ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIX, 2013, f.1

STRONGLY TRANSITIVE GEOMETRIC SPACES ASSOCIATED WITH (m, n)-ARY HYPERMODULES

BY

S.M. ANVARIYEH and B. DAVVAZ

Abstract. In this paper, we define the strongly compatible relation ϵ on the (m, n)-ary hypermodule M, so that the quotient $(M/\epsilon^*, h/\epsilon^*)$ is an (m, n)-ary module over the fundamental (m, n)-ary ring $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$. Also, we determine a family $P(M)$ of subsets of an (m, n)-ary hypermodule M and we give a sufficient condition such that the geometric space $(M, P(M))$ is strongly transitive.
Mathematics Subject Classification 2010: 16Y99, 20N20.
Key words: hypergroup, hyperring, equivalence relation, compatible relation, (m, n)-ary hypermodule.

1. Introduction

Hyperalgebras and power algebras are pairs $(A; (f_i)_{i\in I})$ consisting of a set A and an indexed or non-indexed set of operations $f_i : A \times \ldots \times A \to \mathcal{P}^*(A)$ in the first and $f_i : A \times \ldots \times A \to \mathcal{P}(A)$ in the second case. Here $\mathcal{P}(A)$ is the power set of A and $\mathcal{P}^*(A) = \mathcal{P}(A) \setminus \{\emptyset\}$. The general theory of hyperalgebras, poweralgebras, hyper-coalgebras and power co-algebras can be studied as application of (F_1, F_2)-systems where F_1 and F_2 are appropriate set-valued functors [6]. There are applications in several branches of mathematics and in computer science. For instance, hyperalgebras are used to prove that any non-deterministic automaton is equivalent to a deterministic one. n-ary groups and n-ary semigroups are algebras with one n-ary operation which is associative and invertible (in the first case) in a generalized sense. The idea of investigations of n-ary algebras seems to be going back to Kasner's lecture [11] at the 53rd annual meeting of the

American Association of the Advancement of Science in 1904. But the first paper concerning the theory of n-ary groups was written (under inspiration of Emmy Noether) by DÖRNTE in 1928 (see [7]). Since then many papers concerning various n-ary algebras have appeared in the literature, for example see [8, 9, 17].

n-hyperstructures, recently introduced by DAVVAZ and VOUGIOUKLIS (see [5]) are a nice generalization of the algebraic hyperstructures, which have been studied since 70 years (see [7]), both on the theoretical point of view and for the richness of their applications, especially to computer sciences, but also to fuzzy set theory, graphs and hypergraphs, geometry and others (see [1, 2, 3, 4, 12, 13, 16]).

A generalization of ordinary hyperstructures is studied in this paper, namely (m, n)-ary hypermodules. Using this notion and the concept of geometric spaces, we prove that the fundamental relation ϵ on an (m, n)-ary hypermodule is transitive.

Let H be a non-empty set and h be a mapping $h : H^n \to \wp^*(H)$ where $\wp^*(H)$ is the set of all non-empty subsets of H and H^n the cartesian product $H \times \ldots \times H$, where appears n times and an element of H^n will be denoted by $(x_1, \ldots, x_n)$, such that $x_i \in H$ for any i with $1 \leq i \leq n$. In general, a mapping $h : H^n \to \wp^*(H)$ is called an n-ary hyperoperation and n is called the arity of hyperoperation.

In the following we shall denote the sequence $x_i, x_{i+1}, \ldots, x_j$ by x_i^j. For $j < i$, x_i^j is the empty set. In this convention $h(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, x_{j+1}, \ldots, x_n)$ will be written $h(x_1^i, y_{i+1}^j, x_{j+1}^n)$ and $h(a, \ldots, a)$ denoted by $h(a^{(n)})$.

Let h be an n-ary hyperoperation on H and $A_1, \ldots, A_n$ be non-empty subsets of H. We define $h(A_1^n) := h(A_1, \ldots, A_n) = \bigcup\{h(x_1^n) | x_i \in A_i, i = 1, \ldots, n\}$.

If h is an n-ary groupoid and $t{=}l(n{-}1){+}1$, then the t-ary hyperoperation $h_{(l)}$ given by $h_{(l)}(x_1^{l(n-1)+1}){=}h(h(\ldots, h(h(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(l-1)(n-1)+2}^{l(n-1)+1})$, will be denoted by $h_{(l)}$.

A non-empty set H with an n-ary hyperoperation $h : H^n \to \wp^*(H)$ is called an n-ary hypergroupoid and it is denoted by (H, h). An n-ary hypergroupoid (H, h) is an n-ary semihypergroup if and only if the following associative axiom holds:

$$h(x_1^{i-1}, h(x_i^{n+i-1}), x_{n+i}^{2n-1})) = h(x_1^{j-1}, h(x_j^{n+j-1}), x_{n+j}^{2n-1})),$$

for every $i,j \in \{1,2,\dots,n\}$ and $x_1,x_2,\dots,x_{2n-1} \in H$. An n-ary hypersemigroup (H,h), in which the equation $b \in h(a_1^{i-1},x_i,a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1,\dots,a_{i-1},a_{i+1},\dots,a_n,b \in H$ and $1 \le i \le n$, is called n-ary hypergroup.

An (m,n)-ary hyperring [15] is an algebraic hyperstructure $< R,f,g >$, which satisfies the following axioms:

(1) (R,f) is an m-ary hypergroup,

(2) (R,g) is an n-ary hypersemigroup,

(3) the n-ary hyperoperation g is distributive with respect to the m-ary hyperoperation f, i.e., $g(a_1^{i-1},f(x_1^m),a_{i+1}^n) = f(g(a_1^{i-1},x_1,a_{i+1}^n),\dots, g(a_1^{i-1},x_m,a_{i+1}^n))$, for every $a_1^{i-1},a_{i+1}^n,x_1^m \in R,\ 1 \le i \le n$.

$< R,f,g >$ is called an m-ary hyperring if $m = n$. An m-ary hyperring R is a hyperring if $m = 2$.

Example 1. Let $(R,+,\cdot)$ be a hyperring. Let f be an m-ary hyperoperation and g be an n-ary operation on R as follows:

$$f(x_1^m) = \sum_{i=1}^m x_i, \quad \forall x_1^m \in R,$$
$$g(x_1^n) = \prod_{i=1}^n x_i, \quad \forall x_1^n \in R,$$

then (R,f,g) is a (m,n)-hyperring and denoted by (R,f,g)=$der_{(m,n)}(R,+,\cdot)$.

2. ϵ-relation on (m,n)-ary hypermodules

A non-empty set $M = (M,h,k)$ is an (m,n)-ary hypermodule over an (m,n)-ary hyperring R, if (M,h) is an m-ary hypergroup and there exists the n-ary hyperoperation

$$k : \underbrace{R \times \ldots \times R}_{n-1} \times M \to \wp^*(M)$$

such that

(1) $k(r_1^{n-1},h(x_1^m)) = h(k(r_1^{n-1},x_1),\dots,k(r_1^{n-1},x_m))$,

(2) $k(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x) = h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))$,

(3) $k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x) = k(r_1^{n-1}, k(r_n^{2n-2}, x))$.

If k is a scalar n-ary hyperoperation on M, $S_1, \ldots, S_{n-1}$ be non-empty subsets of R and $M_1 \subseteq M$, we set

$$k(S_1, \ldots, S_{n-1}, M_1) = \bigcup \{k(r_1, \ldots, r_{n-1}, x) \mid r_i \in S_i,\ i = 1, \ldots, n-1,\ x \in M_1\}.$$

An (m, n)-ary hypermodule M is an R-hypermodule, if $m = n = 2$.

Example 2. Let $M = \{0, 1, 2\}$ and $(R, f, g) = der_{(3,2)}(\mathbb{Z}, +, \cdot)$ (see Example 1). We define the commutative hyperoperations h and k as follows:

$$\begin{array}{l} h(0,0,0) = h(0,0,2) = h(0,2,2) = h(2,2,2) = \{0,2\}, \\ h(0,0,1) = h(0,2,1) = h(2,2,1) = 1, \\ h(0,1,1) = h(2,1,1) = \{0,2\}, h(1,1,1) = 1, \end{array}$$

and $k : R \times M \to \wp^*(M)$,

$$k(r, x) = \begin{cases} \{0, 2\}, & \text{if } \ r \in 2\mathbb{Z} \ \text{ or } \ x \in \{0, 2\} \\ 1, & \text{else} \end{cases}$$

Then (M, h, k) is an $(3, 2)$-ary hypermodule over $(3, 2)$-ary hyperring (R, f, g).

Example 3. Let $(R, +, \cdot)$ be a hyperring and $(M, +)$ be an R-hypermodule. If N is a subhypermodule of M, then we set:

$$h(x_1^m) = \sum_{i=1}^{m} x_i + N, \ \ \forall\ x_1^m \in M,$$

$$f(r_1^m) = \sum_{i=1}^{m} r_i, \ \ \forall\ r_1^m \in R,$$

$$g(x_1^n) = \prod_{i=1}^{n} r_i, \ \ \forall\ r_1^n \in R,$$

$$k(r_1^{n-1}, x) = (\sum_{i=1}^{n-1} r_i) \cdot x + N, \ \ \forall\ r_1^{n-1} \in R, \ \forall x \in M.$$

Then $M = (M, h, k)$ is an (m, n)-ary hypermodule over (m, n)-ary hyperring R.

Definition 2.1. Let $M = (M, h, k)$ be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R. An equivalence relation ρ on M is called compatible if $a_1 \ \rho \ b_1, \dots, \ a_m \ \rho \ b_m$, implies that for all $a \in h(a_1, \dots, a_m)$ there exists $b \in h(b_1, \dots, b_m)$ such that $a\rho b$, and if $r_1, \dots, r_{n-1} \in R$, and $x\rho y$, then for all $a \in k(r_1, \dots, r_{n-1}, x)$ there exists $b \in k(r_1, \dots, r_{n-1}, y)$ such that $a\rho b$.

Let $M = (M, h, k)$ be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R and ρ be an equivalence relation on M. Then ρ is a strongly compatible relation if $a_i \rho b_i$, for all $1 \leq i \leq m$ implies that $h(a_1, \dots a_m) \overline{\overline{\rho}} \, h(b_1, \dots, b_m)$, and for every $r_1, \dots, r_{n-1} \in R$ and $x\rho y$, then $k(r_1, \dots, r_{n-1}, x) \overline{\overline{\rho}} k(r_1, \dots, r_{n-1}, y)$.

We recall the following Theorem from [14].

Theorem 2.2. *Let (H, f) be an m-ary hypergroup and let ρ be an equivalence relation on H. Then the relation ρ is strongly compatible if and only if the quotient $(H/\rho, f/\rho)$ is an m-ary group.*

Now, we study the strong compatible relation Γ on an (m, n)-ary hyperring R.

Definition 2.3. Let (R, f, g) be an (m, n)-ary hyperring. For every $k \in \mathbb{N}$ and $l_1^s \in \mathbb{N}$, where $s = k(m-1)+1$, we define the relation $\Gamma_{k;l_1^s}$, as follows:

$x \ \Gamma_{k;l_1^s} \ y$ if and only if there exist $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1)+1$, $i = 1, \dots, s$ such that $\{x, y\} \subseteq f_{(k)}(u_1, \dots, u_s)$, where for every $i = 1, \dots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$.

Now, set $\Gamma_k = \bigcup_{l_1^s \in \mathbb{N}} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup_{k \in \mathbb{N}} \Gamma_k$. Then the relation Γ is reflexive and symmetric. Let Γ^* be the transitive closure of relation Γ.

Theorem 2.4 ([15]). *The relation Γ^* is a strongly compatible relation both on m-ary hypergroup (R, f) and n-ary semihypergroup (R, g) and the quotient $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an (m, n)-ary ring.*

We define $h/\rho(\rho(a_1), \dots, \rho(a_m)) := \{\rho(a) | a \in h(a_1, \dots, a_m)\} = \rho(h(a_1^m))$ and $k/\rho(r_1, \dots, r_{n-1}, \rho(a)) := \{\rho(x) | x \in k(r_1, \dots, r_{n-1}, a)\} = \rho(k((r_1^{n-1}, a))$.

Theorem 2.5. *Let $M = (M, h, k)$ be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R and ρ be an equivalence relation on M. Then the following conditions are equivalent.*

(1) *The relation ρ is strongly compatible.*

(2) *If* $r_1,\ldots,r_{n-1}\in R$, $x_1^m, a, b\in M$ *and* $a\rho b$, *then for every* $(i=1,\ldots,m)$, *we have* $h(x_1^{i-1},a,x_{i+1}^m)\ \overline{\overline{\rho}}\ h(x_1^{i-1},b,x_{i+1}^m)$ *and* $k(r_1,\ldots,r_{n-1},a)\ \overline{\overline{\rho}}\ k(r_1,\ldots,r_{n-1},b)$.

(3) *The quotient* $(M/\rho, h/\rho, k/\rho)$ *is an* (m,n)*-ary module over an* (m,n)*-ary ring* R. *In the other words,* $(M/\rho, h/\rho)$ *is an* m*-ary group, and the scalar* n*-ary hyperoperation* k *is singleton.*

Proof. We show that (2) $\Leftrightarrow$ (1) $\Leftrightarrow$ (3).
(1) $\Rightarrow$ (2) It is straightforward.
(2) $\Rightarrow$ (1) Let $a_i\ \rho\ b_i$, where $i=1,\ldots,m$. By (2) we have

$$\begin{array}{rll} h(a_1,\ldots,a_m) & \overline{\overline{\rho}} & h(a_1,\ldots,a_{m-1},b_m) \\ & \overline{\overline{\rho}} & h(a_1,\ldots,a_{m-2},b_{m-1},b_m) \\ & \vdots & \\ & \overline{\overline{\rho}} & h(a_1,b_2,\ldots,b_m) \\ & \overline{\overline{\rho}} & h(b_1,\ldots,b_m). \end{array}$$

Since $\overline{\overline{\rho}}$ is transitive, thus $\overline{\overline{\rho}}$ is strongly compatible for h. Now, let $r_1,\ldots,r_{n-1}$ $\in R$ and $a\rho b$, hence $k(r_1,\ldots,r_{n-1},a)\ \overline{\overline{\rho}}\ k(r_1,\ldots,r_{n-1},b)$. Since $\overline{\overline{\rho}}$ is transitive, then ρ is strongly compatible.

(1) $\Rightarrow$ (3) Since ρ is a compatible relation, then we conclude that h/ρ and k/ρ are well-defined. Also ρ is strongly compatible, so $(M/\rho, h/\rho)$ is an m-ary group by Theorem 2.2. Now, we have

$$\begin{aligned} & h/\rho(k/\rho(r_1^{n-1},\rho(x_1)),\ldots,k/\rho(r_1^{n-1},\rho(x_m))) \\ & = h/\rho(\rho(k(r_1^{n-1},x_1)),\ldots,\rho(k(r_1^{n-1},x_m))) = \rho(k(r_1^{n-1},h(x_1^m))) \\ & = \bigcup_{x\in h(x_1^m)} \rho(k(r_1^{n-1},x)). \end{aligned}$$

On the other hand

$$\begin{aligned} & k/\rho(r_1^{n-1},h/\rho(\rho(x_1),\ldots,\rho(x_m))) = k/\rho(r_1^{n-1},\rho(h(x_1^m))) \\ & = \rho(k(r_1^{n-1},h(\rho(x_1),\ldots,\rho(x_m))) = \bigcup_{x\in h(x_1^m)} \rho(k(r_1^{n-1},x)). \end{aligned}$$

We have

$$\begin{aligned}&k/\rho(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, \rho(x)) = \rho(k(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x))\\&= \rho(h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))).\end{aligned}$$

On the other hand

$$\begin{aligned}&h/\rho(k/\rho(r_1^{i-1}, s_1, r_{i+1}^{n-1}, \rho(x)), \ldots, k/\rho(r_1^{i-1}, s_m, r_{i+1}^{n-1}, \rho(x)))\\&= h/\rho(\rho(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x)))\\&= \rho(h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))).\end{aligned}$$

Also, we have

$$k/\rho(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, \rho(x)) = \rho(k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+n}^{2n-2}, x)).$$

On the other hand

$$\begin{aligned}&k/\rho(r_1^{n-1}, k/\rho(r_n^{2n-2}, \rho(x))) = k/\rho(r_1^{n-1}, \rho(k(r_n^{2n-2}, x)))\\&= \rho(k(r_1^{n-1}, k(r_n^{2n-2}, x))).\end{aligned}$$

(3) $\Rightarrow$ (1) Now, let $(M/\rho, h/\rho, k/\rho)$ be an (m,n)-ary module.

Let $a_i\rho b_i$, where $i = 1, \ldots, m$, since $(M/\rho, h/\rho)$ is an m-ary group, so $h/\rho(\rho(a_1), \ldots, \rho(a_m)) = \{\rho(x) | x \in h(a_1, \ldots, a_m)\}$ and $h/\rho(\rho(b_1), \ldots, \rho(b_m))$ $= \{\rho(x) | x \in h(b_1, \ldots, a_m)\}$ are singleton. Thus for every $y \in h(a_1, \ldots, a_m)$ and $z \in h(b_1, \ldots, b_m)$ we have $h/\rho(\rho(a_1), \ldots, \rho(a_m)) = \rho(y)$ and $h/\rho(\rho(b_1),$ $\ldots, \rho(b_m)) = \rho(z)$. But $\rho(a_i) = \rho(b_i)$ and so we obtain $\rho(y) = \rho(z)$ for every $y \in h(a_1, \ldots, a_m)$ and $z \in h(b_1, \ldots, b_m)$. Therefore $h(a_1, \ldots, a_m) \,\overline{\overline{\rho}}\, h(b_1, \ldots,$ $b_m)$.

Now, let $r_1, \ldots, r_{n-1} \in R$ and $a\rho b$, since $(M/\rho, h/\rho, k/\rho)$ is an (m,n)-ary module over $(m,n)-ary$ ring R, so $k/\rho(r_1, \ldots, r_{n-1}, \rho(a)) = \{\rho(x) | x \in$ $k(r_1, \ldots, r_{n-1}, a)$ and $k/\rho(r_1, \ldots, r_{n-1}, \rho(b)) = \{\rho(y) | y \in k(r_1, \ldots, r_{n-1}, b)$ are singleton. Thus for every $x \in k(r_1, \ldots, r_{n-1}, a)$ and $y \in k(r_1, \ldots, r_{n-1}, b)$ we have $k/\rho(r_1, \ldots, r_{n-1}, \rho(a)) = \rho(x)$ and $(k/\rho(r_1, \ldots, r_{n-1}, \rho(b)) = \rho(y)$. But $\rho(a) = \rho(b)$ and so $\rho(x) = \rho(y)$ for every $x \in k(r_1, \ldots, r_{n-1}, a)$ and $y \in k(r_1, \ldots, r_{n-1}, b)$. Therefore $k(r_1, \ldots, r_{n-1}, a) \,\overline{\overline{\rho}}\, k(r_1, \ldots, r_{n-1}, b)$. □

Remark 1. Let R be a hyperring and M be a hypermodule over R. We recall the definition of relation ϵ on M as follows [19]:

$$x\epsilon y \ \Leftrightarrow \ \ x, y \in \sum_{i=1}^{n} m'_i; \quad m'_i = m_i \quad \text{or} \quad m'_i = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}} x_{ijk})z_i,$$
$$m_i \in M, \quad x_{ijk} \in R, \quad z_i \in M.$$

The equivalence relation ϵ^* (transitive closure of ϵ) was first introduced by Vougiouklis, and studied by many authors concerning hypermodules. The fundamental relation ϵ^* on M, defined as the smallest equivalence relation such that the quotient M/ϵ^* is a module over the corresponding fundamental ring such that M/ϵ^* is not an abelian group, see [18, 19].

Now, let M be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R. We define the relation ϵ on M.

Definition 2.6. Let $M = (M, h, k)$ be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R. We define

$$x \ \epsilon \ y \Longleftrightarrow \begin{cases} \{x, y\} \subseteq h_{(a)}(u_1^r), r = a(m-1)+1, \ a \in \mathbb{N} & \text{where,} \\ u_i = m_i \ \text{ or } k(v_{i1}^{in-1}, x_i), m_i, x_i \in M & \text{where,} \\ v_{ij} = f_{(b_{ij})}(w_{ij1}^{ijs_{ij}}), s_{ij} = b_{ij}(m-1)+1 & \text{where,} \\ w_{ijk} = g_{(c_{ijk})}(x_{ijk1}^{ijkt_{ijk}}), t_{ijk} = c_{ijk}(n-1)+1, & x_{ijkt} \in R. \end{cases}$$

In the following x_i, y_i and z_i are the notations that defined in Definition 2.6.

Example 4. Let $H = \{a, b, c, d\}$ and $h(a, \ldots, a) = \{b, c\}$ and for every $x_1^m \in M$, $h(x_1^m) = \{c, d\}$, where $x_i \neq a$, and $1 \leq i \leq m$. Then (M, h) is an m-ary semihypergroup. If R be an arbitrary (m, n)-ary hyperring then for every $r_1^{n-1} \in R$ and $x \in M$, we define $k(r_1^{n-1}, x) = \{c, d\}$. Then $M = (M, h, k)$ is an (m, n)-ary hypermodule. We have $b\epsilon c$ and $c\epsilon d$ so $b\epsilon^* d$ but $(b, d) \notin \epsilon$. Hence ϵ is not transitive.

Lemma 2.7. *The relation ϵ^* is a strongly compatible relation on (m, n)-ary hypermodule M, both on m-ary hyperoperation h and scalar n-ary hyperoperation k.*

Proof. If $a_1\ \epsilon^*\ b_1,\ldots,a_m\ \epsilon^*\ b_m$, then $\epsilon^*(a_1)=\epsilon^*(b_1),\ldots,\epsilon^*(a_m)=\epsilon^*(b_m)$. For every $a\in h(a_1,\ldots,a_m)$ and $b\in h(b_1,\ldots,b_m)$ we have

$$\begin{aligned}\epsilon^*(a)&=\epsilon^*(h(a_1,\ldots,a_m))=h/\epsilon^*(\epsilon^*(a_1),\ldots,\epsilon^*(a_m))\\&=h/\epsilon^*(\epsilon^*(b_1),\ldots,\epsilon^*(b_m))=\epsilon^*(h(b_1,\ldots,b_m))=\epsilon^*(b).\end{aligned}$$

Now, let $r_1,\ldots,r_{n-1}\in R$, $a_1,b_1\in M$ and $a_1\epsilon^*b_1$, then for every $a\in k(r_1,\ldots,r_{n-1},a_1)$ and $b\in k(r_1,\ldots,r_{n-1},b_1)$, we have

$$\begin{aligned}\epsilon^*(a)&=\epsilon^*(k(r_1,\ldots,r_{n-1},a_1))=k/\epsilon^*((r_1,\ldots,r_{n-1},\epsilon^*(a_1))\\&=k/\epsilon^*k(r_1,\ldots,r_{n-1},\epsilon^*(b_1))=\epsilon^*(b).\end{aligned}$$

□

Corollary 2.8. Let $M=(M,h,k)$ be an (m,n)-ary hypermodule over an (m,n)-ary hyperring R. Then the quotient $(M/\epsilon^*,h/\epsilon^*,k/\epsilon^*)$ is an (m,n)-ary module over an (m,n)-ary ring R, where

$$h/\epsilon^*(\epsilon^*(a_1),\ldots,\epsilon^*(a_m)):=\{\epsilon^*(a)|a\in h(a_1,\ldots,a_m)\}=\epsilon^*(h(a_1^m))$$

and

$$k/\epsilon^*(r_1,\ldots,r_{n-1},\epsilon^*(a)):=\{\epsilon^*(x)|x\in k(r_1,\ldots,r_{n-1},a)\}=\epsilon^*(k((r_1^{n-1},a)).$$

Proof. Since ϵ^* is a strongly compatible relation by Theorem 2.5, $(M/\epsilon^*,h/\epsilon^*,k/\epsilon^*)$ is an (m,n)-ary module over an (m,n)-ary ring R. □

If $M=(M,h,k)$ is an (m,n)-ary hypermodule, then $\hat{\epsilon}$ denoted the transitive closure of the relation $\epsilon=\bigcup_{a\geq 0}\epsilon_a$, where ϵ_0 is the diagonal, i.e., $\epsilon_0=\{(x,x)|x\in M\}$ and for every integer $a\geq 1$, ϵ_a is the relation defined as follows: $x\epsilon_a y$ if and only if $\{x,y\}\subseteq h_{(a)}$, for some $a\in\mathbb{N}$. If $x\epsilon_0 y$ (i.e., $x=y$) then we write $\{x,y\}\subseteq u_{(0)}$. We define ϵ^* as the smallest equivalence relation such that the quotient $(M/\epsilon^*,h/\epsilon^*,k/\epsilon^*)$ is an (m,n)-ary module over an (m,n)-ary ring R, where M/ϵ^* is the set of all equivalence classes. The relation ϵ^* is called fundamental equivalence relation.

Lemma 2.9. *Let $M=(M,h,k)$ be an (m,n)-ary hypermodule over an (m,n)-ary hyperring R, then for every $a\in\mathbb{N}$, we have $\epsilon_a\subseteq\epsilon_{a+1}$.*

Proof. Let $x\ \epsilon_a\ y$, then there exists $a\in\mathbb{N}$, and $u_1,\ldots,u_r$, where $r=a(m-1)+1$, such that $\{x,y\}\subseteq h_{(a)}(u_1^r)$. By producibility of h, there exist $u'_1,\ldots,u'_m$, such that $u_1\subseteq h(u'_1,\ldots,u'_m)$. So

$$\begin{aligned}\{x,y\}\subseteq h_{(a)}(u_1^r)=h_{(a)}(u_1\ldots,u_r)\ &\subseteq h_{(a)}(h(u'_1,\ldots,u'_m),u_2,\ldots u_r),\\&=h_{(a+1)}(u'^m_1,u_2^r).\end{aligned}$$

This means $x\ \epsilon_{a+1}\ y$. □

Corollary 2.10. Let $M = (M, h, k)$ be an (m, n)-ary hypermodule over an (m, n)-ary hyperring R, then for every $a \in \mathbb{N}$, we have $\epsilon^*_a \subseteq \epsilon^*_{a+1}$.

Theorem 2.11. *The fundamental relation ϵ^* is the transitive closure of the relation ϵ, i.e., ($\epsilon^* = \widehat{\epsilon}$).*

Proof. Suppose that $\widehat{\epsilon}$ is the transitive closure of ϵ. By Theorem 4.1, [5], we know that the quotient $M/\widehat{\epsilon}$ is an m-ary hypergroup, where $h/\widehat{\epsilon}$ is defined in the usual manner

$$h/\widehat{\epsilon}(\widehat{\epsilon}(x_1), \ldots, \widehat{\epsilon}(x_m)) = \{\widehat{\epsilon}(y) | y \in h(\widehat{\epsilon}(x_1), \ldots, \widehat{\epsilon}(x_m))\},$$

for all $x_1, \ldots, x_m \in M$.

Now, we prove that $M/\widehat{\epsilon}$ is an (m, n)-ary module over an (m, n)-ary ring R. The scalar n-ary hyperoperation $k/\widehat{\epsilon}$ on $M/\widehat{\epsilon}$ is defined in the usual manner:

$$k/\widehat{\epsilon}(r_1, \ldots, r_{n-1}, \widehat{\epsilon}(x)) = \{\widehat{\epsilon}(y) | y \in k(r_1, \ldots, r_{n-1}, x)\},$$

for all $r_1, \ldots, r_{n-1} \in R$ and $x \in M$. Suppose that $a \in \widehat{\epsilon}(x)$. Then we have $a\widehat{\epsilon}x$, if there exist $x_1, \ldots, x_m$ such that $x_1 = a, \ldots, x_m = x$ such that $\{x_i, x_{i+1}\} \subseteq h_{(i)}$. So every element $z \in k(r_1, \ldots, r_{n-1}, x_i)$ is equivalent to every element to $k(r_1, \ldots, r_{n-1}, x_{i+1})$. Therefore $k/\widehat{\epsilon}(r_1, \ldots, r_{n-1}, \widehat{\epsilon}(x))$ is singleton. So we can write $k/\widehat{\epsilon}(r_1, \ldots, r_{n-1}, \widehat{\epsilon}(x)) = \widehat{\epsilon}(y)$, for all $y \in k(r_1, \ldots, r_{n-1}, \widehat{\epsilon}(x))$.

Moreover, since k has n-ary hypermodule scalar properties, consequently, $k/\widehat{\epsilon}$ has (m, n)-ary hypermodule scalar properties.

Now, let θ be an equivalence relation on M such that M/θ is (m, n)-ary hypermodule over an (m, n)-ary hyperring R. Then for all $x_1, \ldots, x_m \in M$, we have $h/\theta(\theta(x_1), \ldots, \theta(x_m)) = \theta(y)$ for all $y \in h(\theta(x_1), \ldots, \theta(x_m))$. Also $k/\theta(r_1, \ldots, r_{n-1}, \theta(x)) = \theta(z)$, for all $z \in k(r_1, \ldots, r_{n-1}, \theta(x))$. But also, for every $x_1, \ldots, x_m, x \in M$, $r_1, \ldots, r_{n-1} \in R$, $A_i \subseteq \theta(x_i), (i = 1, \ldots, m)$ and $A \subseteq \theta(x)$, we have

$$h/\theta(\theta(x_1), \ldots, \theta(x_m)) = \theta(h(x_1, \ldots, x_m)) = \theta(h(A_1, \ldots, A_m))$$

and

$$k/\theta((r_1, \ldots, r_{n-1}, \theta(x)) = \theta(k(r_1, \ldots, r_{n-1}, x)) = \theta(k(r_1, \ldots, r_{n-1}, A)).$$

Therefore, $\theta(a) = \theta(h_{(i)})$ for all $i \geq 0$ and for all $a \in h_{(i)}$. So for every $a \in M, x \in \epsilon(a)$ implies $x \in \theta(a)$. But θ is transitively closed, so we obtain $x \in \epsilon^*(a)$ implies $x \in \theta(a)$. Hence, the relation ϵ^* is the smallest equivalence relation on M such that M/ϵ^* is an (m,n)-ary module over an (m,n)-ary ring R. □

Theorem 2.12. *Let $M = (M,h,k)$ be an (m,n)-ary hypermodule on (m,n)-ary hyperring R. Then $(M/\epsilon^*, h/\epsilon^*)$ is an (m,n)-ary module on (m,n)-ary ring $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$.*

Proof. By Theorem 2.7, ϵ^* is a strongly compatible relation on M, and $(M/\epsilon^*, h/\epsilon^*)$ is an m-ary group by Theorem 2.2. Also by Theorem 2.4, $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an (m,n)-ary ring. Now, let $r_1, \ldots, r_{n-1} \in R,\ x \in M$ and define

$$k/\epsilon^*(\Gamma^*(r_1), \ldots, \Gamma^*(r_{n-1}), \epsilon^*(x)) := k(\Gamma^*(r_1), \ldots, \Gamma^*(r_{n-1}), \epsilon^*(x)).$$

If $x \in h_a(u_1, \ldots, u_r)$ and $r_i \in f_{k_i}(u'_1, \ldots, u'_s)$, then

$$\begin{aligned} &k(\Gamma^*(r_1) \ldots, \Gamma^*(r_{n-1}), \epsilon^*(x)) \subseteq k(f_{k_1}, \ldots, f_{k_{n-1}}, h_a(u_1, \ldots, u_r)) \\ &= h_a(k(f_{k_1}, \ldots, f_{k_{n-1}}, u_1), \ldots, k(f_{k_1}, \ldots, f_{k_{n-1}}, u_r)). \end{aligned}$$

So, for every $r'_1\Gamma^* r_1, \ldots, r'_{n-1}\Gamma^* r_{n-1}$ and $y\epsilon^* x$, we have

$$\begin{aligned} &k(\Gamma^*(r'_1), \ldots, \Gamma^*(r'_{n-1}), \epsilon^*(y)) \\ &\subseteq h_a(k(f_{k_1}, \ldots, f_{k_{n-1}}, u_1), \ldots, k(f_{k_1}, \ldots, f_{k_{n-1}}, u_r)). \end{aligned}$$

Since M is an (m,n)-ary hypermodule on (m,n)-ary hyperring R, the properties of M as an (m,n)-ary hypermodule, guarantee that the m-ary group M/ϵ^* is an $(m,n) - ary$ R/Γ^*-module. □

3. Fundamental relation and strongly transitive geometric space

A geometric space is a pair (M, ϵ) such that M is a non-empty set, whose elements we call points, and ϵ is a non-empty family of subsets of M, whose elements we call blocks. The family $\mathcal{F}_\epsilon(M)$ of all ϵ-parts (see [2]) of M is non-empty since $\emptyset$ and M are elements of $\mathcal{F}_\epsilon(M)$. Moreover, the intersection of elements of $\mathcal{F}_\epsilon(M)$ is an element of $\mathcal{F}_\epsilon(M)$, hence $\mathcal{F}_\epsilon(M)$ is a closure system of M. For a subset X of M, we denote by $\Gamma(X)$ the intersection of all ϵ-part of M containing X. The set $\Gamma(X)$ is the smallest ϵ-part of M, called the closure of X.

The following properties are true:

$(P1)$ $X \subseteq \Gamma(X)$.

$(P2)$ $X \subseteq Y \Rightarrow \Gamma(X) \subseteq \Gamma(Y)$.

$(P3)$ $\Gamma(\Gamma(X)) = \Gamma(X)$.

$(P4)$ $\Gamma(X) = \bigcup_{x \in X} \Gamma(x)$, where $\Gamma(x) = \Gamma(\{x\})$.

For all subsets X of M, we can associate an ascending chain of subsets $(\Gamma_n(X))_{n\in\mathbb{N}}$, called *cone* of X, defined by the following conditions: $\Gamma_0(X) = X$; and for every integer $n \geq 0$ $\Gamma_{n+1}(X) = \Gamma_n(X) \cup [\cup\{B \in \epsilon | B \cap \Gamma_n(X) \neq \emptyset\}]$.

Freni [10] used the notion of the cone of X and obtain the closure of X, as it is shown in the next result.

Proposition 3.1. *Let (M, ϵ) be a geometric space. For every $n \in \mathbb{N}$ and for every pair (X, Y) of subsets of M we have:*

1) $X \subseteq Y \Rightarrow \Gamma_n(X) \subseteq \Gamma_n(Y)$.

2) $\Gamma_n(X) = \bigcup_{x\in X} \Gamma_n(x)$, *where* $\Gamma_n(x) = \Gamma_n(\{x\})$.

3) $\Gamma_n(\Gamma_m(X)) = \Gamma_{n+m}(X)$.

4) $\Gamma(X) = \bigcup_{n\in\mathbb{N}} \Gamma_n(X)$.

5) *If the family ϵ is a covering of M, then* $\Gamma_{n+1}(X) = \bigcup\{B \in \epsilon | B \cap \Gamma_n(X) \neq \emptyset\}$.

Remark 2. By property (5) of Proposition 3.1, in a geometric space (M, ϵ) such that ϵ is a covering of M, the cone $(\Gamma_n(X))_{n\in\mathbb{N}}$ of X is defined by two conditions: $\Gamma_0(X) = X$ and $\Gamma_{n+1}(X) = \bigcup\{B \in \epsilon | B \cap \Gamma_n(X) \neq \emptyset\}$, for every integer $n \geq 0$.

If $B_1, B_2, \ldots, B_n$ are n blocks of a geometric space (M, ϵ) such that $B_i \cap B_{i+1} \neq \emptyset$, for any $i \in \{1, 2, \ldots, n-1\}$, then the n-tuple $(B_1, B_2, \ldots, B_n)$ is called a polygonal of (M, ϵ). The concept of polygonal allows us to define on M the following relation:

$x \approx y \Leftrightarrow x = y$ or a polygonal $(B_1, B_2, \ldots, B_n)$ exists such that $x \in B_1$ and $y \in B_n$.

The relation $\approx$ is an equivalence and it is easy to see that it coincides with the transitive closure of the following relation:

$x \sim y \Leftrightarrow x = y$ or there exists $B \in \epsilon$ such that $\{x, y\} \subseteq B$,

so $\approx$ is equal to $\bigcup_{n \geq 1} \sim^n$, where $\sim^n = \sim \circ \sim \circ \ldots \circ \sim$ n times.

If ϵ is a covering of M, the relation $\sim$ and $\approx$ can be defined in the following simply way:

$x \sim y \Leftrightarrow$ there exists $B \in \epsilon$ such that $\{x, y\} \subseteq B$,

$x \approx y \Leftrightarrow$ a polygonal $(B_1, B_2, \ldots, B_n)$ exists such that $x \in B_1$ and $y \in B_n$.

Proposition 3.2. *For every integer $n \geq 1$ and for every pair (x, y) of elements of M, we have:*

1) $y \sim^n x \Leftrightarrow y \in \Gamma_n(x)$.

2) $[x] = \Gamma(x)$.

Proof. (1) We proceed by induction on $n \in \mathbb{N}$. If $n = 1$, we have:

$$\begin{aligned} y \sim x \quad &\Leftrightarrow x = y \quad \text{or} \quad \exists B \in \epsilon : \{x, y\} \subseteq B \\ &\Leftrightarrow x = y \quad \text{or} \quad \exists B \in \epsilon : y \in B, \quad B \cap \{x\} = B \cap \Gamma_0(x) \neq \emptyset \\ &\Leftrightarrow y \in \Gamma_1(x). \end{aligned}$$

Assume now that $y \sim^n x \Leftrightarrow y \in \Gamma_n(x)$. By (1), (2) and (3) of Proposition 3.1, we deduce that

$$\begin{aligned} y \sim^{n+1} x \quad &\Leftrightarrow \exists z \in M : y \sim z, z \sim^n x \\ &\Leftrightarrow \exists z \in M : y \in \Gamma_1(z), \ z \in \Gamma_n(x) \\ &\Leftrightarrow y \in \Gamma_1(\Gamma_n(x)) = \Gamma_{n+1}(x). \end{aligned}$$

(2) By the preceding claim, we have

$$\begin{aligned} y \in [x] \quad &\Leftrightarrow y \approx x \Leftrightarrow \exists n \geq 1 : y \sim^n x \\ &\Leftrightarrow \exists n \geq 1 : y \in \Gamma_n(x) \\ &\Leftrightarrow y \in \Gamma(x). \end{aligned}$$

□

Corollary 3.3. For every integer $n \geq 1$, we have:

1) $\sim^n$ is transitive $\Leftrightarrow \Gamma(x) = \Gamma_n(x)$, for all $x \in M$.

2) $\sim$ is transitive $\Leftrightarrow \Gamma(x) = \Gamma_1(x)$, for all $x \in M$.

Proof. (1) If $\sim^n$ is transitive, then $\sim^{2n}\subseteq\sim^n$ and, by (1) of Proposition 3.1, we have the implications: $y \in \Gamma_{n+1}(x) \subseteq \Gamma_{2n}(x) \Rightarrow y \sim^{2n} x \Rightarrow y \sim^n x \Rightarrow y \in \Gamma_n(x)$. Therefore, we have $\Gamma_{n+1}(x) \subseteq \Gamma_n(x)$ and, using Proposition 4.1, we obtain $\Gamma(x) = \Gamma_n(x)$.

Conversely, if $\Gamma(x) = \Gamma_n(x)$ for every $x \in M$, then, by Proposition 3.1, we have the equality $\Gamma_n(x) = \Gamma_{2n}(x)$ and the implications: $y \sim^{2n} x \Rightarrow y \in \Gamma_{2n}(x) \Rightarrow y \in \Gamma_n(x) \Rightarrow y \sim^n x$. Hence, the relation $\sim^n$ is transitive. □

Theorem 3.4 ([10]). *For every pair (A, B) of blocks of a geometric space (M, ϵ) and for any $n \in \mathbb{N}$, the following conditions are equivalent:*

1) $A \cap B \neq \emptyset,\ x \in B \Rightarrow \exists C \in \epsilon :\ (A \cup \{x\}) \subseteq C$.

2) $A \cap B \neq \emptyset,\ x \in \Gamma_n(B) \Rightarrow \exists C \in \epsilon :\ (A \cup \{x\}) \subseteq C$.

3) $A \cap \Gamma_n(B) \neq \emptyset,\ x \in \Gamma_n(B) \Rightarrow \exists C \in \epsilon :\ (A \cup \{x\}) \subseteq C$.

A geometric space (M, ϵ) is strongly transitive if the family ϵ is a covering of M and moreover one of the three equivalent conditions of Theorem 3.4, is satisfied.

Let $M = (M, h, k)$ be an (m, n)-ary hypermodule on an (m, n)-ary hyperring (R, f, g) and $P(H)$ be the family of subsets of M defined as follows: for every integer $n \geq 1$ and for every n-tuple $(u_1, u_2, \ldots, u_r)$, where $r = a(m-1) + 1$, we set:

1) $B(u_1) = \{u_1\}$,

2) If $k \in \mathbb{N}$ then $B(u_1^r) = h_{(k)}(u_1^r)$.

Lemma 3.5. *Let $M = (M, h, k)$ be an (m, n)-ary hypermodule on an (m, n)-ary hyperring R, if there exist u_1^r and $l \in \{1, \ldots, r\}$, where $r = a(m-1) + 1$ such that $u_l \subseteq B(z_1^r)$, then*

1) $h(y_1^{l-1}, B(u_1, \ldots, u_r), y_{l+1}^m) = B(y_1^{l-1}, u_1, \ldots, u_r, y_{l+1}^m)$,

2) $k(r_1^{n-1}, B(z_1^r)) = B(k(r_1^{n-1}, z_1), \ldots, k(r_1^{n-1}, z_r))$.

Proof.

1) It gets easily from definition.

2)

$$\begin{aligned} k(r_1^{n-1}, B(z_1^r)) &= k(r_1^{n-1}, h_{(k)}(z_1^r)) \\ &= h(k(r_1^{n-1}, h_{(k-1)}(z_1^r)) \\ &\vdots \\ &= B(k(r_1^{n-1}, z_1), \ldots, k(r_1^{n-1}, z_r)). \end{aligned}$$

□

Lemma 3.6. *Let $M = (M, h, k)$ be an (m, n)-ary hypermodule on an (m, n)-ary hyperring R. If $k_1^n \in \mathbb{N}^n$, $1 \leq i \leq n$ and $r_k = k_i(m-1)+1$, then for every $r_1 + \ldots + r_m$-ary $(x_1^{r_1}, y_1^{r_2}, \ldots, z_1^{r_m})$, we have*

1) $h(B(x_1^{r_1}), B(y_1^{r_2}), \ldots, B(z_1^{r_m})) = B(x_1^{r_1}, y_1^{r_2}, \ldots, z_1^{r_m})$

2) $k(r_1^{n-1}, h(B(x_1^{r_1}), B(y_1^{r_2}), \ldots, B(z_1^{r_m})))$
$= h(k(r_1^{n-1}, B(x_1^{r_1})), k(r_1^{n-1}, B(y_1^{r_2})), \ldots, k(r_1^{n-1}, B(z_1^{r_m})).$

Proof.

1) $h(B(x_1^{r_1}), B(y_1^{r_2}), \ldots, B(z_1^{r_m})) = h(h_{(k_1)}(x_1^{r_1}), h_{(k_2)}(y_1^{r_2}), \ldots, h_{(k_m)}(z_1^{r_m})$
$= h_{(1+r_1+\ldots+r_m)}(x_1^{r_1}, y_1^{r_2}, \ldots, z_1^{r_m}) = B(x_1^{r_1}, y_1^{r_2}, \ldots, z_1^{r_m})$

2) $k(r_1^{n-1}, h(B(x_1^{r_1}), B(y_1^{r_2}), \ldots, B(z_1^{r_m})))$
$= h(k(r_1^{n-1}, h_{(k_1)}(x_1^{r_1})), k(r_1^{n-1}, h_{(k_2)}(y_1^{r_2})), \ldots, k(r_1^{n-1}, h_{(k_m)}(z_1^{r_m})))$
$= h(k(r_1^{n-1}, B(x_1^{r_1})), k(r_1^{n-1}, B(y_1^{r_2})), \ldots, k(r_1^{n-1}, B(z_1^{r_m}))).$

□

Lemma 3.7. *Let $M = (M, h, k)$ be an (m, n)-ary hypermodule on an (m, n)-ary hyperring R. If there exists $z_1^{r'}$, where $r' = a'(m-1)+1$ and there exist $x_1^{r'}$ and $l \in \{1, \ldots, r'\}$, such that $z_l \subseteq B(x_1^{r'})$, then*

1) $B(z_1^r) \subseteq B(z_1^{l-1}, x_1^{r'}, z_{l+1}^m)$

2) $k(r_1^{n-1}, B(z_1^r)) \subseteq k(r_1^{n-1}, B(z_1^{l-1}, x_1^{r'}, z_{l+1}^m))$

Proof. It gets easily from definition. □

Lemma 3.8. *Let $M = (M, h, k)$ be an (m, n)-ary hypermodule on an (m, n)-ary hyperring R, $q \in \mathbb{N}$ and $l = q(m-1)+1$. For $k_1^l \in \mathbb{N}$ and $1 \le i \le l$, we set $r_i = k_i(m-1)+1$, then for every $r_1 + \ldots + r_l$-ary $(x_1^{r_1}, y_1^{r_2}, \ldots, z_1^{r_l})$, we have*

$$B(B(x_1^{r_1}), B(y_1^{r_2}), \ldots, B(z_1^{r_l})) = B(x_1^{r_1}, y_1^{r_2}, \ldots, z_1^{r_l}).$$

Proof. Since $M = (M, h, k)$ is an (m, n)-ary hypermodule, we have

$$\begin{aligned} B(B(x_1^{r_1}), B(y_1^{r_2}), \ldots, B(z_1^{r_l})) &= h_{(q)}(h_{(k_1)}(x_1^{r_1}), h_{(k_2)}(x_1^{r_2}), \ldots, h_{(k_l)}(x_1^{r_m})) \\ &= h_{(q+k_1+k_2+\ldots+k_l)}(x_1^{r_1}, y_1^{r_2}, \ldots, z_1^{r_l}) \\ &= B(x_1^{r_1}, y_1^{r_2}, \ldots, z_1^{r_l}). \end{aligned}$$

□

Theorem 3.9. *Let $M = (M, h, k)$ be an (m, n)-ary hypermodule on an (m, n)-ary hyperring R. If for every $x \in M$ there exist v_1^{n-1} (see Definition 2.6), and $y \in M$ such that $k(v_1^{n-1}, x) = h(x^{(m-2)}, y, x)$, then the geometric space $(M, P(M))$ is strongly transitive.*

Proof. Let $B(z_1^r)$ and $B(x_1^{r'})$ be two blocks such that

$$B(z_1^r) \cap B(x_1^{r'}) \neq \emptyset \quad \text{and} \quad x \in B(x_1^{r'}),$$

where $r = k(m-1)+1$ and $r' = k'(m-1)+1$, for some $k, k' \in \mathbb{N}$.

Let $b \in B(z_1^r) \cap B(x_1^{r'})$, since there exist v_1^{n-1} and $x_1 \in M$ such that $z_r = k(v_1^{n-1}, x_1)$, then there exist $c, y \in M$ such that

$$x \in h(b, c, b^{(m-2)}) \quad \text{and} \quad z_r = h(x^{(m-2)}, y, x).$$

Now, since $x \in B(x_1^{r'})$, we have

$$\begin{aligned} & x \in h(b, c, b^{(m-2)}) \subseteq h(B(z_1^r), c, b^{(m-2)})) = h(h_{(k)}(z_1^r), c, b^{(m-2)})) \\ & \subseteq h_{(k+1)}(z_1^r, c, b^{(m-2)}) = B(z_1^r, c, b^{(m-2)}) \\ & \subseteq B(z_1^{r-1}, h(x^{(m-2)}, y, x), c, b^{(m-2)}) = B(z_1^{r-1}, x^{(m-2)}, y, x, c, b^{(m-2)}) \\ & \subseteq B(z_1^{r-1}, x^{(m-2)}, y, x_1^{r'}, c, b^{(m-2)}) = B. \end{aligned}$$

Since $b \in B(x_1^{r'})$,

$$\begin{aligned} B(z_1^r) &\subseteq B(z_1^{r-1}, h(x^{(m-2)}, y, x)) = B(z_1^{r-1}, x^{(m-2)}, y, x) \\ &\subseteq B(z_1^{r-1}, x^{(m-2)}, y, h(b, c, b^{(m-2)})) = B(z_1^{r-1}, x^{(m-2)}, y, b, c, b^{(m-2)})) \\ &\subseteq B(z_1^{r-1}, x^{(m-2)}, y, x_1^{r'}, c, b^{(m-2)}) = B. \end{aligned}$$

Hence $B(z_1^r) \cup \{x\} \subseteq B(z_1^{r-1}, x^{(m-2)}, y, x_1^{r'}, c, b^{(m-2)}) = B$, therefore $(M, P(M))$ is a transitive geometric space. □

Remark 3. If M is an (m,n)-ary hypermodule, the relation $\sim$ defined on the geometric space $(M, P(M))$ coincides with the relation ϵ used in this paper. In fact, the relation ϵ is defined as follows

$$x\epsilon y \Leftrightarrow \exists n \in \mathbb{N},\ \exists (u'_1, \ldots, u'_r): \quad \{x,y\} \subseteq B(u'_1, u'_2, \ldots, u'_r),$$

thus $x\epsilon y \Leftrightarrow x \sim y$.

Theorem 3.10. *Let $M = (M,h,k)$ be an (m,n)-ary hypermodule on an (m,n)-ary hyperring R. If for every $x \in M$ there exist v_1^{n-1} (see Definition* 2.6*), and $y \in M$ such that $k(v_1^{n-1}, x) = h(x^{(m-2)}, y, x)$, then the relation ϵ is a compatible and transitive relation.*

Proof. Since $(M, P(M))$ is a transitive geometric space, by Remark 3, the relation ϵ is transitive. □

Corollary 3.11. Let M be an R-hypermodule and for every $x \in M$, $R.x = M$. Then the relation ϵ introduced in Remark 1, is a transitive relation on hypermodules.

REFERENCES

1. Anvariyeh, S.M.; Mirvakili, S.; Davvaz, B. – *Fundamental relation on (m,n)-ary hypermodules over (m,n)-ary hyperrings*, Ars Combin., 94 (2010), 273–288.
2. Anvariyeh, S.M.; Davvaz, B. – *Strongly transitive geometric spaces associated to hypermodules*, J. Algebra, 322 (2009), 1340–1359.
3. Corsini, P.; Leoreanu, V. – *Applications of Hyperstructure Theory*, Advances in Mathematics (Dordrecht), 5, Kluwer Academic Publishers, Dordrecht, 2003.
4. Davvaz, B.; Dudek, W.A.; Vougiouklis, T. – *A generalization of n-ary algebraic systems*, Comm. Algebra, 37 (2009), 1248–1263.
5. Davvaz, B.; Vougiouklis, T. – *n-ary hypergroups*, Iran. J. Sci. Technol. Trans. A Sci., 30 (2006), 165–174.
6. Denecke, K.; Wismath, S.L. – *Universal Algebra and Coalgebra*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.

7. Dörnte, W. – *Untersuchungen ber einen verallgemeinerten Gruppenbegriff*, Math. Z., 29 (1929), 1–19.
8. Dudek, W.A. – *Remarks on n-groups*, Demonstratio Math., 13 (1980), 165–181.
9. Dudek, W.A. – *Varieties of polyadic groups. Algebra, logic & discrete mathematics,* Filomat, 9 (1995), 657–674.
10. Freni, D. – *Strongly transitive geometric spaces: applications to hypergroups and semigroups theory*, Comm. Algebra, 32 (2004), 969–988.
11. Kasner, E. – *An extension of the group concept*, Bull. Amer. Math. Soc., 10 (1904), 290–291.
12. Leoreanu-Fotea, V.; Davvaz, B. – *n-hypergroups and binary relations*, European J. Combin., 29 (2008), 1207–1218.
13. Leoreanu-Fotea, V.; Davvaz, B. – *Roughness in n-ary hypergroups*, Inform. Sci., 178 (2008), 4114–4124.
14. Mirvakili, S.; Davvaz, B. – *n-ary hypergroups and strongly transitive geometric spaces,* Ars Combinatoria, (in print).
15. Mirvakili, S.; Davvaz, B. – *Constructions of* (m, n)*-ary hyperring*, (submitted).
16. Pelea, C. – *On the fundamental relation of a multialgebra*, Ital. J. Pure Appl. Math., 10 (2001), 141–146.
17. Post, E.L. – *Polyadic groups*, Trans. Amer. Math. Soc., 48 (1940), 208–350.
18. Vougiouklis, T. – *Hyperstructures and their Representations*, Hadronic Press Monographs in Mathematics, Hadronic Press, Inc., Palm Harbor, FL, 1994.
19. Vougiouklis, T. – *The fundamental relation in hyperrings. The general hyperfield*, Algebraic hyperstructures and applications, 203–211, World Sci. Publ., Teaneck, NJ, 1991.

Received: 8.III.2010
Revised: 26.X.2010

Department of Mathematics,
Yazd University,
Yazd,
IRAN
davvaz@yazduni.ac.ir