

CURVATURE OF CR MANIFOLDS

BY

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Abstract. We prove the existence and uniqueness of a torsion-free and h -metric linear connection ∇ (CR connection) on the horizontal distribution of a CR manifold M . Then we define the CR sectional curvature of M and obtain a characterization of the CR space forms. Also, by using the CR Ricci tensor and the CR scalar curvature we define the CR Einstein gravitational tensor field on M . Thus, we can write down Einstein equations on the horizontal distribution of the 5-dimensional CR manifold involved in the Penrose correspondence. Finally, some CR differential operators are defined on M and two examples are given to illustrate the theory developed in the paper. Most of the results are obtained for CR manifolds that do not satisfy the integrability conditions.

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Introduction

The concept of CR manifold has its roots in the seminal paper of POINCARÉ [15], where he showed that two real hypersurfaces in C^2 are in general biholomorphically inequivalent. Since then, CR manifolds have been studied from both the analytic point of view and the geometric point of view (cf. CHIRKA [6], DRAGOMIR and TOMASSINI [8], JACOBOWITZ [10]). From the abstract concept of CR manifold (cf. GREENFIELD [9]) emerged the geometry of CR submanifolds, which in fact is concerned with embedded CR manifolds in Kähler manifolds (cf. YANO and KON [21], BEJANCU [1]). Also, an important research work was carried out on the applications of the theory of CR manifolds to physics (cf. PENROSE [13], [14], WELLS [19], [20]).

The purpose of the present paper is to bring a contribution to the geometry of CR manifolds and to apply it to physics. More precisely, we introduce and study the CR sectional curvature of a CR manifold. This is done by introducing a CR connection on the horizontal distribution HM of a CR manifold M . Note that, in contrast with some other studies on CR manifolds (cf. TANAKA [16], [17], WEBSTER [18]), we concentrate our study on the geometry of HM , instead of the geometry of the whole tangent bundle TM . Thus our study does not depend on any extension of the Levi form to a Riemannian metric on M . The CR connection on HM enables us to define the CR Einstein gravitational tensor field on a CR manifold. This is raising the idea of a general relativity theory on the horizontal distribution of a 5-dimensional CR manifold, that is, the main object from Penrose correspondence.

Now, we outline the content of the paper. In the first section we prove an existence and uniqueness theorem (cf. Theorem 1.1) for a torsion-free and h -metric linear connection ∇ on the horizontal distribution HM of the CR manifold M . We call ∇ the CR connection on HM and concentrate our study on the geometry of M via ∇ . We should note that, in contrast with the conditions required for the well known Tanaka–Webster connection (the integrability condition for the almost complex structure on HM), the CR connection is defined on any nearly CR manifold, which does not require the integrability condition (1.1b). By using the Darboux coordinate system on a CR manifold, we obtain several characterizations of nearly CR manifolds whose Reeb vector field is a CR Killing vector field (cf. Theorem 1.3). In Section 2, we present the main properties of the curvature tensor field of the CR connection (Theorems 2.1 and 2.2) and then we introduce the CR sectional curvature of a CR manifold. We prove a Schur Theorem for CR manifolds and find the form of the curvature tensor field of a CR space form (cf. Theorems 2.4 and 2.5). The CR differential operators like CR differential, CR gradient, CR divergence and CR Laplacian are introduced and studied in Section 3. The CR Ricci tensor and the CR scalar curvature are the main objects for an application to Penrose correspondence which is presented in Section 4. Here, we introduce the CR Einstein gravitational tensor field and prove that it is symmetric and its CR divergence vanishes identically on the 5-dimensional CR manifold PV_0^* , which is the main object from the Penrose correspondence. Finally, in the last section we apply the theory we developed in the paper for the Heisenberg group and the sphere S^3 .

1. The CR connection on a nearly CR manifold

Let M be a real $(2m + 1)$ -dimensional manifold with tangent bundle TM and complexified tangent bundle $TM \otimes C$. Suppose that there exists a complex subbundle \mathcal{H} of $TM \otimes C$ such that $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$, and \mathcal{H} is involutive, that is, for complex vector fields U and V in \mathcal{H} , the Lie bracket $[U, V]$ is also in \mathcal{H} . Then \mathcal{H} defines a CR structure on M , and M is called a CR manifold (cf. GREENFIELD [9]). The concept of CR structure on M can also be presented by using a real distribution on M , as follows. Suppose that HM is a real $2m$ -dimensional distribution on M endowed with an almost complex structure $J : HM \rightarrow HM$, $J^2 = -I$. Then M is a CR manifold if and only if the following conditions are satisfied (cf. BEJANCU [1], p. 129)

$$(1.1) \quad \begin{aligned} (a) \quad & [X, JY] + [JX, Y] \in \Gamma(HM) \\ (b) \quad & N(X, Y) = [JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0, \end{aligned}$$

for all $X, Y \in \Gamma(HM)$.

Here and in the sequel, $\Gamma(HM)$ stands for the $\mathcal{F}(M)$ -module of smooth sections of HM , where $\mathcal{F}(M)$ is the algebra of smooth functions on M . Also, throughout the paper we use the Einstein convention, that is, repeated indices with one upper index and one lower index denotes the summation over their range. If not stated otherwise, we use indices $i, j, k, \dots \in \{1, \dots, 2m\}$ and $a, b, c, \dots \in \{1, \dots, m\}$.

In the present paper we study a structure which is weaker than a CR structure. Let M be a real $(2m + 1)$ -dimensional manifold endowed with a contact form η , that is, $\eta \wedge (d\eta)^n \neq 0$ on M . It follows that $\eta \wedge (d\eta)^n$ is a volume element on M , so M is orientable. Also, $d\eta$ is of rank $2m$, and thus there exists a global vector field ξ on M satisfying

$$(1.2) \quad (a) \quad \eta(\xi) = 1, \quad (b) \quad d\eta(\xi, X) = 0, \quad \forall X \in \Gamma(TM).$$

ξ is called the *characteristic vector field* or the *Reeb vector field* of the contact structure defined by η . Denote by HM the *horizontal (contact) distribution* defined by η , that is, we have $HM_x = \{X \in T_x M : \eta(X) = 0\}$, $\forall x \in M$. As a consequence of (1.2b) we obtain $\eta([\xi, X]) = 0$, for any $X \in \Gamma(HM)$, that is,

$$(1.3) \quad [\xi, X] \in \Gamma(HM), \quad \forall X \in \Gamma(HM).$$

Next, we suppose that on HM there exists an almost complex structure J , and consider the *Levi form* g defined by

$$(1.4) \quad g(X, Y) = d\eta(X, JY), \quad \forall X, Y \in \Gamma(HM).$$

Now, we can prove the following.

Proposition 1.1. *Let (M, η) be a contact manifold endowed with an almost complex structure J on the horizontal distribution HM . Then the following assertions are equivalent:*

- (i) *The Levi form is a symmetric tensor field on HM .*
- (ii) *The Levi form is almost Hermitian, that is, we have*

$$(1.5) \quad g(JX, JY) = g(X, Y), \quad \forall X, Y \in \Gamma(HM).$$

- (iii) *The condition (1.1a) is satisfied.*

Proof. By (1.4) it is easy to see that both (i) and (ii) are equivalent to $d\eta(X, JY) + d\eta(JX, Y) = 0$, $\forall X, Y \in \Gamma(HM)$, which actually is just (1.1a). \square

Next, we suppose that (M, η) is a contact manifold endowed with an almost complex structure J on HM , satisfying the conditions

- a. The Levi form is nondegenerate on M .
- b. One of the conditions in Proposition 1.1 is satisfied (and, therefore, all).

Such a manifold M is called a *nearly CR manifold*. We gave this name to M because just one condition, namely (1.1b), is not satisfied on M to be a *CR manifold*. Under the name of *nondegenerate pseudo-Hermitian manifold*, the same concept has been introduced before by CHERN and HAMILTON [5]. The geometry of (possibly non integrable) almost *CR* structures on contact Riemannian manifolds have been studied by BLAIR and DRAGOMIR [4].

The geometry of *CR* manifolds has been intensively studied (cf. CIRKA [6], DRAGOMIR and TOMASSINI [8], JACOBOWITZ [10]). The main geometric object in these studies was the Tanaka–Webster connection on M (cf. TANAKA [16], [17] and WEBSTER [18]). The construction of this connection is tied to both integrability conditions in (1.1), so in case of nearly *CR* manifolds is unavailable. On the other hand, the Tanaka–Webster connection depends on the Webster metric which is an extension of the Levi form to the whole TM .

The purpose of this section is to construct a linear connection on the contact distribution HM of a nearly *CR* manifold M which does not depend

on any extension of the Levi form to TM . In the next sections we will see that this connection enables us to define some CR objects like: CR sectional curvature, CR Ricci tensor, CR scalar curvature, CR gradient, CR divergence and CR Laplacian on M .

First, we write down the direct sum decomposition

$$(1.6) \quad TM = HM \oplus \{\xi\},$$

where $\{\xi\}$ is the line distribution spanned by ξ . Then we denote by h the projection morphism of TM on HM with respect to (1.6). Next, we consider a linear connection ∇ on HM and define its *torsion tensor field* as an $\mathcal{F}(M)$ -bilinear mapping

$$(1.7) \quad \begin{aligned} T : \Gamma(TM) \times \Gamma(HM) &\rightarrow \Gamma(HM), \\ T(X, hY) &= \nabla_X hY - \nabla_{hY} hX - h[X, hY], \quad \forall X, Y \in \Gamma(TM). \end{aligned}$$

If T vanishes identically on M , we say that ∇ is a *torsion-free connection* on HM . Also, we say that ∇ is an *h -metric connection* if we have

$$(1.8) \quad \begin{aligned} (\nabla_{hX} g)(hY, hZ) &= hX(g(hY, hZ)) - g(\nabla_{hX} hY, hZ) \\ &\quad - g(hY, \nabla_{hX} hZ) = 0, \quad \forall X, Y \in \Gamma(TM). \end{aligned}$$

Now, we can prove the following.

Theorem 1.1. *Let M be a nearly CR manifold. Then there exists a unique torsion-free and h -metric linear connection ∇ on the horizontal distribution HM of M .*

Proof. First, we define the differential operator $\nabla : \Gamma(TM) \times \Gamma(HM) \rightarrow \Gamma(HM)$ as follows:

$$(1.9) \quad \begin{aligned} (a) \quad &2g(\nabla_{hX} hY, hZ) = hX(g(hY, hZ)) + hY(g(hZ, hX)) \\ &\quad - hZ(g(hX, hY)) + g(h[hX, hY], hZ) - g(h[hY, hZ], hX) \\ &\quad + g(h[hZ, hX], hY), \\ (b) \quad &\nabla_{\xi} hY = [\xi, hY], \quad \forall X, Y, Z \in \Gamma(TM). \end{aligned}$$

Note that, by (1.3), the right part of (1.9b) lies in $\Gamma(HM)$. Then, by using (1.7), (1.8) and (1.9), it is easy to check that ∇ is a torsion-free and h -metric linear connection on HM . Next, suppose that ∇^* is another torsion-free

and h -metric linear connection on HM . Then, by (1.3) and (1.7), we must have

$$(1.10) \quad \begin{aligned} (a) \quad & \nabla_{\xi}^* hY = [\xi, hY], \\ (b) \quad & \nabla_{hX}^* hY - \nabla_{hY}^* hX - h[hX, hY] = 0. \end{aligned}$$

Next, by using (1.8) for ∇^* and (1.10b), we obtain

$$(1.11) \quad \begin{aligned} 0 &= (\nabla_{hX}^* g)(hY, hZ) + (\nabla_{hY}^* g)(hZ, hX) - (\nabla_{hZ}^* g)(hX, hY) \\ &= hX(g(hY, hZ)) + hY(g(hZ, hX)) - hZ(g(hX, hY)) \\ &\quad + g(h[hX, hY], hZ) - g(h[hY, hZ], hX) + g(h[hZ, hX], hY) \\ &\quad - 2g(\nabla_{hX}^* hY, hZ). \end{aligned}$$

Finally, taking into account (1.9), (1.10a) and (1.11), we deduce that $\nabla^* = \nabla$, which proves the uniqueness of ∇ given by (1.9). \square

The linear connection ∇ given by (1.9) is called the *CR connection* on HM . In order to define a covariant derivative for more general objects, we consider the dual vector bundle HM^* of HM . Any section of HM (resp. HM^*) is called a *horizontal vector field* (resp. *horizontal 1-form*) on M . For any $\omega \in \Gamma(HM^*)$ we define

$$(\nabla_X \omega)(hY) = X(\omega(hY)) - \omega(\nabla_X hY), \quad \forall X, Y \in \Gamma(TM).$$

More general, a *horizontal tensor field* of type (p, q) is a $(p+q)$ - $\mathcal{F}(M)$ -multilinear mapping $S : \Gamma(HM^*)^p \times \Gamma(HM)^q \rightarrow \mathcal{F}(M)$. By using the covariant derivatives of horizontal vector fields and horizontal 1-forms, as in case of a linear connection on M , we can extend the covariant derivative to S . Details can be found in BEJANCU-FARRAN [2].

In particular, J is a horizontal tensor field of type $(1, 1)$ and we have

$$(1.12) \quad (\nabla_X J)(hY) = \nabla_X JhY - J\nabla_X hY, \quad \forall X, Y \in \Gamma(TM).$$

Then we say that J is *h -parallel* (resp. *ξ -parallel*) with respect to the *CR connection* ∇ if we have

$$(1.13) \quad (\nabla_{hX} J)(hY) = 0 \text{ (resp. } (\nabla_{\xi} J)(hY) = 0), \quad \forall X, Y \in \Gamma(TM).$$

We are now able to prove the following important result.

Theorem 1.2. *A nearly CR manifold is a CR manifold if and only if J is h -parallel with respect to the CR connection on HM .*

Proof. First, according to (1.6) we can write

$$(1.14) \quad X = hX + \eta(X)\xi, \quad \forall X \in \Gamma(TM).$$

Then, taking into account that $d^2\eta = 0$ and using (1.14) and (1.2b) we deduce that

$$(1.15) \quad \begin{aligned} hX(d\eta(hY, hZ)) &+ hY(d\eta(hZ, hX)) + hZ(d\eta(hX, hY)) \\ &- d\eta(h[hX, hY], hZ) - d\eta(h[hY, hZ], hX) \\ &- d\eta(h[hZ, hX], hY) = 0, \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. As the CR connection ∇ is torsion-free, by (1.7) we have

$$(1.16) \quad \nabla_{hX}hY - \nabla_{hY}hX - h[hX, hY] = 0.$$

Then, by using (1.4), (1.16), (1.8), (1.5) and (1.12) in (1.15), we infer that

$$(1.17) \quad g((\nabla_{hX}J)hZ, hY) + g((\nabla_{hY}J)hX, hZ) + g((\nabla_{hZ}J)hY, hX) = 0.$$

Next, replace hY and hZ from (1.17) by JhY and JhZ respectively, and taking into account that

$$(1.18) \quad (\nabla_{hU}J)JhV + J(\nabla_{hU}J)hV = 0, \quad \forall U, V \in \Gamma(TM),$$

and by using (1.5) we obtain

$$(1.19) \quad g(JhZ, (\nabla_{JhY}J)hX) + g(JhX, (\nabla_{JhZ}J)hY) - g(hY, (\nabla_{hX}J)hZ) = 0.$$

By subtracting (1.19) from (1.17) and using (1.5) we deduce that

$$(1.20) \quad \begin{aligned} 2g(hY, (\nabla_{hX}J)hZ) + g(JhX, J(\nabla_{hZ}J)hY - (\nabla_{JhZ}J)hY) \\ + g(J(\nabla_{hY}J)hX - (\nabla_{JhY}J)hX, JhZ) = 0. \end{aligned}$$

Taking into account that

$$g((\nabla_{hU}J)hV, hW) + g(hV, (\nabla_{hU}J)hW) = 0, \quad \forall U, V, W \in \Gamma(TM),$$

and by using (1.5) and (1.18) we infer that

$$\begin{aligned} g(J(\nabla_{hY}J)hX - (\nabla_{JhY}J)hX, JhZ) &= g(hX, (\nabla_{JhY}J)JhZ - (\nabla_{hY}J)hZ) \\ &= g(JhX, (\nabla_{JhY}J)hZ - J(\nabla_{hY}J)hZ). \end{aligned}$$

Thus, (1.20) becomes

$$(1.21) \quad 2g(hY, (\nabla_{hX} J)hZ) = g(JhX, (\nabla_{JhZ} J)hY - J(\nabla_{hZ} J)hY - (\nabla_{JhY} J)hZ + J(\nabla_{hY} J)hZ).$$

On the other hand, taking into account (1.1a), (1.16) and (1.12) we obtain

$$\begin{aligned} N(hY, hZ) &= h[JhY, JhZ] - h[hY, hZ] - J(h[hY, JhZ] + h[JhY, hZ]) \\ &= (\nabla_{JhY} J)hZ - (\nabla_{JhZ} J)hY + J(\nabla_{hZ} J)hY - J(\nabla_{hY} J)hZ. \end{aligned}$$

Hence (1.21) becomes

$$(1.22) \quad 2g(hY, (\nabla_{hX} J)hZ) = g(JhX, N(hZ, hY)).$$

Finally, from (1.22) we see that (1.1b) and the first equality in (1.13) are equivalent. Thus the proof is complete. \square

Next we want to express the CR connection and some of its properties by using some special local coordinate systems. First, we recall that by Darboux Theorem (cf. BLAIR [3], p. 18), there exist local coordinates (x^i, x^0) , $i \in \{1, \dots, 2m\}$, on M , such that the contact form is expressed as follows

$$(1.23) \quad \eta = dx^0 - \sum_{a=1}^m x^{m+a} dx^a.$$

Then the contact distribution HM is locally spanned by the vector fields

$$(1.24) \quad \left\{ \frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} + x^{m+a} \frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^{m+a}} = \frac{\partial}{\partial x^{m+a}}, a \in \{1, \dots, m\} \right\},$$

and $\xi = \partial/\partial x^0$. Thus, by direct calculations we obtain

$$(1.25) \quad \text{(a) } \left[\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i} \right] = 0, \quad \text{(b) } \left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right] = \Omega_{ij} \frac{\partial}{\partial x^0},$$

for all $i, j \in \{1, \dots, 2m\}$, where all Ω_{ij} vanish, except

$$\Omega_{a(m+b)} = -\Omega_{(m+b)a} = \delta_{ab}, \quad a, b \in \{1, \dots, m\}.$$

We call (x^i, x^0) , $i \in \{1, \dots, 2m\}$, the *Darboux coordinate system* and $\{\delta/\delta x^i, \partial/\partial x^0\}$ the *local Darboux frame field* on M . Next, we put

$$(1.26) \quad \text{(a) } g_{ij} = g \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right), \quad \text{(b) } J \left(\frac{\delta}{\delta x^i} \right) = J_i^k \frac{\delta}{\delta x^k},$$

and by using (1.26), (1.4), (1.25b) and (1.2a), we deduce that

$$(1.27) \quad \bar{g}_{ij} = \Omega_{ik} J_j^k.$$

Now, we put

$$(1.28) \quad \text{(a) } \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = \Gamma_i^{kj} \frac{\delta}{\delta x^k}, \quad \text{(b) } \nabla_{\frac{\partial}{\partial x^0}} \frac{\delta}{\delta x^i} = \Gamma_i^{k0} \frac{\delta}{\delta x^k},$$

and by using (1.9), (1.28), (1.26a) and (1.25) we obtain

$$(1.29) \quad \text{(a) } \Gamma_i^{kj} = \frac{1}{2} g^{kh} \left\{ \frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \right\}, \quad \text{(b) } \Gamma_i^{k0} = 0,$$

where g^{kh} are the entries of the inverse of the matrix $[g_{kh}]$. Then, by using (1.12) and (1.28b), we infer that

$$(1.30) \quad \left(\nabla_{\frac{\partial}{\partial x^0}} J \right) \left(\frac{\delta}{\delta x^i} \right) = \frac{\partial J_i^k}{\partial x^0} \frac{\delta}{\delta x^k}.$$

Next, taking into account (1.3) we define the *Lie derivative* of the Levi form g with respect to ξ , as the horizontal tensor field of type $(0, 2)$ given by

$$(1.31) \quad (L_\xi g)(hX, hY) = \xi(g(hX, hY)) - g([\xi, hX], hY) - g([\xi, hY], hX),$$

for all $X, Y \in \Gamma(TM)$. If $L_\xi g$ vanishes identically on M , we say that ξ is a *CR Killing vector field*. Now, we recall that the natural extension of the Levi form to a semi-Riemannian metric in sense of O'NEILL [12] is the so called *Webster metric*. More precisely, such a metric \tilde{g} is given by

$$(1.32) \quad \text{(a) } \tilde{g}(\xi, hX) = 0, \quad \text{(b) } \tilde{g}(\xi, \xi) = 1,$$

and it coincides with g on HM . Clearly, if ξ is a Killing vector field with respect to \tilde{g} , it is a *CR Killing vector field*. By (1.32) and (1.3) it is easy to see that the converse is also true. Thus ξ is a *CR Killing vector field* if and only if it is a Killing vector field with respect to the Webster metric.

Now, we say that the *CR connection* ∇ is a ξ -*metric connection* on HM if

$$(1.33) \quad \begin{aligned} (\nabla_\xi g)(hX, hY) &= \xi(g(hX, hY)) - g(\nabla_\xi hX, hY) \\ &\quad - g(hX, \nabla_\xi hY) = 0, \quad \forall X, Y \in \Gamma(TM). \end{aligned}$$

Then, we can prove the following.

Theorem 1.3. *Let M be a nearly CR manifold. Then the following assertions are equivalent.*

- (i) *The CR connection is a ξ -metric connection.*
- (ii) *The Reeb vector field is a CR Killing vector field.*
- (iii) *The local components of the Levi form with respect to the Darboux frame field on HM are functions of (x^1, \dots, x^{2m}) alone, that is, we have*

$$(1.34) \quad \frac{\partial g_{ij}}{\partial x^0} = 0, \quad \forall i, j \in \{1, \dots, 2m\}.$$

- (iv) *The almost complex structure J on HM is ξ -parallel with respect to the CR connection.*
- (v) *The local components of J with respect to the Darboux frame field are functions of (x^1, \dots, x^{2m}) alone, that is, we have*

$$(1.35) \quad \frac{\partial J_j^i}{\partial x^0} = 0, \quad \forall i, j \in \{1, \dots, 2m\}.$$

Proof. First, by using (1.31), (1.33) and (1.9b) we deduce that (i) and (ii) are equivalent. Then, by using the Darboux coordinates, (1.27b) and (1.28b) we infer that (1.33) is equivalent to

$$\left(\nabla_{\frac{\partial}{\partial x^0}} g \right) \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = \frac{\partial g_{ij}}{\partial x^0} = 0.$$

Thus (i) and (iii) are equivalent. Similarly, by using (1.13) and (1.30) we obtain the equivalence of (iv) and (v). Finally, taking into account that Ω_{ij} are constant functions, from (1.27) we deduce that (1.34) and (1.35) are equivalent. This proves the equivalence of (iii) and (v), and completes the proof of the theorem. \square

Now, we denote by R the curvature tensor field of the CR connection ∇ , that is, we have

$$(1.36) \quad R(X, Y)hZ = \nabla_X \nabla_Y hZ - \nabla_Y \nabla_X hZ - \nabla_{[X, Y]} hZ,$$

for all $X, Y, Z \in \Gamma(TM)$. Then we put

$$(1.37) \quad \begin{aligned} (a) \quad & R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^i} = R_i^h{}_{jk} \frac{\delta}{\delta x^h}, \\ (b) \quad & R\left(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^j}\right) \frac{\delta}{\delta x^i} = R_i^h{}_{j0} \frac{\delta}{\delta x^h}, \end{aligned}$$

and by using (1.36), (1.37), (1.28) and (1.25) we obtain

$$(1.38) \quad \begin{aligned} (a) \quad & R_i^h{}_{jk} = \frac{\delta \Gamma_i^h{}_{jk}}{\delta x^k} - \frac{\delta \Gamma_i^h{}_{kj}}{\delta x^j} + \Gamma_i^t{}_j \Gamma_t^h{}_{k} - \Gamma_i^t{}_k \Gamma_t^h{}_{j}, \\ (b) \quad & R_i^h{}_{j0} = \frac{\partial \Gamma_i^h{}_{jk}}{\partial x^0}. \end{aligned}$$

In particular, we suppose that ξ is a CR Killing vector field. Then, by using (1.34) in (1.29a), we deduce that $\Gamma_i^k{}_j$ are functions of (x^1, \dots, x^{2m}) alone. Thus (1.38) becomes

$$(1.39) \quad \begin{aligned} (a) \quad & R_i^h{}_{jk} = \frac{\partial \Gamma_i^h{}_{jk}}{\partial x^k} - \frac{\partial \Gamma_i^h{}_{kj}}{\partial x^j} + \Gamma_i^t{}_j \Gamma_t^h{}_{k} - \Gamma_i^t{}_k \Gamma_t^h{}_{j}, \\ (b) \quad & R_i^h{}_{j0} = 0. \end{aligned}$$

Finally, we note that (1.39b) is equivalent to

$$(1.40) \quad R(\xi, hX)hY = 0, \quad \forall X, Y \in \Gamma(TM).$$

2. CR sectional curvature of a CR manifold

As it is well known, the holomorphic sectional curvature plays an important role in studying Kähler geometry. Such an object was introduced by using some special properties of the curvature tensor field of the Levi–Civita connection. As the horizontal distribution HM and the CR connection ∇ play respectively the role of the tangent bundle and of the Levi–Civita connection from the Kähler geometry point of view, it is natural to ask for a sectional curvature in the direction of almost complex planes on a CR manifold. The purpose of this section is to define such a sectional curvature and to show the similitude between the CR geometry and the Kähler geometry. We also define a CR Ricci tensor field and a CR scalar curvature which enable us to introduce in Section 4 the CR Einstein gravitational tensor field.

First, we define the horizontal tensor field R of type $(0, 4)$ by

$$(2.1) \quad R(hX, hY, hZ, hU) = g(R(hX, hY)hU, hZ),$$

where R from the right hand side is the curvature tensor field of the CR connection ∇ . Then we prove the following.

Theorem 2.1. *Let M be a nearly CR manifold such that ξ is a CR Killing vector field. Then we have the following identities:*

$$(2.2) \quad \begin{aligned} (a) \quad & R(hX, hY, hZ, hU) + R(hY, hX, hZ, hU) = 0, \\ (b) \quad & R(hX, hY, hZ, hU) + R(hX, hY, hU, hZ) = 0, \\ (c) \quad & \sum_{(hX, hY, hZ)} \{R(hX, hY, hZ, hU)\} = 0, \\ (d) \quad & R(hX, hY, hZ, hU) = R(hZ, hU, hX, hY), \\ (e) \quad & \sum_{(hX, hY, hZ)} \{(\nabla_{hX} R)(hY, hZ, hU, hV)\} = 0, \end{aligned}$$

for all $X, Y, Z, U, V \in \Gamma(TM)$, where $\sum_{(hX, hY, hZ)}$ denotes the cyclic sum with respect to (hX, hY, hZ) .

Proof. First, (2.2a) is a consequence of the skew-symmetry of R given by (1.36). Then, by direct calculations taking into account that ∇ is h -metric and using (1.26a), we obtain

$$(2.3) \quad \begin{aligned} g\left(\nabla_{\frac{\delta}{\delta x^h}} \nabla_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) &= \frac{\delta}{\delta x^h} \frac{\delta}{\delta x^k} (g_{ij}) - \frac{\delta}{\delta x^h} \left(g\left(\frac{\delta}{\delta x^i}, \nabla_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j}\right)\right) \\ &\quad - \frac{\delta}{\delta x^k} \left(g\left(\frac{\delta}{\delta x^i}, \nabla_{\frac{\delta}{\delta x^h}} \frac{\delta}{\delta x^j}\right)\right) + g\left(\frac{\delta}{\delta x^i}, \nabla_{\frac{\delta}{\delta x^k}} \nabla_{\frac{\delta}{\delta x^h}} \frac{\delta}{\delta x^j}\right). \end{aligned}$$

Also, by using (1.25b), (1.28b) and (1.29b) we deduce that

$$(2.4) \quad \nabla_{\left[\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}\right]} \frac{\delta}{\delta x^i} = 0.$$

Thus, by using (2.1), (1.36), (2.3), (2.4), (1.25b) and (1.34), we infer that

$$\begin{aligned}
R\left(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) &= g\left(\nabla_{\frac{\delta}{\delta x^h}} \nabla_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^i} - \nabla_{\frac{\delta}{\delta x^k}} \nabla_{\frac{\delta}{\delta x^h}} \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) \\
&= \left[\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}\right](g_{ij}) + g\left(\frac{\delta}{\delta x^i}, \nabla_{\frac{\delta}{\delta x^k}} \nabla_{\frac{\delta}{\delta x^h}} \frac{\delta}{\delta x^j} - \nabla_{\frac{\delta}{\delta x^h}} \nabla_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j}\right) \\
&= \Omega_{kh} \frac{\partial g_{ij}}{\partial x^0} - g\left(R\left(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}\right) \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) \\
&= -R\left(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right),
\end{aligned}$$

which proves (2.2b). Next, we consider the linear connection $\tilde{\nabla}$ on M defined as follows

$$(2.5) \quad (a) \tilde{\nabla}_X hY = \nabla_X hY, \quad (b) \tilde{\nabla}_X \xi = 0, \quad \forall X, Y \in \Gamma(TM).$$

Then, we recall the Bianchi identities for $\tilde{\nabla}$ (cf. KOBAYASHI and NOMIZU [11], p. 135)

$$\begin{aligned}
(2.6) \quad (a) \quad &\sum_{(X,Y,Z)} \left\{ (\tilde{\nabla}_X \tilde{T})(Y, Z) + \tilde{T}(\tilde{T}(X, Y), Z) - \tilde{R}(X, Y)Z \right\} = 0, \\
(b) \quad &\sum_{(X,Y,Z)} \left\{ (\tilde{\nabla}_X \tilde{R})(Y, Z)U + \tilde{R}(\tilde{T}(X, Y), Z)U \right\} = 0,
\end{aligned}$$

for all $X, Y, Z, U \in \Gamma(TM)$, where \tilde{T} and \tilde{R} are the torsion tensor field and curvature tensor field of $\tilde{\nabla}$, respectively. By using (1.14) and (1.16), we obtain

$$\begin{aligned}
(2.7) \quad \tilde{T}(hX, hY) &= \tilde{\nabla}_{hX} hY - \tilde{\nabla}_{hY} hX - [hX, hY] \\
&= \nabla_{hX} hY - \nabla_{hY} hX - h[hX, hY] - \eta([hX, hY])\xi \\
&= \eta([hY, hX])\xi.
\end{aligned}$$

Thus, by using (2.5) and (1.9b), we deduce that

$$\begin{aligned}
\tilde{T}(\tilde{T}(hX, hY), hZ) &= \eta([hY, hX])(\tilde{\nabla}_\xi hZ - \tilde{\nabla}_{hZ} \xi - [\xi, hZ]) = 0, \\
(\tilde{\nabla}_{hX} \tilde{T})(hY, hZ) &= \{hX(\eta([hZ, hY])) - \eta([hZ, \nabla_{hX} hY]) - \eta([\nabla_{hX} hZ, hY])\}\xi.
\end{aligned}$$

Then, we obtain (2.2c) by taking the horizontal part in (2.6a), since

$$(2.8) \quad \tilde{R}(hX, hY)hZ = R(hX, hY)hZ.$$

Now, by using (2.7), (2.5a) and (1.40), we infer that

$$(2.9) \quad \tilde{R}(\tilde{T}(hX, hY), hZ)hU = \eta([hY, hX])R(\xi, hZ)hU = 0.$$

Then, we obtain (2.2e) from (2.6b) by using (2.5a), (2.8) and (2.9). Finally, following O'NEILL [12], p. 75, by a combinatorial exercise using (2.2a), (2.2b) and (2.2c), we obtain (2.2d). Thus the proof is complete. \square

Theorem 2.2. *Let M be a nearly CR manifold such that ξ is a CR Killing vector field. Then we have the following identities:*

$$(2.10) \quad \begin{aligned} (a) \quad & R(hX, hY, JhZ, JhU) = R(hX, hY, hZ, hU), \\ (b) \quad & R(JhX, JhY, hZ, hU) = R(hX, hY, hZ, hU), \end{aligned}$$

for all $X, Y, Z, U \in \Gamma(TM)$.

Proof. First, taking into account that J is h -parallel with respect to the CR connection ∇ (see Theorem 1.1) and using (1.5), we obtain (2.10a). Then, (2.10b) is obtained by using (2.2d) and (2.10a). \square

Now, we define the CR Ricci tensor field Ric of the nearly CR manifold M as follows

$$(2.11) \quad \text{Ric}(hX, hY) = \sum_{i=1}^{2m} \varepsilon_i \{R(E_i, hX, E_i, hY)\}, \quad \forall X, Y \in \Gamma(TM),$$

where $\{E_i\}$, $i \in \{1, \dots, 2m\}$, is an orthonormal basis in $\Gamma(HM)$ and $\varepsilon_i = g(E_i, E_i)$. By (2.2d) we see that the CR Ricci tensor field is a symmetric horizontal tensor field on M . Also, we define the CR scalar curvature S of M by

$$(2.12) \quad S = \sum_{i=1}^{2m} \varepsilon_i \{\text{Ric}(E_i, E_i)\}.$$

Next, we put

$$(2.13) \quad (a) \quad \frac{\delta}{\delta x^i} = A_i^j E_j, \quad (b) \quad E_i = B_i^j \frac{\delta}{\delta x^j},$$

and obtain

$$(2.14) \quad (a) \ g_{ij} = \sum_{k=1}^{2m} \varepsilon_k A_i^k A_j^k, \quad (b) \ g^{ij} = \sum_{k=1}^{2m} \varepsilon_k B_k^i B_k^j.$$

Then, by using (2.11), (2.1), (2.13b), (2.14b), (1.38a) and (2.12), we deduce that

$$(2.15) \quad (a) \ R_{ij} = \text{Ric} \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = R_i^h{}_{jh}, \quad (b) \ S = g^{ij} R_{ij}.$$

Theorem 2.3. *Let M be a connected real $(2m+1)$ -dimensional, $m > 1$, nearly CR manifold such that ξ is a CR Killing vector field. If the CR Ricci tensor field satisfies*

$$\text{Ric}(hX, hY) = fg(hX, hY), \quad \forall X, Y \in \Gamma(TM),$$

then the function f must be a constant on M .

Proof. Consider the Darboux frame field $\{\delta/\delta x^i, \partial/\partial x^0\}$ and express (2.2a), (2.2b) and (2.2d) as follows

$$(2.16) \quad R_{ijkh} = -R_{ijhk} = -R_{jikh} = R_{khij},$$

where we put

$$R_{ijkh} = R_i^t{}_{kh} g_{tj}.$$

Next, by using (2.15a) and the hypothesis from the theorem, we deduce that

$$(2.17) \quad R_{ij} = g^{hk} R_{ikjh} = fg_{ij}.$$

Now, we put

$$R_{ijkhlt} = \frac{\delta R_{ijkh}}{\delta x^t} - R_{sjkh} \Gamma_i^s{}_t - R_{iskh} \Gamma_j^s{}_t - R_{ijsh} \Gamma_k^s{}_t - R_{ijks} \Gamma_h^s{}_t,$$

and infer that

$$\left(\nabla_{\frac{\delta}{\delta x^t}} R \right) \left(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i} \right) = R_{ijkhlt}.$$

Then (2.2e) is expressed as follows

$$(2.18) \quad R_{ijkhlt} + R_{ijhtlk} + R_{ijtklh} = 0.$$

Also, since ∇ is h -metric, we have

$$(2.19) \quad (a) \ g_{ij|k} = 0, \quad \text{and} \quad (b) \ g^{ij}|_k = 0.$$

Now, contracting (2.18) by $g^{ik}g^{jh}$ and using (2.16), (2.17) and (2.19), we obtain

$$2(m-1) \frac{\delta f}{\delta x^i} = 0, \quad i \in \{1, \dots, 2m\}.$$

As $m > 1$, we deduce that

$$(2.20) \quad \frac{\delta f}{\delta x^i} = 0, \quad i \in \{1, \dots, 2m\}.$$

Finally, from (2.17) we obtain $\partial f / \partial x^0 = 0$, since both g_{ij} and R_{ij} are functions of (x^i) alone. Then, by (1.24) and (2.20), we infer that

$$\frac{\partial f}{\partial x^i} = 0, \quad \forall i \in \{1, \dots, 2m\}.$$

As M is a connected manifold, we conclude that f must be constant on M . \square

Remark 2.1. It is easy to see that f from Theorem 2.3 is given by $f = S/2m$. So, under the conditions of this theorem, we conclude that the CR scalar curvature of M is constant on M . \square

Next, we proceed with some algebraic results on quadrilinear mappings on a real $2m$ -dimensional vector space V endowed with an almost complex structure J (cf. YANO and KON [22], pp. 131–132). Suppose that

$$A : V \times V \times V \times V \rightarrow \mathbb{R},$$

is a quadrilinear mapping on V , satisfying the conditions:

$$(2.21) \quad \begin{aligned} (a) \quad & A(X, Y, Z, U) + A(Y, X, Z, U) = 0, \\ (b) \quad & A(X, Y, Z, U) + A(X, Y, U, Z) = 0, \\ (c) \quad & \sum_{(X, Y, Z)} \{A(X, Y, Z, U)\} = 0, \\ (d) \quad & A(X, Y, Z, U) = A(Z, U, X, Y), \\ (e) \quad & A(JX, JY, Z, U) = A(X, Y, JZ, JU) = A(X, Y, Z, U), \end{aligned}$$

for all $X, Y, Z, U \in V$. Then, we recall the following.

Lemma 2.1 (YANO and KON [22], p. 132). *Let A and B be two quadrilinear mappings satisfying (2.21). If $A(X, JX, X, JX) = B(X, JX, X, JX)$, $\forall X \in V$, then $A = B$.*

Now, let g be a Hermitian inner product on V . Then we consider the quadrilinear mapping

$$A_0(X, Y, Z, U) = \frac{1}{4} \{g(X, Z)g(Y, U) - g(X, U)g(Y, Z) + g(X, JZ)g(Y, JU) - g(X, JU)g(Y, JZ) + 2g(X, JY)g(Z, JU)\}.$$

It is easy to see that A_0 satisfies (2.21) and $A_0(X, JX, X, JX) = 1$, for any unit vector $X \in V$. Then we consider a complex plane Π in V , that is, Π is a real 2-dimensional subspace of V such that $J(\Pi) = \Pi$. An orthonormal basis in Π can be $\{X, JX\}$, where X is a unit vector in Π . It is easy to check that the real number, $K(\Pi) = A(X, JX, X, JX)$, does not depend on the unit vector $X \in \Pi$, where A is satisfying (2.21).

Moreover, from Lemma 2.1 we deduce the following

Lemma 2.2. *Let A be a quadrilinear mapping satisfying (2.21). If $K(\Pi) = c$ for all complex planes Π , then $A = cA_0$.*

Next, we consider a complex plane Π_x in HM_x , $x \in M$. Take a unit vector $X \in \Pi_x$ and denote by $H(\Pi_x)$ the real number $H(\Pi_x) = R_x(X, JX, X, JX)$, where R is the curvature tensor field of type $(0, 4)$ of the CR connection ∇ (see (2.1)). By Theorems 2.1 and 2.2 we see that R_x is a quadrilinear mapping on HM_x satisfying (2.21), provided M is a CR manifold and ξ is a CR Killing vector field. Thus $H(\Pi_x)$ is independent of the unit vector $X \in \Pi_x$. We call $H(\Pi_x)$ the CR sectional curvature of M at the point $x \in M$, determined by the complex plane Π_x . If $H(\Pi_x)$ is independent of both the point $x \in M$ and the complex plane Π_x , then we say that M is a CR space form. The next two theorems entitle us to claim that CR space forms are those objects in CR geometry corresponding to complex space forms from complex geometry.

Theorem 2.4 (Schur theorem for CR manifolds). *Let M be a connected real $(2m + 1)$ -dimensional CR manifold with $m > 1$ and with CR Killing vector field ξ . If the CR sectional curvature is independent of the complex planes, then M is a CR space form.*

Proof. First, we consider the horizontal tensor field R_0 of type $(0, 4)$ given by

$$(2.22) \quad \begin{aligned} R_0(hX, hY, hZ, hU) = & \frac{1}{4} \{g(hX, hZ)g(hY, hU) - g(hX, hU)g(hY, hZ) \\ & + g(hX, JhZ)g(hY, JhU) - g(hX, JhU)g(hY, JhZ) \\ & + 2g(hX, JhY)g(hZ, JhU)\}, \end{aligned}$$

for all $X, Y, Z, U \in \Gamma(TM)$. Taking into account that both R_0 and the curvature tensor field R of ∇ satisfy (2.21) at any point $x \in M$, and by using the hypothesis and Lemma 2.2, we deduce that

$$(2.23) \quad R(hX, hY, hZ, hU) = f(x)R_0(hX, hY, hZ, hU).$$

Here, $f(x)$ represents the CR sectional curvature of M at the point x , since $R_0(hX, JhX, hX, JhX) = 1$ for any unit vector $hX \in HM_x$. Then, by using (2.23), (2.22) and (2.11), we obtain

$$\text{Ric}(hX, hY) = \frac{f}{2} (m+1)g(hX, hY).$$

Finally, we apply Theorem 2.3 and deduce that f is a constant function on M . \square

From the proof of Theorem 2.4 we deduce the following interesting characterization of CR space forms.

Theorem 2.5. *A CR manifold M is a CR space form of constant CR sectional curvature c if and only if the curvature tensor field R of the CR connection ∇ is given by*

$$(2.24) \quad \begin{aligned} (a) \quad R(hX, hY)hZ = & \frac{c}{4} \{g(hY, hZ)hX - g(hX, hZ)hY \\ & + g(JhY, hZ)JhX - g(JhX, hZ)JhY \\ & + 2g(hX, JhY)JhZ\}, \\ (b) \quad R(\xi, hY)hZ = & 0, \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$.

3. Some CR differential operators

Let M be a nearly CR manifold and f be a smooth function on M . Then the CR differential of f is a horizontal covector field denoted $d_h f$ and defined by

$$(3.1) \quad d_h f(hX) = hX(f) = X^i E_i(f),$$

where $\{E_i\}$, $i \in \{1, \dots, 2m\}$, is an orthonormal basis in $\Gamma(HM)$ and $hX = X^i E_i$. This enables us to define the CR gradient of f as the horizontal vector field $\text{grad}_h f$ given by

$$(3.2) \quad g(\text{grad}_h f, hX) = d_h f(hX).$$

Then, by using (3.1) and (3.2), we obtain

$$(3.3) \quad \text{grad}_h f = \sum_{i=1}^{2m} \varepsilon_i E_i(f) E_i.$$

Moreover, we can state the following.

Proposition 3.1. *Let M be a nearly CR manifold and (x^i, x^0) be the Darboux coordinates on M . Then the CR differential and the CR gradient of f are expressed as follows:*

$$(3.4) \quad \begin{aligned} \text{(a)} \quad d_h f &= \frac{\delta f}{\delta x^j} dx^j, \\ \text{(b)} \quad \text{grad}_h f &= g^{jk} \frac{\delta f}{\delta x^j} \frac{\delta}{\delta x^k}. \end{aligned}$$

Proof. First, by using (3.1) and (2.13b), we obtain

$$d_h f(E_i) = B_i^j \frac{\delta f}{\delta x^j}.$$

Then we consider the dual frame field

$$\left\{ dx^i, \delta x^0 = dx^0 - \sum_{a=1}^m x^{m+a} dx^a \right\}$$

to the Darboux frame field $\{\delta/\delta x^i, \partial/\partial x^0\}$, and deduce that

$$\frac{\delta f}{\delta x^j} dx^j(E_i) = \frac{\delta f}{\delta x^j} B_i^k dx^j \left(\frac{\delta}{\delta x^k} \right) = B_i^j \frac{\delta f}{\delta x^j}.$$

Thus (3.4a) is proved. Next, by using (3.3), (2.13b) and (2.14b), we obtain

$$\text{grad}_h f = \left(\sum_{i=1}^{2m} \varepsilon_i B_i^j B_i^k \right) \frac{\delta f}{\delta x^j} \frac{\delta}{\delta x^k} = g^{jk} \frac{\delta f}{\delta x^j} \frac{\delta}{\delta x^k},$$

which proves (3.4b). \square

Now, we consider a horizontal vector field hX and a symmetric horizontal tensor field A of type $(0, 2)$ on a nearly CR manifold. Then we define the CR divergence of hX and A as follows

$$(3.5) \quad \begin{aligned} \text{(a)} \quad \text{div}_h(hX) &= \sum_{i=1}^{2m} \varepsilon_i g(\nabla_{E_i} hX, E_i), \\ \text{(b)} \quad (\text{div}_h A)(hY) &= \sum_{i=1}^{2m} \varepsilon_i (\nabla_{E_i} A)(E_i, hY), \quad \forall Y \in \Gamma(TM), \end{aligned}$$

where ∇ is the CR connection on M . Then, we put

$$(3.6) \quad \text{(a)} \quad hX = X^i \frac{\delta}{\delta x^i}, \quad \text{(b)} \quad A_{ij} = A \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right), \quad \text{(c)} \quad A = A_{ij} dx^i \otimes dx^j,$$

and state the following.

Proposition 3.2. *Let M be a nearly CR manifold. Then we have*

$$(3.7) \quad \text{(a)} \quad \text{div}_h(hX) = X^i \lrcorner_i, \quad \text{(b)} \quad \text{div}_h A = g^{ij} A_{jkl} dx^k = A^i \lrcorner_{kl} dx^k,$$

where " \lrcorner " denotes the covariant derivative defined by ∇ .

Proof. First, by using (3.5a), (2.13b), (2.14b) and (1.26a), we obtain

$$\begin{aligned} \text{div}_h(hX) &= \left(\sum_{i=1}^{2m} \varepsilon_i B_i^j B_i^k \right) g \left(\nabla_{\frac{\delta}{\delta x^j}} \left(X^h \frac{\delta}{\delta x^h} \right), \frac{\delta}{\delta x^k} \right) \\ &= g^{jk} g \left(X^h \lrcorner_j \frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k} \right) = X^j \lrcorner_j. \end{aligned}$$

Similarly, by using (3.5b), (2.13b), (2.14b), (3.6b) and (2.19b), we deduce that

$$(\text{div}_h A) \left(\frac{\delta}{\delta x^i} \right) = g^{jk} \left(\nabla_{\frac{\delta}{\delta x^j}} A \right) \left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i} \right) = g^{jk} A_{kij} = A^j \lrcorner_{ij}.$$

Thus, the proof is complete. \square

Finally, for a smooth function f on a nearly CR manifold M , we define the CR Laplacian $\Delta_h f$ by

$$(3.8) \quad \Delta_h f = \operatorname{div}_h(\operatorname{grad}_h f).$$

Then, by using (3.3), (3.5a) and (3.8), we obtain

$$(3.9) \quad \Delta_h f = \sum_{i=1}^{2m} \varepsilon_i \{E_i^2 + \operatorname{div}(E_i)E_i\}(f).$$

Now, consider the Darboux coordinates (x^i, x^0) on M and, by using (3.4b), (3.7a) and (2.19b), we deduce that

$$(3.10) \quad \begin{aligned} \Delta_h f &= \left(g^{jk} \frac{\delta f}{\delta x^j} \right)_{|k} = g^{jk} \left\{ \frac{\delta}{\delta x^k} \left(\frac{\delta f}{\delta x^j} \right) - \Gamma_{j^h k} \frac{\delta f}{\delta x^h} \right\} \\ &= g^{jk} \left\{ \frac{\delta^2 f}{\delta x^k \delta x^j} - \Gamma_{j^h k} \frac{\delta f}{\delta x^h} \right\}. \end{aligned}$$

If, in particular, $\{\delta/\delta x^i\}$, $i \in \{1, \dots, 2m\}$, is an orthonormal basis with respect to g , then from (1.29a) we deduce that $\Gamma_{j^h k} = 0$ for all $h, j, k \in \{1, \dots, 2m\}$. Hence, in this case, the CR Laplacian is expressed as follows

$$(3.11) \quad \Delta_h f = \sum_{i=1}^{2m} \frac{\delta^2 f}{(\delta x^i)^2}.$$

Remark 3.1. We should observe a striking similitude between the formulas (3.4), (3.7) and (3.10) and the well-known formulas for gradient, divergence and Laplacian from Riemannian geometry, respectively. The only difference is that the natural vector fields $\partial/\partial x^i$ and the Levi-Civita connection from Riemannian geometry, are replaced in CR geometry by the nonholonomic vector fields $\delta/\delta x^i$ and the CR connection, respectively. \square

Remark 3.2. All the formulas we present in the paper by using the Darboux coordinates are still valid with respect to any coordinate system on M induced by the flow of ξ . \square

4. An application to Penrose correspondence

First, we prove a general result on the CR Ricci tensor field on a nearly CR manifold.

Theorem 4.1. *Let M be a nearly CR manifold such that ξ is a CR Killing tensor field. Then the CR Ricci tensor field and the CR scalar curvature of M satisfy the identity*

$$(4.1) \quad 2 \operatorname{div}_h \operatorname{Ric} = d_h S.$$

Proof. First, by using (2.19a) into (2.18), we deduce that $R_i^h{}_{jk}{}_{lh} + R_i^h{}_{khlj} - R_i^h{}_{jh}{}_{lk} = 0$, which by (2.15a) is equivalent to

$$(4.2) \quad R_i^h{}_{jk}{}_{lh} + R_{ik}{}_{lj} - R_{ij}{}_{lk} = 0.$$

Contracting (4.2) by g^{ij} and using (2.19b), (2.15b) and (2.16), we infer that

$$(4.3) \quad 2R_i^h{}_{lh} = S_{lh} = \frac{\delta S}{\delta x^h}.$$

Finally, (4.3) represents just (4.1) via (3.4a) and (3.7b). \square

Next, we present the Penrose correspondence and then show an application of the above theorem to the physical phenomenon described by this correspondence.

Let V be a 4-dimensional complex vector space and let (z^0, z^1, z^2, z^3) be the coordinates of a vector z of V . Suppose that V is endowed with the Hermitian form Φ of signature $(2, 2)$ given by $\Phi(z) = z^0 \bar{z}^2 + z^1 \bar{z}^3 + z^2 \bar{z}^0 + z^3 \bar{z}^1$. The pair (V, Φ) is called the *twistor space* and plays an important role in physics (cf. PENROSE [13], [14] and Wells [19], [20]). Now, we consider the real 4-dimensional Minkowski space M with a flat semi-Riemannian metric of Lorentzian signature $(1, 3)$. For the real coordinates (x^0, x^1, x^2, x^3) of $x \in M$ we consider the coordinates

$$\begin{aligned} u &= \frac{1}{\sqrt{2}} (x^0 + x^1), & v &= \frac{1}{\sqrt{2}} (x^0 - x^1), \\ \alpha &= \frac{1}{\sqrt{2}} (x^2 + ix^3), & \bar{\alpha} &= \frac{1}{\sqrt{2}} (x^2 - ix^3), \end{aligned}$$

on $M \otimes C$. Then the points $z \in V$ and $x \in M$ are said to be incident if we have

$$(4.4) \quad (z^2, z^3) = \frac{1}{i\sqrt{2}} (z^0, z^1) \begin{bmatrix} u & \alpha \\ \bar{\alpha} & v \end{bmatrix},$$

which can only hold if $\Phi(z) = 0$.

Now, we denote by PV the complex 3-dimensional projective space associated with V . Then consider the open complex submanifolds

$$\begin{aligned} PV_+ &= \{z \in PV : \Phi(z) > 0\}, & \text{and} \\ PV_- &= \{z \in PV : \Phi(z) < 0\}, \end{aligned}$$

and the real 5-dimensional manifold

$$PV_0 = \{z \in PV : \Phi(z) = 0\}.$$

As PV_0 is the common boundary of PV_+ and PV_- , it inherits a natural CR structure as a real hypersurface of the complex manifold PV . We have to note that PV_0 contains the points of PV for which (4.4) holds for a certain x , but not only them. As we can easily see, the points of the projective line L , given by $z^0 = z^1 = 0$ admit no solution for x in (4.4). Thus, for each $z \in PV_0 \setminus L$, the equation (4.4) can be solved for $x \in M$. Moreover, all the solutions of this equation are null vectors in M , and therefore the points x that are incident with a given z form a null geodesic. Thus $PV_0 \setminus L$ can be considered as the space of null geodesics in M . On the other hand, for each $x \in M$ fixed in (4.4), there exists a complex projective line in $PV_0 \setminus L$ and each line lying entirely in $PV_0 \setminus L$ is obtained from points in M . Therefore, we have the *Penrose correspondence*

$$\begin{aligned} &\{\text{points of the Minkowski space } M\} \\ &\quad \Downarrow \\ &\{\text{projective lines lying entirely in } PV_0 \setminus L\}. \end{aligned}$$

Next, we put $PV_0^* = PV_0 \setminus L$ and consider the CR structure on PV_0^* induced by the complex structure of PV . Moreover, we suppose that PV_0^* satisfies the hypothesis in Theorem 4.1, that is, ξ is a CR Killing vector field. Then we define the *CR Einstein gravitational tensor field*

$$(4.5) \quad G = \text{Ric} - \frac{S}{2} g,$$

where Ric and S are the CR Ricci tensor and the CR scalar curvature on PV_0^* , respectively. Now, we prove the following.

Theorem 4.2. *The CR Einstein gravitational tensor field is a symmetric horizontal tensor field of type $(0, 2)$ on the CR manifold PV_0^* and its divergence vanishes identically on PV_0^* .*

Proof. As Ric and g are both symmetric horizontal tensor fields of type $(0, 2)$, from (4.5) we deduce that G is so. Then, by direct calculations using (4.5), (3.7b), (2.19a) and (4.3), we obtain

$$\text{div}_h G = \text{div}_h \text{Ric} - \frac{1}{2} \text{div}_h (Sg) = R_k^i \lrcorner_i dx^k - \frac{1}{2} S \lrcorner_k dx^k = 0.$$

Thus the proof is complete. \square

As it is well known, the general relativity flows from the following law.

”If M is a spacetime containing matter with stress-energy tensor T , then

$$(4.6) \quad G = 8\pi T,$$

where G is the Einstein gravitational tensor field on M .”

Recall that G from (4.6) is given by (4.5), but Ric and S are the Ricci tensor and the scalar curvature of the Levi-Civita connection on the spacetime M . Also, we note that the Einstein equation (4.6) is considered on the tangent bundle TM of M , and it is mainly based on the curvature of M . Thus, if M is a relativistic model (Robertson-Walker spacetime, Schwarzschild spacetime etc.), then the Einstein equation is considered on the tangent bundles of such models.

However, the theory presented in this section shows that Einstein equations can be considered on the vector bundle $\Pi : HPV_0^* \rightarrow PV_0^*$, where PV_0^* is the CR manifold from Penrose correspondence and HPV_0^* is the horizontal distribution on PV_0^* . In this way, we have a relativistic model which grows up directly from a Minkowski space via Penrose correspondence. Moreover, the Einstein equations on HPV_0^* can be thought as induced equations from Einstein equations on the 5-dimensional manifold PV_0^* . In this way, the study can be related to Kaluza-Klein theories for $5D$. We hope that in the future we will come with more details about a relativistic theory on the above vector bundle and based on the curvature tensor field of the CR connection.

5. Examples

In this section we present two examples of CR manifolds on which ξ is a CR Killing vector field. Both CR manifolds have constant CR sectional curvature, so they are examples of CR space forms. Also, we present explicitly the CR differential operators we defined in Section 3.

Example 5.1 (Heisenberg group). Let $M = \mathbb{R}^{2m+1}$ considered as a Lie group with the noncommutative law

$$\begin{aligned} x * \bar{x} &= (x^a, x^{m+a}, x^0) * (\bar{x}^a, \bar{x}^{m+a}, \bar{x}^0) \\ &= \left(x^a + \bar{x}^a, x^{m+a} + \bar{x}^{m+a}, x^0 + \bar{x}^0 + \frac{1}{2} \sum_{a=1}^m (x^a \bar{x}^{m+a} - \bar{x}^a x^{m+a}) \right). \end{aligned}$$

Then $(M, *)$ is called the *Heisenberg group* and it is one of the important examples of sub-Riemannian manifolds (cf. DANIELLY-GAROFALO-NHIEU [7]). Denote by $(L_x)_*$ the differential of the left translation defined by $x \in M$, and take the real distribution HM of rank $2m$ spanned by the vector fields

$$(5.1) \quad \begin{aligned} \frac{\delta}{\delta x^a} &= (L_x)_* \left(\frac{\partial}{\partial x^a} \right) = \frac{\partial}{\partial x^a} - \frac{x^{m+a}}{2} \frac{\partial}{\partial x^0}, \\ \frac{\delta}{\delta x^{m+a}} &= (L_x)_* \left(\frac{\partial}{\partial x^{m+a}} \right) = \frac{\partial}{\partial x^{m+a}} + \frac{x^a}{2} \frac{\partial}{\partial x^0}, \quad a \in \{1, \dots, m\}. \end{aligned}$$

The only nonzero Lie brackets of vector fields in (5.1) are the following

$$(5.2) \quad \left[\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^{m+b}} \right] = \delta_{ab} \frac{\partial}{\partial x^0}.$$

Now, we define the natural almost complex structure J on HM by

$$(5.3) \quad J \left(\frac{\delta}{\delta x^a} \right) = -\frac{\delta}{\delta x^{m+a}}, \quad J \left(\frac{\delta}{\delta x^{m+a}} \right) = \frac{\delta}{\delta x^a}.$$

Then, by direct calculations using (5.2) and (5.3), we obtain

$$\begin{aligned} \left[\frac{\delta}{\delta x^i}, J \left(\frac{\delta}{\delta x^j} \right) \right] + \left[J \left(\frac{\delta}{\delta x^i} \right), \frac{\delta}{\delta x^j} \right] &= 0, \\ \left[J \left(\frac{\delta}{\delta x^i} \right), J \left(\frac{\delta}{\delta x^j} \right) \right] - \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] &= 0, \quad \forall i, j \in \{1, \dots, 2m\}. \end{aligned}$$

Thus the conditions in (1.1) are satisfied, and therefore the Heisenberg group M is a CR manifold.

Alternatively, HM is the kernel of the 1-form

$$(5.4) \quad \eta = dx^0 + \frac{1}{2} \sum_{a=1}^m (x^{m+a} dx^a - x^a dx^{m+a}).$$

Then, η is a contact form with

$$d\eta = - \sum_{a=1}^m dx^a \wedge dx^{m+a}.$$

The real Levi form is given by

$$g(X, Y) = d\eta(X, JY) = \sum_{a=1}^m (X^a Y^a + X^{m+a} Y^{m+a}),$$

where

$$X = X^a \frac{\delta}{\delta x^a} + X^{m+a} \frac{\delta}{\delta x^{m+a}}, \quad Y = Y^a \frac{\delta}{\delta x^a} + Y^{m+a} \frac{\delta}{\delta x^{m+a}}.$$

It is easy to see that

$$(5.5) \quad g_{ij} = g \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = \delta_{ij}, \quad \forall i, j \in \{1, \dots, 2m\}.$$

Taking into account (5.5), we deduce that (1.34) is satisfied, so ξ is a *CR* Killing vector field. Also, by using (5.5) and (5.1) into (1.29), we obtain

$$(5.6) \quad \Gamma_i^k{}_j = 0, \quad \forall i, j, k \in \{1, \dots, 2m\}.$$

Thus by (1.39a) we infer that $R_i^h{}_{jk} = 0$, and therefore we can state the following.

Proposition 5.1. *The Heisenberg group is a CR space form of constant CR sectional curvature $c = 0$.*

Thus we may say that *the Heisenberg group is for CR geometry exactly what the Euclidean space \mathbb{R}^{2m} is for Kähler geometry.*

Next, we write down explicitly the *CR* differential operators which have been defined in Section 3. First, by using (3.4), (5.1) and (5.4), and taking into account that $g^{ij} = \delta^{ij}$, we deduce that

$$d_h f = df - \frac{\partial f}{\partial x^0} \eta \quad \text{and} \quad \text{grad}_h f = \sum_{i=1}^{2m} \frac{\delta f}{\delta x^i} \frac{\delta}{\delta x^i}.$$

Then, we take a horizontal vector field $hX = X^i \delta / \delta x^i$, and by using (3.7a) and (5.6) we obtain

$$\operatorname{div}_h(hX) = \sum_{i=1}^{2m} \frac{\delta X^i}{\delta x^i}.$$

Finally, by using (5.6) into (3.10) and taking into account that $g^{jk} = \delta^{jk}$, we deduce that the CR Laplace operator on the Heisenberg group is given by

$$\Delta_h f = \sum_{i=1}^{2m} \frac{\delta^2 f}{(\delta x^i)^2}.$$

The above formulas entitle us to say that the CR operators on Heisenberg group are obtained from usual operators on \mathbb{R}^{2m} by replacing the partial derivatives $\partial / \partial x^i$ by nonholonomic derivatives $\delta / \delta x^i$. \square

Example 5.2 (The sphere S^3). Let S^3 be the 3-dimensional sphere in \mathbb{R}^4 given by the equation $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1$. As it is well known, S^3 is a parallelizable manifold, and

$$(5.7) \quad \begin{aligned} E_1 &= x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}, \\ E_2 &= x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^4}, \\ \xi &= 2 \left(x^4 \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3} - x^1 \frac{\partial}{\partial x^4} \right), \end{aligned}$$

are the linearly independent vector fields globally defined on S^3 . Consider the distribution HS^3 on S^3 spanned by $\{E_1, E_2\}$ and define the almost complex structure $J : HS^3 \rightarrow HS^3$ by

$$(5.8) \quad J(E_1) = E_2, \quad J(E_2) = -E_1.$$

It is easy to see that (1.1) is satisfied and therefore S^3 is a CR manifold.

Alternatively, HS^3 is given by the contact form

$$\eta = \frac{1}{2} (x^4 dx^1 - x^3 dx^2 + x^2 dx^3 - x^1 dx^4).$$

Then we have

$$(5.9) \quad d\eta = -dx^1 \wedge dx^4 - dx^3 \wedge dx^2,$$

and by using (1.4), (5.9) and (5.8) we deduce that the Levi form is given by

$$(5.10) \quad g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_1, E_2) = 0.$$

It is easy to check that g given by (5.10) is just the induced metric on HS^3 by the Euclidean metric of \mathbb{R}^4 . Also, we have

$$(5.11) \quad (a) [E_1, E_2] = -\xi, \quad (b) [\xi, E_1] = -4E_2, \quad (c) [\xi, E_2] = 4E_1.$$

Then, by using (1.9), (5.10) and (5.11), we deduce that the CR connection ∇ on HS^3 is given by

$$(5.12) \quad \begin{aligned} (a) \quad & \nabla_{E_1} E_1 = \nabla_{E_2} E_2 = \nabla_{E_1} E_2 = \nabla_{E_2} E_1 = 0, \\ (b) \quad & \nabla_{\xi} E_1 = -4E_2, \quad (c) \quad \nabla_{\xi} E_2 = 4E_1. \end{aligned}$$

By using (5.11b), (5.11c) and (5.10) in (1.31), we infer that $\mathcal{L}_{\xi} g = 0$, that is, ξ is a CR Killing vector field. Also, by using (1.36), (5.8), (5.11a) and (5.12c), we obtain $R(E_1, JE_1)JE_1 = 4E_1$. Then, by using (2.1), we deduce that $R(E_1, JE_1, E_1, JE_1) = 4$, which enables us to state the following.

Proposition 5.2. *The sphere S^3 is a CR space form of constant CR sectional curvature $c = 4$.*

Finally, by using (3.5a) and (5.12a), we obtain

$$\operatorname{div}_h(X^1 E_1 + X^2 E_2) = E_1(X^1) + E_2(X^2).$$

In particular, we have $\operatorname{div}_h(E_1) = \operatorname{div}_h(E_2) = 0$, and therefore by (3.9) the CR Laplacian on the sphere S^3 becomes $\Delta_h f = \{(E_1)^2 + (E_2)^2\}(f)$. \square

REFERENCES

1. BEJANCU, A. – *Geometry of CR-Submanifolds*, Mathematics and its Applications (East European Series), 23, D. Reidel Publishing Co., Dordrecht, 1986.
2. BEJANCU, A.; FARRAN, H.R. – *Foliations and Geometric Structures*, Mathematics and Its Applications (Springer), 580, Springer, Dordrecht, 2006.
3. BLAIR, D.E. – *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2002.

4. BLAIR, D.E.; DRAGOMIR, S. – *Pseudohermitian geometry on contact Riemannian manifolds*, Rend. Mat. Appl., 22 (2002), 275–341 (2003).
5. CHERN, S.S.; HAMILTON, R.S. – *On Riemannian metrics adapted to three-dimensional contact manifolds*, Lecture Notes in Math., 1111, Workshop Bonn 1984 (Bonn, 1984), 279–308, Springer, Berlin, 1985.
6. CHIRKA, E.M. – *An introduction to the geometry of CR manifolds*, Uspekhi Mat. Nauk, 46 (1991), 1 (277), 81–164, 240; translation in Russian Math. Surveys 46 (1991), no. 1, 95–197.
7. DANIELLI, D.; GAROFALO, N.; NHIEU, D.M. – *Sub-Riemannian calculus on hypersurfaces in Carnot groups*, Adv. Math., 215 (2007), 292–378.
8. DRAGOMIR, S.; TOMASSINI, G. – *Differential Geometry and Analysis on CR Manifolds*, Progress in Mathematics, 246, Birkhäuser Boston, Inc., Boston, MA, 2006.
9. GREENFIELD, S.J. – *Cauchy-Riemann equations in several variables*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 275–314.
10. JACOBOWITZ, H. – *An introduction to CR structures*, Mathematical Surveys and Monographs, 32, American Mathematical Society, Providence, RI, 1990.
11. KOBAYASHI, S.; NOMIZU, K. – *Foundations of differential geometry*, Vol I. Interscience Publishers, New York-London, 1963.
12. O’NEILL, B. – *Semi-Riemannian geometry. With applications to relativity*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.
13. PENROSE, R. – *The complex geometry of the natural world*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), 189–194, Acad. Sci. Fennica, Helsinki, 1980.
14. PENROSE, R. – *Physical space-time and nonrealizable CR-structures*, Bull. Amer. Math. Soc. (N.S.), 8 (1983), 427–448.
15. POINCARÉ, H. – *Les fonctions analytiques de deux variables et la représentation conforme*, Rend. Circ. Mat. Palermo, 1907, 185–220.
16. TANAKA, N. – *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan, 14 (1962), 397–429.
17. TANAKA, N. – *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math. (N.S.), 2 (1976), 131–190.
18. WEBSTER, S.M. – *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geom., 13 (1978), 25–41.
19. WELLS, R.O., JR. – *Complex manifolds and mathematical physics*, Bull. Amer. Math. Soc. (N.S.), 1 (1979), 296–336.
20. WELLS, R. O., JR. – *The Cauchy-Riemann equations and differential geometry*, Bull. Amer. Math. Soc. (N.S.), 6 (1982), 187–199.
21. YANO, K.; KON, M. – *CR submanifolds of Kaehlerian and Sasakian manifolds*, Progress in Mathematics, 30, Birkhäuser, Boston, Mass., 1983.

22. YANO, K.; KON, M. – *Structures on manifolds*, Series in Pure Mathematics, 3, World Scientific Publishing Co., Singapore, 1984.

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