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# ON $\mathcal{A}_I^*$-SETS, $\mathcal{C}_I$-SETS, $\mathcal{C}_I^*$-SETS AND DECOMPOSITIONS OF CONTINUITY IN IDEAL TOPOLOGICAL SPACES

BY

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**Abstract.** The aim of this paper is to introduce and study the notions of $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces. Properties of $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets are investigated. Moreover, decompositions of continuous functions and decompositions of $\mathcal{A}_I^*$-continuous functions via $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces are established.

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**Key words:** $\mathcal{A}_I^*$-set, $\mathcal{C}_I$-set, $\mathcal{C}_I^*$-set, pre-$I$-regular set, ideal topological space, decomposition, $\star$-extremally disconnected ideal space, $\star$-hyperconnected ideal space, $I$-submaximal ideal space.

## 1. Introduction and preliminaries

In this paper, $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces are introduced and studied. The relationships and properties of $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces are investigated. Futhermore, decompositions of continuous functions and decompositions of $\mathcal{A}_I^*$-continuous functions via $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets and $\mathcal{C}_I^*$-sets in ideal topological spaces are provided.

In the present paper, $(X, \tau)$ or $(Y, \sigma)$ will denote topological spaces with no separation properties assumed. For a subset $V$ of $X$, let $Cl(V)$ and $Int(V)$ denote the closure and the interior of $V$, respectively, with respect to the topological space $(X, \tau)$.

An ideal $I$ on a set $X$ is a nonempty collection of subsets of $X$ which satisfies

(1) $V \in I$ and $G \subset V$ implies $G \in I$.

(2) $V \in I$ and $G \in I$ implies $V \cup G \in I$ [13].

If $I$ is an ideal on $X$ and $X \notin I$, then $\mathcal{F} = \{X \backslash G : G \in I\}$ is a filter [11].

For an ideal $I$ on $(X, \tau)$, $(X, \tau, I)$ is said to be an ideal topological space or briefly an ideal space. Let $P(X)$ be the set of all subsets of $X$. For an ideal topological space $(X, \tau, I)$, a set operator $(.)^* : P(X) \to P(X)$, which will be said the local function [13] of $G \subset X$ with respect to $\tau$ and $I$, is defined as follows: $G^*(I, \tau) = \{x \in X : H \cap G \notin I$ for every $H \in \tau(x)\}$ where $\tau(x) = \{H \in \tau : x \in H\}$. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, said to be the $\star$-topology, finer than $\tau$, is defined by $Cl^*(G) = G \cup G^*(I, \tau)$ [11]. We will briefly write $G^*$ for $G^*(I, \tau)$ and $\tau^*$ for $\tau^*(I, \tau)$.

**Remark 1.** The $\star$-topology is generated by $\tau$ and by the filter $\mathcal{F}$. Also, the family $\{H \cap G : H \in \tau,\ G \in \mathcal{F}\}$ is a basis for this topology [11].

**Lemma 2** ([10])**.** *Let $K$ be a subset of an ideal topological space $(X, \tau, I)$. If $N$ is open, then $N \cap Cl^*(K) \subset Cl^*(N \cap K)$.*

**Definition 3.** A subset $K$ of an ideal topological space $(X, \tau, I)$ is called pre-$I$-open [3] (resp. semi-$I$-open [8], $\alpha$-$I$-open [8], strongly $\beta$-$I$-open [9], $\star$-dense [4], $t$-$I$-set [8], semi*-$I$-open [5, 6]) if $K \subset Int(Cl^*(K))$ (resp. $K \subset Cl^*(Int(K))$, $K \subset Int(Cl^*(Int(K)))$, $K \subset Cl^*(Int(Cl^*(K)))$, $Cl^*(K) = X$, $Int(K) = Int(Cl^*(K))$, $K \subset Cl(Int^*(K))$).

**Lemma 4** ([6])**.** *Every semi-$I$-open set is semi*-$I$-open in an ideal topological space.*

**Remark 5.** The reverse implication of Lemma 4 is not true in general as shown in [5, 6].

**Definition 6.** The complement of a pre-$I$-open (resp. semi-$I$-open, $\alpha$-$I$-open, semi*-$I$-open) set is called pre-$I$-closed [3] (resp. semi-$I$-closed [8], $\alpha$-$I$-closed [8], semi*-$I$-closed [5, 6]).

**Definition 7.** The pre-$I$-closure of a subset $K$ of an ideal topological space $(X, \tau, I)$, denoted by $p_ICl(K)$, is defined as the intersection of all pre-$I$-closed sets of $X$ containing $K$ [6].

**Lemma 8** ([6])**.** *For a subset $K$ of an ideal topological space $(X, \tau, I)$, $p_ICl(K) = K \cup Cl(Int^*(K))$.*

**Definition 9.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be pre-$I$-continuous [3] (resp. $\alpha$-$I$-continuous [8]) if $f^{-1}(K)$ is pre-$I$-open (resp. $\alpha$-$I$-open) in $X$ for each open set $K$ in $Y$.

**Definition 10.** A subset $K$ of an ideal topological space $(X, \tau, I)$ is called

(1) an $\eta\zeta$-set [14] if $K = L \cap M$, where $L$ is open and $M$ is clopen in $X$.

(2) locally closed [2] if $K = L \cap M$, where $L$ is open and $M$ is closed in $X$.

(3) a $B_I$-set [8] if $K = L \cap M$, where $L$ is open and $M$ is a $t$-$I$-set in $X$.

(4) semi-$I$-regular [12] if $K$ is a $t$-$I$-set and semi-$I$-open in $X$.

(5) an $AB_I$-set [12] if $K = L \cap M$, where $L$ is open and $M$ is a semi-$I$-regular set in $X$.

## 2. $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets, $\mathcal{C}_I^*$-sets in ideal topological spaces

A subset $K$ of an ideal topological space $(X, \tau, I)$ is called pre-$I$-regular if $K$ is pre-$I$-open and pre-$I$-closed in $(X, \tau, I)$.

**Definition 11.** Let $(X, \tau, I)$ be an ideal topological space and $K \subset X$. $K$ is said to be a $\mathcal{C}_I^*$-set if $K = L \cap M$, where $L$ is an open set and $M$ is a pre-$I$-regular set in $X$.

**Theorem 12.** *Let $(X, \tau, I)$ be an ideal topological space. Then each $\mathcal{C}_I^*$-set in $X$ is a pre-$I$-open set.*

**Proof.** Let $K$ be a $\mathcal{C}_I^*$-set in $X$. It follows that $K = L \cap M$, where $L$ is an open set and $M$ is a pre-$I$-regular set in $X$. Since $M$ is a pre-$I$-open set, then by Proposition 2.10 of [3], $K = L \cap M$ is a pre-$I$-open set in $X$.□

**Remark 13.** The converse of Theorem 12 need not be true in general as shown in the following example.

**Example 14.** Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $K = \{a, b, c\}$ is a pre-$I$-open set but it is not a $\mathcal{C}_I^*$-set.

**Remark 15.** In an ideal topological space, every open set and every pre-$I$-regular set is a $\mathcal{C}_I^*$-set. The converse of this implication is not true in general as shown in the following example.

**Example 16.** Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $K = \{a, b, c\}$ is a $\mathcal{C}_I^*$-set but it is not pre-$I$-regular. The set $L = \{a, c, d\}$ is a $\mathcal{C}_I^*$-set but it is not open.

**Remark 17.** By Remark 15 and Theorem 12, the following diagram holds for a subset $K$ of an ideal topological space $(X, \tau, I)$:

$$\begin{array}{ccc} & & \text{pre-}I\text{-open} \\ & & \Uparrow \\ \text{pre-}I\text{-regular} & \Longrightarrow & \mathcal{C}_I^*\text{-set} \end{array}$$

**Definition 18.** A subset $K$ of an ideal topological space $(X, \tau, I)$ is said to be

(1) a $\mathcal{C}_I$-set if $K = L \cap M$, where $L$ is an open set and $M$ is a pre-$I$-closed set in $X$.

(2) an $\eta_I$-set if $K = L \cap M$, where $L$ is an open set and $M$ is an $\alpha$-$I$-closed set in $X$.

(3) an $\mathcal{A}_I^*$-set if $K = L \cap M$, where $L$ is an open set and $M = Cl(Int^*(M))$.

**Remark 19.** Let $(X, \tau, I)$ be an ideal topological space and $K \subset X$. The following diagram holds for $K$:

$$\begin{array}{ccc} \mathcal{C}_I^*\text{-set} & \Longrightarrow & \mathcal{C}_I\text{-set} \\ & & \Uparrow \\ \mathcal{A}_I^*\text{-set} & \Longrightarrow & \eta_I\text{-set} \end{array}$$

The following examples show that these implications are not reversible in general.

**Example 20.** Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $K = \{b, c, d\}$ is a $\mathcal{C}_I$-set and an $\mathcal{A}_I^*$-set but it is not a $\mathcal{C}_I^*$-set. The set $L = \{a, b, d\}$ is a $\mathcal{C}_I$-set but it is not an $\eta_I$-set.

**Example 21.** Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $K = \{d\}$ is an $\eta_I$-set but it is not an $\mathcal{A}_I^*$-set. The set $L = \{a, b, d\}$ is a $\mathcal{C}_I^*$-set but it is not an $\eta_I$-set.

**Theorem 22.** *For a subset $K$ of an ideal topological space $(X, \tau, I)$, the following properties are equivalent*:

(1) *$K$ is a $\mathcal{C}_I$-set and a semi*-$I$-open set in $X$.*

(2) *$K = L \cap Cl(Int^*(K))$ for an open set $L$.*

**Proof.** (1) $\Rightarrow$ (2): Suppose that $K$ is a $\mathcal{C}_I$-set and a semi*-$I$-open set in $X$. Since $K$ is a $\mathcal{C}_I$-set, then we have $K = L \cap M$, where $L$ is an open set and $M$ is a pre-$I$-closed set in $X$. We have $K \subset M$, so $Cl(Int^*(K)) \subset Cl(Int^*(M))$. Since $M$ is a pre-$I$-closed set in $X$, we have $Cl(Int^*(M)) \subset M$. Since $K$ is a semi*-$I$-open set in $X$, we have $K \subset Cl(Int^*(K))$. It follows that $K = K \cap Cl(Int^*(K)) = L \cap M \cap Cl(Int^*(K)) = L \cap Cl(Int^*(K))$.

(2) $\Rightarrow$ (1): Let $K = L \cap Cl(Int^*(K))$ for an open set $L$. We have $K \subset Cl(Int^*(K))$. It follows that $K$ is a semi*-$I$-open set in $X$. Since $Cl(Int^*(K))$ is a closed set, then $Cl(Int^*(K))$ is a pre-$I$-closed set in $X$. Hence, $K$ is a $\mathcal{C}_I$-set in $X$. □

**Theorem 23.** *For a subset $K$ of an ideal topological space $(X, \tau, I)$, the following properties are equivalent*:

(1) *$K$ is an $\mathcal{A}_I^*$-set in $X$.*

(2) *$K$ is an $\eta_I$-set and a semi*-$I$-open set in $X$.*

(3) *$K$ is a $\mathcal{C}_I$-set and a semi*-$I$-open set in $X$.*

**Proof.** (1) $\Rightarrow$ (2): Suppose that $K$ is an $\mathcal{A}_I^*$-set in $X$. It follows that $K = L \cap M$, where $L$ is an open set and $M = Cl(Int^*(M))$. This implies $K = L \cap M = L \cap Cl(Int^*(M)) \subset Cl(L \cap Int^*(M)) = Cl(Int^*(L \cap M)) = Cl(Int^*(K))$. Thus, $K \subset Cl(Int^*(K))$ and hence $K$ is a semi*-$I$-open set in $X$. Moreover, by Remark 19, $K$ is an $\eta_I$-set in $X$.

(2) $\Rightarrow$ (3): It follows from the fact that every $\eta_I$-set is a $\mathcal{C}_I$-set in $X$ by Remark 19.

(3) $\Rightarrow$ (1): Suppose that $K$ is a $\mathcal{C}_I$-set and a semi*-$I$-open set in $X$. By Theorem 22, $K = L \cap Cl(Int^*(K))$ for an open set $L$. We have $Cl(Int^*(Cl(Int^*(K))))$=$Cl(Int^*(K))$. It follows that $K$ is an $\mathcal{A}_I^*$-set in $X$.□

**Theorem 24** ([5])**.** *A subset $K$ of an ideal topological space $(X, \tau, I)$ is semi*-$I$-open if and only if $Cl(K) = Cl(Int^*(K))$.*

**Theorem 25.** *A subset $K$ of an ideal topological space $(X, \tau, I)$ is semi*-$I$-closed if and only if $K$ is a $t$-$I$-set.*

**Proof.** Let $K$ be a semi*-$I$-closed set in $X$. Then $X \backslash K$ is semi*-$I$-open. By Theorem 24, we have $Cl(X \backslash K) = Cl(Int^*(X \backslash K))$. It follows that $Cl(X \backslash K) = X \backslash Int(K) = Cl(Int^*(X \backslash K)) = X \backslash Int(Cl^*(K))$. Thus, $Int(K) = Int(Cl^*(K))$ and hence $K$ is a $t$-$I$-set in $X$. The converse is similar. □

**Theorem 26.** *Let $(X, \tau, I)$ be an ideal topological space and $K \subset X$. The following properties are equivalent*:

(1) *$K$ is an open set.*

(2) *$K$ is an $\alpha$-$I$-open set and an $\mathcal{A}_I^*$-set.*

(3) *$K$ is a pre-$I$-open set and an $\mathcal{A}_I^*$-set.*

**Proof.** (1) $\Rightarrow$ (2): It follows from the fact that every open set is an $\alpha$-$I$-open set and an $\mathcal{A}_I^*$-set.

(2) $\Rightarrow$ (3): It follows from the fact that every $\alpha$-$I$-open set is pre-$I$-open.

(3) $\Rightarrow$ (1): Suppose that $K$ is a pre-$I$-open set and an $\mathcal{A}_I^*$-set. Since $K$ is an $\mathcal{A}_I^*$-set, then we have $K = L \cap M$, where $L$ is an open set and $M = Cl(Int^*(M))$. It follows that $Int(Cl^*(M)) \subset Cl^*(M) \subset Cl(M) = Cl(Int^*(M)) = M$. Since $Int(Cl^*(M)) \subset M$, then $M$ is a semi$^*$-$I$-closed set. By Theorem 25, $M$ is a $t$-$I$-set. Hence, $K$ is a $B_I$-set. Since $K$ is a $B_I$-set and a pre-$I$-open set, then by Proposition 3.3 of [8], $K$ is an open set in $X$. □

**Theorem 27.** *Let $(X, \tau, I)$ be an ideal topological space and $K \subset X$. The following properties are equivalent*:

(1) *$K$ is an open set.*

(2) *$K$ is a $\mathcal{C}_I^*$-set and a semi$^*$-$I$-open set.*

**Proof.** (1) $\Rightarrow$ (2): It follows from the fact that every open set is a $\mathcal{C}_I^*$-set and a semi$^*$-$I$-open set.

(2) $\Rightarrow$ (1): Let $K$ be a $\mathcal{C}_I^*$-set and a semi$^*$-$I$-open set. Since $K$ is a $\mathcal{C}_I^*$-set, then $K$ is a $\mathcal{C}_I$-set. Since $K$ is a $\mathcal{C}_I$-set and a semi$^*$-$I$-open set in $X$, then by Theorem 23, $K$ is an $\mathcal{A}_I^*$-set. Moreover, since $K$ is a $\mathcal{C}_I^*$-set, then $K$ is a pre-$I$-open by Theorem 12. Hence, by Theorem 26, $K$ is an open set in $X$. □

**Theorem 28.** *Let $(X, \tau, I)$ be an ideal topological space and $K \subset X$. The following properties are equivalent*:

(1) *$K$ is an open set.*

(2) *$K$ is an $\alpha$-$I$-open set and a $\mathcal{C}_I^*$-set.*

(3) *$K$ is an $\alpha$-$I$-open set and a $\mathcal{C}_I$-set.*

**Proof.** (1) $\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (3): It follows from the fact that every $\mathcal{C}_I^*$-set is a $\mathcal{C}_I$-set.

(3) $\Rightarrow$ (1): Let $K$ be an $\alpha$-$I$-open set and a $\mathcal{C}_I$-set. It follows that $K$ is a semi$^*$-$I$-open set and a $\mathcal{C}_I$-set. By Theorem 23, $K$ is an $\mathcal{A}_I^*$-set. Since $K$

is an $\alpha$-$I$-open set and an $\mathcal{A}_I^*$-set, then by Theorem 26, $K$ is an open set in $X$. □

**Definition 29.** A subset $K$ of an ideal topological space $(X, \tau, I)$ is said to be *gp*-$I$-open [15] if $N \subset p_I Int(K)$ whenever $N \subset K$ and $N$ is a closed set in $X$, where $p_I Int(K) = K \cap Int(Cl^*(K))$.

**Definition 30.** A subset $K$ of an ideal topological space $(X, \tau, I)$ is said to be generalized pre-$I$-closed ($gp_I$-closed) in $(X, \tau, I)$ if $X \backslash K$ is *gp*-$I$-open.

**Theorem 31.** *For a subset $K$ of an ideal topological space $(X, \tau, I)$, $K$ is $gp_I$-closed if and only if $p_I Cl(K) \subset N$ whenever $K \subset N$ and $N$ is an open set in $(X, \tau, I)$.*

**Proof.** Let $K$ be a $gp_I$-closed set in $X$. Suppose that $K \subset N$ and $N$ is an open set in $(X, \tau, I)$. Then $X \backslash K$ is *gp*-$I$-open and $X \backslash N \subset X \backslash K$ where $X \backslash N$ is closed. Since $X \backslash K$ is *gp*-$I$-open, then we have $X \backslash N \subset p_I Int(X \backslash K)$, where $p_I Int(X \backslash K) = (X \backslash K) \cap Int(Cl^*(X \backslash K))$. Since $(X \backslash K) \cap Int(Cl^*(X \backslash K)) = (X \backslash K) \cap (X \backslash Cl(Int^*(K))) = X \backslash (K \cup Cl(Int^*(K)))$, then by Lemma 8, $(X \backslash K) \cap Int(Cl^*(X \backslash K)) = X \backslash (K \cup Cl(Int^*(K))) = X \backslash p_I Cl(K)$. It follows that $p_I Int(X \backslash K) = X \backslash p_I Cl(K)$. Thus, $p_I Cl(K) = X \backslash p_I Int(X \backslash K) \subset N$ and hence $p_I Cl(K) \subset N$. The converse is similar.□

**Theorem 32.** *Let $(X, \tau, I)$ be an ideal topological space and $V \subset X$. Then $V$ is a $\mathcal{C}_I$-set in $X$ if and only if $V = G \cap p_I Cl(V)$ for an open set $G$ in $X$.*

**Proof.** If $V$ is a $\mathcal{C}_I$-set, then $V = G \cap M$ for an open set $G$ and a pre-$I$-closed set $M$. But then $V \subset M$ and so $V \subset p_I Cl(V) \subset M$. It follows that $V = V \cap p_I Cl(V) = G \cap M \cap p_I Cl(V) = G \cap p_I Cl(V)$. Conversely, it is enough to prove that $p_I Cl(V)$ is a pre-$I$-closed set. But $p_I Cl(V) \subset M$, for any pre-$I$-closed set $M$ containing $V$. So, $Cl(Int^*(p_I Cl(V))) \subset Cl(Int^*(M)) \subset M$. It follows that $Cl(Int^*(p_I Cl(V))) \subset \cap_{V \subset M,\ M \text{ is pre-}I\text{-closed}} M = p_I Cl(V)$.□

**Theorem 33.** *Let $(X, \tau, I)$ be an ideal topological space and $N \subset X$. The following properties are equivalent:*

(1) *$N$ is a pre-$I$-closed set in $X$.*

(2) *$N$ is a $\mathcal{C}_I$-set and a $gp_I$-closed set in $X$.*

**Proof.** (1) ⇒ (2): It follows from the fact that any pre-$I$-closed set in $X$ is a $\mathcal{C}_I$-set and a $gp_I$-closed set in $X$.

(2) $\Rightarrow$ (1): Suppose that $N$ is a $\mathcal{C}_I$-set and a $gp_I$-closed set in $X$. Since $N$ is a $\mathcal{C}_I$-set, then by Theorem 32, $N = G \cap p_I Cl(N)$ for an open set $G$ in $(X, \tau, I)$. Since $N \subset G$ and $N$ is a $gp_I$-closed set in $X$, then $p_I Cl(N) \subset G$. It follows that $p_I Cl(N) \subset G \cap p_I Cl(N) = N$. Thus, $N = p_I Cl(N)$ and hence $N$ is pre-$I$-closed. □

**Theorem 34.** *Let $(X, \tau, I)$ be an ideal topological space and $K \subset X$. If $K$ is a $\mathcal{C}_I$-set in $X$, then $p_I Cl(K) \backslash K$ is a pre-$I$-closed set and $K \cup (X \backslash p_I Cl(K))$ is a pre-$I$-open set in $X$.*

**Proof.** Suppose that $K$ is a $\mathcal{C}_I$-set in $X$. By Theorem 32, we have $K = L \cap p_I Cl(K)$ for an open set $L$ in $X$. It follows that $p_I Cl(K) \backslash K = p_I Cl(K) \backslash (L \cap p_I Cl(K)) = p_I Cl(K) \cap (X \backslash (L \cap p_I Cl(K))) = p_I Cl(K) \cap ((X \backslash L) \cup (X \backslash p_I Cl(K))) = (p_I Cl(K) \cap (X \backslash L)) \cup (p_I Cl(K) \cap (X \backslash p_I Cl(K))) = (p_I Cl(K) \cap (X \backslash L)) \cup \emptyset = p_I Cl(K) \cap (X \backslash L)$. Thus, $p_I Cl(K) \backslash K = p_I Cl(K) \cap (X \backslash L)$ and hence $p_I Cl(K) \backslash K$ is pre-$I$-closed. Moreover, since $p_I Cl(K) \backslash K$ is a pre-$I$-closed set in $X$, then $X \backslash (p_I Cl(K) \backslash K) = X \backslash (p_I Cl(K) \cap (X \backslash K)) = (X \backslash p_I Cl(K)) \cup K$ is a pre-$I$-open set.
Thus, $X \backslash (p_I Cl(K) \backslash K) = (X \backslash p_I Cl(K)) \cup K$ is a pre-$I$-open set in $X$. □

## 3. Further properties

**Definition 35.** Let $(X, \tau, I)$ be an ideal topological space. $(X, \tau, I)$ is said to be pre-$I$-connected if $X$ can not be expressed as the disjoint union of two nonvoid pre-$I$-open sets.

**Theorem 36.** *Let $(X, \tau, I)$ be an ideal topological space. The following properties are equivalent*:

(1) $(X, \tau, I)$ *is pre-$I$-connected.*

(2) $(X, \tau, I)$ *can not be expressed as the disjoint union of two nonvoid $\mathcal{C}_I^*$-sets.*

**Proof.** (1) $\Rightarrow$ (2): Suppose that $(X, \tau, I)$ can be expressed as the disjoint union of two nonvoid $\mathcal{C}_I^*$-sets. Since any $\mathcal{C}_I^*$-set is a pre-$I$-open set, then $(X, \tau, I)$ can be expressed as the disjoint union of two nonvoid pre-$I$-open sets. So, $(X, \tau, I)$ is not pre-$I$-connected. This is a contradiction.

(2) $\Rightarrow$ (1): Suppose that $(X, \tau, I)$ is not pre-$I$-connected. Then, $X$ can be expressed as the disjoint union of two nonvoid pre-$I$-open sets. It follows that $X$ has a nontrivial pre-$I$-regular subset $A$. Moreover, $A$ and

$B = X \backslash A$ are pre-$I$-regular. Then $A$ and $B$ are $\mathcal{C}_I^*$-sets. Hence, $(X, \tau, I)$ can be expressed as the disjoint union of two nonvoid $\mathcal{C}_I^*$-sets. This is a contradiction. □

**Definition 37.** An ideal topological space $(X, \tau, I)$ is called $I$-submaximal [1, 7] if every $\star$-dense subset of $X$ is open.

**Theorem 38** ([7])**.** *For an ideal topological space* $(X, \tau, I)$*, the following properties are equivalent*:

(1) $X$ *is* $I$*-submaximal.*

(2) *Every pre-*$I$*-open set is open.*

(3) *Every pre-*$I$*-open set is semi-*$I$*-open and every* $\alpha$*-*$I$*-open set is open.*

**Theorem 39.** *In an* $I$*-submaximal ideal space* $(X, \tau, I)$*, the following properties hold*:

(1) *Any* $\mathcal{C}_I^*$*-set is an* $\eta\zeta$*-set and an* $AB_I$*-set.*

(2) *Any* $\eta_I$*-set is a locally closed set.*

**Proof.** (1): Suppose that $K$ is a $\mathcal{C}_I^*$-set in $X$. It follows that $K = L \cap M$, where $L$ is an open set and $M$ is a pre-$I$-regular set in $X$. By Theorem 38, $M$ is semi-$I$-open and semi-$I$-closed. It follows from Lemma 4 that $M$ is semi-$I$-open and semi$^*$-$I$-closed. By Theorem 25, $M$ is semi-$I$-open and a $t$-$I$-set in $X$. Thus, $K$ is an $AB_I$-set in $X$. Furthermore, by Theorem 38, $K$ is an $\eta\zeta$-set in $X$.

(2): It follows from Theorem 38. □

**Definition 40.** An ideal topological space $(X, \tau, I)$ is said to be $\star$-hyperconnected [6] if $A$ is $\star$-dense for every open subset $A \neq \emptyset$ of $X$.

**Theorem 41** ([6])**.** *The following properties are equivalent for an ideal topological space* $(X, \tau, I)$:

(1) $X$ *is* $\star$*-hyperconnected.*

(2) $A$ *is* $\star$*-dense for every strongly* $\beta$*-*$I$*-open subset* $\emptyset \neq A \subset X$*.*

**Theorem 42.** *For an ideal topological space* $(X, \tau, I)$*, the following properties are equivalent*:

(1) $(X, \tau, I)$ *is* $\star$*-hyperconnected.*

(2) *any* $\mathcal{C}_I^*$*-set in* $X$ *is* $\star$*-dense.*

**Proof.** (1) $\Rightarrow$ (2): Let $K$ be a $\mathcal{C}_I^*$-set in $X$. By Theorem 12, $K$ is pre-$I$-open. Since $(X, \tau, I)$ is a $\star$-hyperconnected ideal topological space, then by Theorem 41, $K$ is $\star$-dense.

(2) $\Rightarrow$ (1): Suppose that any $\mathcal{C}_I^*$-set in $(X,\tau,I)$ is $\star$-dense in $X$. Since an open set $K$ in $X$ is a $\mathcal{C}_I^*$-set, then $K$ is $\star$-dense. Thus, $(X,\tau,I)$ is $\star$-hyperconnected. $\square$

## 4. Decompositions of continuity and $\mathcal{A}_I^*$-continuity

**Definition 43.** A function $f:(X,\tau,I)\to(Y,\sigma)$ is said to be

(1) $\mathcal{C}_I^*$-continuous if $f^{-1}(A)$ is a $\mathcal{C}_I^*$-set in $X$ for every open set $A$ in $Y$.

(2) $PR_I$-continuous if $f^{-1}(A)$ is a pre-$I$-regular set in $X$ for every open set $A$ in $Y$.

**Remark 44.** For a function $f:(X,\tau,I)\to(Y,\sigma)$, the following diagram holds. The reverses of these implications are not true in general as shown in the following example

$$\begin{array}{ccc} & & \text{pre-}I\text{-continuous} \\ & & \Uparrow \\ PR_I\text{-continuous} & \Longrightarrow & \mathcal{C}_I^*\text{-continuous} \end{array}$$

**Example 45.** Let $X=\{a,b,c,d\}$, $\tau=\{X,\emptyset,\{a\},\{a,b\},\{c,d\},\{a,c,d\}\}$ and $I=\{\emptyset,\{a\},\{d\},\{a,d\}\}$. The function $f:(X,\tau,I)\to(X,\tau)$, defined by $f(a)=c$, $f(b)=d$, $f(c)=a$, $f(d)=b$ is pre-$I$-continuous but it is not $\mathcal{C}_I^*$-continuous. The identity function $i:(X,\tau,I)\to(X,\tau)$ is $\mathcal{C}_I^*$-continuous but it is not $PR_I$-continuous.

**Definition 46.** A function $f:(X,\tau,I)\to(Y,\sigma)$ is said to be

(1) $\mathcal{C}_I$-continuous if $f^{-1}(A)$ is a $\mathcal{C}_I$-set in $X$ for every open set $A$ in $Y$.

(2) $\mathcal{A}_I^*$-continuous if $f^{-1}(A)$ is an $\mathcal{A}_I^*$-set in $X$ for every open set $A$ in $Y$.

(3) $\eta_I$-continuous if $f^{-1}(A)$ is an $\eta_I$-set in $X$ for every open set $A$ in $Y$.

**Remark 47.** For a function $f:(X,\tau,I)\to(Y,\sigma)$, the following diagram holds. The reverses of these implications are not true in general as shown in the following example

$$\begin{array}{ccc} \mathcal{C}_I^*\text{-continuous} & \Longrightarrow & \mathcal{C}_I\text{-continuous} \\ & & \Uparrow \\ \mathcal{A}_I^*\text{-continuous} & \Longrightarrow & \eta_I\text{-continuous} \end{array}$$

**Example 48.** Let $X=\{a,b,c,d\}$, $\tau=\{X,\emptyset,\{a\},\{b,c\},\{a,b,c\}\}$ and $I=\{\emptyset,\{a\},\{d\},\{a,d\}\}$. The function $f:(X,\tau,I)\to(X,\tau)$, defined by

$f(a) = b$, $f(b) = c$, $f(c) = c$, $f(d) = a$ is $\eta_I$-continuous but it is not $\mathcal{A}_I^*$-continuous. The function $g : (X,\tau,I) \to (X,\tau)$, defined by $g(a) = b$, $g(b) = c$, $g(c) = a$, $g(d) = c$ is $\mathcal{C}_I^*$-continuous but it is not $\eta_I$-continuous.

**Example 49.** Let $X = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}\}$ and $I = \{\emptyset, \{a\}, \{d\}, \{a,d\}\}$. The function $f : (X,\tau,I) \to (X,\tau)$, defined by $f(a)=b, f(b)=a, f(c)=c, f(d)=d$ is $\mathcal{C}_I$-continuous and $\mathcal{A}_I^*$-continuous but it is not $\mathcal{C}_I^*$-continuous. The function $g : (X,\tau,I) \to (X,\tau)$, defined by $g(a) = a$, $g(b) = a$, $g(c) = b$, $g(d) = a$ is $\mathcal{C}_I$-continuous but it is not $\eta_I$-continuous.

**Definition 50.** A function $f : (X,\tau,I) \to (Y,\sigma)$ is said to be semi$^*$-$I$-continuous if $f^{-1}(V)$ is a semi$^*$-$I$-open set in $X$ for every open set $V$ in $Y$.

**Theorem 51.** *The following properties are equivalent for a function* $f : (X,\tau,I) \to (Y,\sigma)$:
(1) $f$ *is* $\mathcal{A}_I^*$*-continuous.*
(2) $f$ *is* $\eta_I$*-continuous and semi*$^*$*-*$I$*-continuous.*
(3) $f$ *is* $\mathcal{C}_I$*-continuous and semi*$^*$*-*$I$*-continuous.*

**Proof.** It follows from Theorem 23. □

**Theorem 52.** *The following properties are equivalent for a function* $f : (X,\tau,I) \to (Y,\sigma)$:
(1) $f$ *is continuous.*
(2) $f$ *is* $\alpha$*-*$I$*-continuous and* $\mathcal{A}_I^*$*-continuous.*
(3) $f$ *is pre-*$I$*-continuous and* $\mathcal{A}_I^*$*-continuous.*
(4) $f$ *is semi*$^*$*-*$I$*-continuous and* $\mathcal{C}_I^*$*-continuous.*
(5) $f$ *is* $\alpha$*-*$I$*-continuous and* $\mathcal{C}_I^*$*-continuous.*
(6) $f$ *is* $\alpha$*-*$I$*-continuous and* $\mathcal{C}_I$*-continuous.*

**Proof.** It follows from Theorem 26, 27 and 28. □

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