viscosity solutions. Indeed, we can prove the following:

Proposition 6.1. *A function $v \in USC_p((0,T]\times\mathbb{R}^n)$ ($v \in LSC_p((0,T]\times\mathbb{R}^n)$) is a viscosity subsolution (supersolution) for equation (1.6) if and only if $v(T,\cdot) \le h$ ($v(T,\cdot)\ge h$) on $\mathbb{R}^n$ and, whenever $\phi \in C^{1,2}((0,T)\times\mathbb{R}^n)$ such that $v-\phi$ has a global maximum (minimum) at $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$, we have:*

$$\min\left\{-cv + \tfrac{\partial\phi}{\partial t} + \inf_{u\in U}\mathcal{H}_\eta(\phi,v)(\cdot,\cdot,u); h-v\right\}(t_0,x_0) \ge 0 \qquad (\le 0),$$

for every $\eta\in(0,1)$.

Now we are able to state and prove the comparison principle; for this purpose we will need supplementary Lipschitz conditions on the coefficients:

(A1') $|b(t,x,u)-b(s,x,u)| \le L|t-s|$;

(A2') $|\gamma(t,x,u,z)-\gamma(s,x,u,z)| \le \rho(z)|t-s|$;

(B3) $|c(t,x)-c(s,y)| + |g(t,x,u)-g(s,y,u)| + |h(x)-h(y)| \le L(|t-s|+|x-y|)$,

for all $(s,x),(t,y)\in[0,T]\times\mathbb{R}^n$, and $z\in\mathbb{R}^d$.

Remark. Clearly, if $p>1$, then (B3) implies condition (B1).

Proposition 6.2. *Suppose that assumptions (A0)–(A3), (A1′), (A2′), (B0), (B2), (B3) hold. Let v and w be a subsolution (respectively, a supersolution) of equation (1.6). If v and w have p'-growth for some $p'<p$, then $v\le w$ on $(0,T]\times\mathbb{R}^n$.*

Proof. We will use the function β and the constant C_β from Lemma 2.2. Let

$$K := \sup_{(t,x,u)\in[0,T]\times\mathbb{R}^n\times U} \tfrac{|b(t,x,u)|}{1+|x|}, \qquad \tilde{C}_\beta := C_\beta \sup_{|z|\le 1} \left(1+\rho\left(z\right)_p\right),$$

and $\tilde{K} := \tilde{C}_\beta\{K + 2\int_{\mathbb{R}^d}[\rho(z)^2\mathbf{1}_{\{|z|<1\}} + (1+\rho(z)^p)\mathbf{1}_{\{|z|\ge 1\}}]\nu(\mathrm{d}z)\}$. Without loss of generality, we can assume that the constant c_0 from (B2) is greater than $\tilde{K}$. Indeed, the transformation $v \to \mathrm{e}^{(\tilde{K}-c_0)t}v$ converts equation (1.6) into the following one:

$$\begin{cases} \min\left\{-\left(c\left(t,x\right)-c_0+\tilde{K}\right)v+\dfrac{\partial v}{\partial t}+\inf\limits_{u\in U}\tilde{\mathcal{H}}v\left(t,x,u\right);h\left(x\right)-v\right\}=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T)\times\mathbb{R}^n; \\ v\left(T,x\right)=\mathrm{e}^{(\tilde{K}-c_0)T}h\left(x\right),\ x\in\mathbb{R}^n, \end{cases}$$

where $\tilde{\mathcal{H}}\phi = \mathrm{e}^{(\tilde{K}-c_0)t}g + \mathrm{D}\phi\cdot b + \mathcal{I}_1\phi + \mathcal{I}_2\phi$.

Since, from the definition of viscosity subsolution (respectively, supersolution) we have that $v\left(T,x\right)\le h\left(x\right)\le w\left(T,x\right),\ \forall x\in\mathbb{R}^n$, we must only prove that $v\le w$ on $(0,T)\times\mathbb{R}^n$. We will show that inequality by assuming the contrary, *i.e.*, that there exists $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$ such that $\theta := v\left(t_0,x_0\right)-w\left(t_0,x_0\right)>0$.

For ε, $\delta>0$, we set

$$\phi_{\delta,\varepsilon}\left(t,x,s,y\right) := \tfrac{1}{2\delta}\left(|x-y|^2+|t-s|^2\right)+\tfrac{\varepsilon}{2}\left(\beta\left(x\right)+\beta\left(y\right)\right)+\tfrac{\theta t_0}{8}\left(\tfrac{1}{t}+\tfrac{1}{s}\right);$$
$$\psi_{\delta,\varepsilon}\left(t,x,s,y\right) := v\left(t,x\right)-w\left(s,y\right)-\phi_{\delta,\varepsilon}\left(t,x,s,y\right),$$

for $(t,x,s,y)\in((0,T]\times\mathbb{R}^n)^2$. Clearly, $\phi_{\delta,\varepsilon}\in C^{1,2}\left(((0,T]\times\mathbb{R}^n)^2\right)$.

Let $M_{\delta,\varepsilon} := \sup\limits_{\zeta\in((0,T]\times\mathbb{R}^n)^2}\psi_{\delta,\varepsilon}\left(\zeta\right)$. From the p'-growth condition on v and w, it is clear that $M_{\delta,\varepsilon}$ is finite. We have that

$$M_{\delta,\varepsilon}\ge v\left(t_0,x_0\right)-w\left(t_0,x_0\right)-\varepsilon\beta\left(x_0\right)-\tfrac{\theta}{4}\ge\tfrac{\theta}{2}, \tag{6.1}$$

the last inequality holding for $\varepsilon<\varepsilon_0 := \frac{\theta}{4\beta(x_0)}$.

Moreover, since $\psi_{\delta,\varepsilon}\left(\zeta\right)<0$ for ζ outside of a compact (depending only on ε) of $(0,T]\times\mathbb{R}^n$, there exists $\zeta_{\delta,\varepsilon}=(t_{\delta,\varepsilon},x_{\delta,\varepsilon},s_{\delta,\varepsilon},y_{\delta,\varepsilon})\in((0,T]\times\mathbb{R}^n)^2$ with $\psi_{\delta,\varepsilon}\left(\zeta_{\delta,\varepsilon}\right)=M_{\delta,\varepsilon}$ and the set $\{(x_{\delta,\varepsilon},y_{\delta,\varepsilon})\}_{\delta>0}$ is bounded for every $\varepsilon<\varepsilon_0$.

For every $\varepsilon < \varepsilon_0$, the net $(M_{\delta,\varepsilon})_{\delta>0}$ is increasing, and from (6.1) it follows that $\lim_{\delta\searrow 0} M_{\delta,\varepsilon}$ is finite. Furthermore,

$$\begin{aligned} M_{2\delta,\varepsilon} &\geq \psi_{2\delta,\varepsilon}\left(\zeta_{\delta,\varepsilon}\right) = \psi_{\delta,\varepsilon}\left(\zeta_{\delta,\varepsilon}\right) + \tfrac{1}{4\delta}\left(|x_{\varepsilon,\delta} - y_{\varepsilon,\delta}|^2 + |t_{\varepsilon,\delta} - s_{\varepsilon,\delta}|^2\right) \\ &= M_{\delta,\varepsilon} + \tfrac{1}{4\delta}\left(|x_{\varepsilon,\delta} - y_{\varepsilon,\delta}|^2 + |t_{\varepsilon,\delta} - s_{\varepsilon,\delta}|^2\right), \end{aligned}$$

hence

$$\lim_{\delta\searrow 0} \tfrac{1}{\delta}\left(|x_{\varepsilon,\delta} - y_{\varepsilon,\delta}|^2 + |t_{\varepsilon,\delta} - s_{\varepsilon,\delta}|^2\right) = 0 \tag{6.2}$$

for every $\varepsilon < \varepsilon_0$.

From relation (6.1), (6.2), and the semicontinuity of v and w, it follows that T cannot be an accumulation point for the sequence $(t_{\delta,\varepsilon} \vee s_{\varepsilon,\delta})_{\delta>0}$, if $\varepsilon < \varepsilon_0$. This implies, of course, that $\zeta_{\delta,\varepsilon} \in ((0,T)\times\mathbb{R}^n)^2$, for every δ smaller than a certain $\delta_\varepsilon > 0$.

From now on, in order to simplify notations, we will write $\hat{\phi}$, $\hat{\zeta}$, $\hat{x}$, $\hat{y}$, $\hat{t}$, $\hat{s}$, instead of $\phi_{\delta,\varepsilon}$, $\zeta_{\varepsilon,\delta}$, $x_{\varepsilon,\delta}$, $y_{\varepsilon,\delta}$, $t_{\varepsilon,\delta}$, $s_{\varepsilon,\delta}$, respectively, whenever no confusion can arise.

Let us fix for the moment $\varepsilon < \varepsilon_0$ and $\delta < \delta_\varepsilon$. Then the function $v - [w(\hat{s},\hat{y}) + \hat{\phi}(\cdot,\cdot,\hat{s},\hat{y})]$ has a maximum at $(\hat{t},\hat{x}) \in (0,T)\times\mathbb{R}^n$. Since v is a viscosity subsolution of equation (1.6), it follows from Proposition 6.1 that

$$v(\hat{t},\hat{x}) \leq h\left(\hat{x}\right), \tag{6.3}$$

and, for every $\eta \in (0,1)$ and $u \in U$,

$$\begin{aligned} &-c(\hat{t},\hat{x})v(\hat{t},\hat{x}) + \tfrac{\partial\hat{\phi}}{\partial t}(\hat{t},\hat{x},\hat{s},\hat{y}) + g(\hat{t},\hat{x},u) + \mathrm{D}_x\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y})\cdot b(\hat{t},\hat{x},u) \\ &\qquad + \mathcal{J}^\eta\hat{\phi}\left(\cdot,\cdot,\hat{s},\hat{y}\right)(\hat{t},\hat{x},u) + \mathcal{J}_\eta v\left(\hat{t},\hat{x},u,\mathrm{D}_x\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y})\right) \geq 0. \end{aligned} \tag{6.4}$$

On the other hand, the function $w - [v(\hat{t},\hat{x}) - \hat{\phi}(\hat{t},\hat{x},\cdot,\cdot)]$ has a minimum at $(\hat{s},\hat{y}) \in (0,T)\times\mathbb{R}^n$. Since w is a viscosity supersolution of equation (1.6), we have, for every $\eta \in (0,1)$,

$$w(\hat{s},\hat{y}) \geq h\left(\hat{y}\right), \text{ or}$$

$$-c(\hat{s},\hat{y})v(\hat{s},\hat{y}) - \tfrac{\partial\hat{\phi}}{\partial s}(\hat{t},\hat{x},\hat{s},\hat{y}) + \inf_{u\in U}\mathcal{H}_\eta(w,-\mathrm{D}_y\hat{\phi}(\hat{t},\hat{x},\cdot,\cdot))(\hat{s},\hat{y},u) \leq 0. \tag{6.5}$$

In the first case, we would have, by (6.3), (B3), and (6.1),

$$\begin{aligned} 0 &\geq v\left(\hat{t},\hat{x}\right) - h\left(\hat{x}\right) \geq v\left(\hat{t},\hat{x}\right) - h\left(\hat{y}\right) - L\left|\hat{x}-\hat{y}\right| \\ &\geq v\left(\hat{t},\hat{x}\right) - w(\hat{s},\hat{y}) - L\left|\hat{x}-\hat{y}\right| \geq \tfrac{\theta}{2} - L\left|\hat{x}-\hat{y}\right|. \end{aligned}$$

But by (6.2), $\lim_{\delta\searrow 0}|x_{\varepsilon,\delta} - y_{\varepsilon,\delta}| = 0$; hence there exists $\tilde{\delta}_\varepsilon \in (0,\delta_\varepsilon)$ such that, if $\delta < \tilde{\delta}_\varepsilon$, (6.5) holds. So, from now on, we suppose that $\delta < \tilde{\delta}_\varepsilon$. Moreover, since U is compact and the function $u \to \mathcal{H}_\eta(w, -\mathrm{D}_y\hat{\phi}(\hat{t},\hat{x},\cdot,\cdot))(\hat{s},\hat{y},u)$ is lower semi-continuous, there exists an $\hat{u} \in U$ such that this function attains its minimum in $\hat{u}$. Therefore, we have:

$$\begin{aligned} c(\hat{s},\hat{y})w(\hat{s},\hat{y}) + \tfrac{\partial\hat{\phi}}{\partial s}(\hat{t},\hat{x},\hat{s},\hat{y}) - g(\hat{s},\hat{y},\hat{u}) + \mathrm{D}_y\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y})\cdot b(\hat{s},\hat{y},\hat{u}) \\ + \mathcal{J}^\eta\hat{\phi}\left(\hat{t},\hat{x},\cdot,\cdot\right)(\hat{s},\hat{y},\hat{u}) + \mathcal{J}_\eta w\left(\hat{s},\hat{y},\hat{u},\mathrm{D}_y\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y})\right) \geq 0. \end{aligned}$$

Summing this inequality with (6.4), we obtain

$$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 \geq 0, \tag{6.6}$$

where

$$\begin{aligned} T_1 &:= -c(\hat{t},\hat{x})v(\hat{t},\hat{x}) + c(\hat{s},\hat{y})w(\hat{s},\hat{y}); \\ T_2 &:= \tfrac{\partial\hat{\phi}}{\partial t}(\hat{t},\hat{x},\hat{s},\hat{y}) + \tfrac{\partial\hat{\phi}}{\partial s}(\hat{t},\hat{x},\hat{s},\hat{y}); \\ T_3 &:= g(\hat{t},\hat{x},u) - g(\hat{s},\hat{y},\hat{u}); \\ T_4 &:= \mathrm{D}_x\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y})\cdot b(\hat{t},\hat{x},u) + \mathrm{D}_y\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y})\cdot b(\hat{s},\hat{y},\hat{u}); \\ T_5 &:= \mathcal{J}^\eta\hat{\phi}\left(\cdot,\cdot,\hat{s},\hat{y}\right)(\hat{t},\hat{x},u) + \mathcal{J}^\eta\hat{\phi}\left(\hat{t},\hat{x},\cdot,\cdot\right)(\hat{s},\hat{y},\hat{u}); \\ T_6 &:= \mathcal{J}_\eta v\left(\hat{t},\hat{x},u,\mathrm{D}_x\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y})\right) - \mathcal{J}_\eta w\left(\hat{s},\hat{y},\hat{u},-\mathrm{D}_y\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y})\right). \end{aligned}$$

We estimate now every term. From (B2), (B3), and the p'-growth of w,

$$\begin{aligned} T_1 &\leq -c_0(v(\hat{t},\hat{x}) - w(\hat{s},\hat{y})) + CL(|\hat{t}-\hat{s}| + |\hat{x}-\hat{y}|)(1+|\hat{y}|^p) \\ &\leq -c_0(v(\hat{t},\hat{x}) - w(\hat{s},\hat{y})) + \delta^{1/2}CL(1+|\hat{y}|^p)\{\tfrac{1}{\delta}(|\hat{t}-\hat{s}|^2 + |\hat{x}-\hat{y}|^2)\}^{1/2}. \end{aligned}$$

Here and the sequel C denotes positive constants not depending on L, K, ρ, ε, δ, or η, and they may change from line to line.

Since $\frac{\partial\hat{\phi}}{\partial t}(\hat{t},\hat{x},\hat{s},\hat{y}) \leq \frac{1}{\delta}(\hat{t}-\hat{s})$ and $\frac{\partial\hat{\phi}}{\partial s}(\hat{t},\hat{x},\hat{s},\hat{y}) \leq \frac{1}{\delta}(\hat{s}-\hat{t})$, it follows that $T_2 \leq 0$.

By (B3), $T_3 \leq L(|\hat{t}-\hat{s}|+|\hat{x}-\hat{y}|)$. We have that

$$\mathrm{D}_x\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y}) = \tfrac{1}{\delta}(\hat{x}-\hat{y}) + \tfrac{\varepsilon}{2}\mathrm{D}\beta\left(\hat{x}\right);$$
$$\mathrm{D}_y\hat{\phi}(\hat{t},\hat{x},\hat{s},\hat{y}) = \tfrac{1}{\delta}(\hat{y}-\hat{x}) + \tfrac{\varepsilon}{2}\mathrm{D}\beta\left(\hat{y}\right).$$

Therefore, by (A1), (A1'), and (2.2),

$$T_4 \leq \tfrac{L}{\delta}\left|\hat{x}-\hat{y}\right|\left(\left|\hat{t}-\hat{s}\right|+\left|\hat{x}-\hat{y}\right|\right) + \tfrac{\varepsilon}{2}KC_\beta\left(1+\beta\left(\hat{x}\right)+\beta\left(\hat{y}\right)\right).$$

By (A3) and (2.5) we get

$$\begin{aligned}
T_5 \leq & \tfrac{1}{2\delta}\int_{\{|z|<\eta\}}\left(\left|\gamma(\hat{t},\hat{x},\hat{u},z)\right|^2+\left|\gamma(\hat{s},\hat{y},\hat{u},z)\right|^2\right)\nu\left(\mathrm{d}z\right)\\
& + \tfrac{\varepsilon}{2}\int_{\{|z|<\eta\}}\left|\beta\left(\hat{x}+\gamma(\hat{t},\hat{x},\hat{u},z)\right)-\beta\left(\hat{x}\right)-\mathrm{D}\beta\left(\hat{x}\right)\cdot\gamma(\hat{t},\hat{x},\hat{u},z)\right|\nu\left(\mathrm{d}z\right)\\
& + \tfrac{\varepsilon}{2}\int_{\{|z|<\eta\}}\left|\beta\left(\hat{y}+\gamma(\hat{s},\hat{y},\hat{u},z)\right)-\beta\left(\hat{y}\right)-\mathrm{D}\beta\left(\hat{y}\right)\cdot\gamma(\hat{s},\hat{y},\hat{u},z)\right|\nu\left(\mathrm{d}z\right)\\
\leq & \int_{\{|z|<\eta\}}\rho\left(z\right)^2\nu\left(\mathrm{d}z\right)\left(\tfrac{2}{\delta}\left(1+\left|\hat{x}\right|^2+\left|\hat{y}\right|^2\right)+\varepsilon\tilde{C}_\beta\left(1+\beta\left(\hat{x}\right)+\beta\left(\hat{y}\right)\right)\right).
\end{aligned}$$

To shorten notation, we define

$$\hat{\xi}\left(z\right) := v\left(\hat{t},\hat{x}+\gamma\left(\hat{t},\hat{x},\hat{u},z\right)\right)-w\left(\hat{s},\hat{y}+\gamma(\hat{s},\hat{y},\hat{u},z)\right)-\left(v\left(\hat{t},\hat{x}\right)-w(\hat{s},\hat{y})\right).$$

We have that

$$\begin{aligned}
T_6 \leq & \int_{\{|z|\geq 1\}}\hat{\xi}\left(z\right)\nu\left(\mathrm{d}z\right)+\tfrac{1}{2\delta}\int_{\eta\leq|z|<1}\left|\gamma(\hat{t},\hat{x},\hat{u},z)-\gamma(\hat{s},\hat{y},\hat{u},z)\right|^2\nu\left(\mathrm{d}z\right)\\
& + \tfrac{\varepsilon}{2}\int_{\eta\leq|z|<1}\left|\beta\left(\hat{x}+\gamma(\hat{t},\hat{x},\hat{u},z)\right)-\beta\left(\hat{x}\right)-\mathrm{D}\beta\left(\hat{x}\right)\cdot\gamma(\hat{t},\hat{x},\hat{u},z)\right|\nu\left(\mathrm{d}z\right)\\
& + \tfrac{\varepsilon}{2}\int_{\eta\leq|z|<1}\left|\beta\left(\hat{y}+\gamma(\hat{s},\hat{y},\hat{u},z)\right)-\beta\left(\hat{y}\right)-\mathrm{D}\beta\left(\hat{y}\right)\cdot\gamma(\hat{s},\hat{y},\hat{u},z)\right|\nu\left(\mathrm{d}z\right)\\
\leq & \int_{\{|z|\geq 1\}}\hat{\xi}\left(z\right)\nu\left(\mathrm{d}z\right)\\
& + \int_{\{|z|<1\}}\rho\left(z\right)^2\nu\left(\mathrm{d}z\right)\left(\tfrac{1}{\delta}\left(|\hat{t}-\hat{s}|^2+|\hat{x}-\hat{y}|^2\right)+\varepsilon\tilde{C}_\beta\left(1+\beta\left(\hat{x}\right)+\beta\left(\hat{y}\right)\right)\right).
\end{aligned}$$

For ε fixed, $1+|\hat{x}|^p+|\hat{y}|^p$ is bounded and $\hat{\xi}(z) \leq C\rho(z)^p(1+|\hat{x}|^p+|\hat{y}|^p)$. Hence, by Fatou's lemma,

$$\limsup_{\delta\searrow 0}\int_{\{|z|\geq 1\}}\hat{\xi}\left(z\right)\nu\left(\mathrm{d}z\right) \leq \int_{\{|z|\geq 1\}}\limsup_{\delta\searrow 0}\hat{\xi}\left(z\right)\nu\left(\mathrm{d}z\right).$$

On the other hand, by (2.3),

$$\hat{\xi}(z) \leq \hat{\phi}\left(\hat{t}, \hat{x} + \gamma\left(\hat{t}, \hat{x}, \hat{u}, z\right), \hat{s}, \hat{y} + \gamma(\hat{s}, \hat{y}, \hat{u}, z)\right) - \hat{\phi}(\hat{\zeta})$$
$$\leq \frac{1}{\delta}(|\hat{t} - \hat{s}|^2 + |\hat{x} - \hat{y}|^2)\rho(z)(2 + \rho(z)) + \varepsilon\tilde{C}_\beta(1 + \beta(\hat{x}) + \beta(\hat{y}))(1 + \rho(z)^p).$$

Consequently,

$$\limsup_{\delta \searrow 0} \int_{\{|z| \geq 1\}} \hat{\xi}(z)\nu(\mathrm{d}z) \leq \varepsilon\tilde{C}_\beta(1 + \beta(\hat{x}) + \beta(\hat{y})) \int_{\{|z| \geq 1\}} (1 + \rho(z)^p)\nu(\mathrm{d}z).$$

Using now all the estimates on $T_1, \ldots, T_6$, and relation (6.2), we can pass to the limit in (6.6), for $\eta \searrow 0$, and $\delta \searrow 0$ (in this order):

$$-c_0\left(v(\hat{t}, \hat{x}) - w\left(\hat{s}, \hat{y}\right)\right) + \tfrac{\varepsilon}{2}\tilde{K} \limsup_{\delta \searrow 0} \left(1 + \beta\left(\hat{x}\right) + \beta\left(\hat{y}\right)\right) \geq 0.$$

Relation (6.1) implies $\frac{\varepsilon}{2}\left(\beta\left(\hat{x}\right) + \beta\left(\hat{y}\right)\right) \leq v\left(\hat{t}, \hat{x}\right) - w(\hat{s}, \hat{y}) - \frac{\theta}{2}$. Hence, letting $\varepsilon \searrow 0$, we obtain

$$(\tilde{K} - c_0) \limsup_{\varepsilon \searrow 0} \limsup_{\delta \searrow 0} \left(v(\hat{t}, \hat{x}) - w\left(\hat{s}, \hat{y}\right)\right) \geq \tfrac{1}{2}\tilde{K}\theta,$$

which is, of course a contradiction, since we assumed that $c_0 \geq \tilde{K}$. □

Combining the existence result with the comparison principle, we get:

Corollary 6.3. *Under assumptions* $(A0)$-$(A3)$, $(A1')$, $(A2')$, $(B0)$-$(B3)$, V *is the unique viscosity solution of equation* (1.6) *in the class* $\bigcup_{p' < p} C_{p'}((0, T] \times \mathbb{R}^n)$.

REFERENCES

1. Fleming, W.H.; Soner, H.M. – *Controlled Markov Processes and Viscosity Solutions*, Applications of Mathematics (New York), 25, Springer-Verlag, New York, 1993.
2. Gihman, I.; Skorohod, A.V. – *Stochastic Differential Equations*, Springer-Verlag, New York-Heidelberg, 1972.
3. Jacod, J. –*Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Mathematics, 714, Springer, Berlin, 1979.

4. Jacod, J.; Shiryaev, A. – *Limit Theorems for Stochastic Processes*, second edition, Grundlehren der Mathematischen Wissenschaften, 288, Springer-Verlag, Berlin, 2003.

5. Jakobsen, E.R.; Karlsen, K.H. – *Continuous dependence estimates for viscosity solutions of integro-PDEs*, J. Differential Equations, 212 (2005), 278–318.

6. Meerschaert, M.M.; Benson, D.A.; Baeumer, B. – *Multidimensional advection and fractional dispersion,* Phys. Rev., 59 (1999), 5026-5028.

7. Pham, H. – *Optimal stopping of controlled jump diffusion processes: a viscosity solution approach,* J. Math. Systems Estim. Control, 8 (1998), 1–27.

8. Sayah, A. – *Equations d'Hamilton-Jacobi du premier ordre avec termes intégro-differentiels: Parties I et II*, Comm. P.D.E, 10 (1991), 1057–1093.

9. Yong, J.; Zhou, X.Y. – *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Applications of Mathematics (New York), 43, Springer-Verlag, New York, 1999.

Received: 17.IX.2009

"O. Mayer" Institute of Mathematics,
Romanian Academy,
Bd. Carol I, no. 8, 700506, Iaşi,
ROMANIA
adrian.zalinescu@googlemail.com