

ABSTRACT REGULAR NULL-NULL-ADDITIVE SET MULTIFUNCTIONS IN HAUSDORFF TOPOLOGY

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Abstract. In this paper we introduce and study various types of abstract regularity in Hausdorff topology for monotone set multifunctions, with direct applications in well-known situations such as the Borel σ -algebra of a Hausdorff space and/or the Borel (Baire, respectively) δ -ring or σ -ring of a locally compact Hausdorff space. Some relationships among abstract regularities are investigated.

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1. Introduction and preliminary

As it is well-known, regularity is an important property of continuity, which connects measure theory and topology. It gives us an important tool to approximate general Borel sets by more tractable sets, such as, for instance, compact and/or open sets. That is why, many authors gave a special interest in the last years to different problems and applications of regularity, for instance, in order to obtain Alexandroff and Lusin type theorems, in convergence problems for Choquet integrals etc. We mention in this sense the works of DINCULEANU [3] (for real normed space-valued measures), HA and WANG [11], LI and YASUDA [18], LI, YASUDA and SONG [19], NARUKAWA [20], NARUKAWA, MUROFUSHI and SUGENO [21], SONG and LI [27], WU and HA [28], WU and WU [29] (for fuzzy measures) etc. We also mention the contributions [14-17] of Kawabe, who generalized many classical fuzzy measure theory problems to Riesz space-valued measures.

On the other hand, by theoretical and practical necessities, a set-valued measure theory became to develop (see, for instance, GUO and ZHANG [10], GAVRILUȚ [4-9], PRECUPANU [23, 24], PRECUPANU, GAVRILUȚ and CROITORU [25], PRECUPANU and GAVRILUȚ [26], ZHANG and GUO [30], ZHANG and WANG [31] and many others).

In this paper, we focus on the study of different abstract regularities for fuzzy set multifunctions. Some relationships among these types of regularity are presented.

The results we obtained generalize other previous results of GAVRILUȚ [4-9], KAWABE [16], WU and WU [29], LI and YASUDA [18], NARUKAWA [20], NARUKAWA ET. AL [21], JIANG and SUZUKI [13] etc. Our generalization is made in two directions: firstly, since here we deal with abstract regularity and secondly, because we work in the set-valued case.

We mention that regularity in an abstract sense was also treated for topological group-valued measures by BELLEY and MORALES [2], by PRECUPANU [24] for additive multimeasures etc.

2. Terminology, notations and basic results

Let T be an abstract set, \mathcal{C} a ring of subsets of T , X a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X , $\mathcal{P}_f(X)$ the family of closed, nonvoid sets of X , $\mathcal{P}_{bf}(X)$ the family of all bounded, closed, nonvoid sets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$.

As it is well-known, $h(M, N) = \max\{e(M, N), e(N, M)\}$, for every $M, N \in \mathcal{P}_f(X)$, where $e(M, N) = \sup_{x \in M} d(x, N)$ is the excess of M over N .

On $\mathcal{P}_{bf}(X)$, h becomes a metric [12].

We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X . If, in addition, X is complete, then the same is $\mathcal{P}_f(X)$ [12].

We observe that $e(N, M) = h(M, N)$, for every $M, N \in \mathcal{P}_f(X)$, with $M \subseteq N$. Also, $e(M, N) \leq e(M, P)$, for every $M, N, P \in \mathcal{P}_f(X)$, with $P \subseteq N$ and $e(M, P) \leq e(N, P)$, for every $M, N, P \in \mathcal{P}_f(X)$, with $M \subseteq N$.

On $\mathcal{P}_0(X)$ we introduce the Minkowski addition " $\overset{\bullet}{+}$ " defined by:

$$M \overset{\bullet}{+} N = \overline{M + N}, \text{ for every } M, N \in \mathcal{P}_0(X),$$

where $M + N = \{x + y; x \in M, y \in N\}$ and $\overline{M + N}$ is the closure of $M + N$ with respect to the topology induced by the norm of X .

We denote by \mathbb{N} the set of all naturals, by \mathbb{R} the set of all real numbers and by \mathbb{R}_+ the set $[0, \infty)$.

Also, by cA we usually mean $T \setminus A$, where $A \subset T$.

Throughout the paper we shall use the following notions in the set-valued case:

Definition 2.1 ([4]-[9], [25, 26]). A set multifunction $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ is said to be:

- I) *increasing convergent* (with respect to h) if $\lim_{n \rightarrow \infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing sequence of sets $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$.
- II) *decreasing convergent* (with respect to h) if $\lim_{n \rightarrow \infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$, with $A_n \searrow A \in \mathcal{C}$.
- III) i) *fuzzy* (or, *monotone*) if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.
 ii) *fuzzy in the sense of Sugeno* (briefly, *(S)-fuzzy*) if it is fuzzy, increasing convergent, decreasing convergent and $\mu(\emptyset) = \{0\}$.
- IV) *exhaustive* (with respect to h) if $\lim_{n \rightarrow \infty} |\mu(A_n)| = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$.
- V) *order continuous* (with respect to h) if $\lim_{n \rightarrow \infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$, with $A_n \searrow \emptyset$.
- VI) *autocontinuous from above* if for every $A \in \mathcal{C}$ and every $(B_n)_{n \in \mathbb{N}} \subset \mathcal{C}$, with $\lim_{n \rightarrow \infty} |\mu(B_n)| = 0$, we have $\lim_{n \rightarrow \infty} h(\mu(A \cup B_n), \mu(A)) = 0$.
- VII) *uniformly autocontinuous* if for every $A \in \mathcal{C}$ and every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ so that for every $B \in \mathcal{C}$, with $|\mu(B)| < \delta$, we have $h(\mu(A \cup B), \mu(A)) < \varepsilon$.
- VIII) i) *null-additive* if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{C}$, with $\mu(B) = \{0\}$.
 ii) *null-null-additive* if $\mu(A \cup B) = \{0\}$, for every $A, B \in \mathcal{C}$, with $\mu(A) = \mu(B) = \{0\}$.
- IX) *single asymptotic null-additive* if for every $A \in \mathcal{C}$ with $\mu(A) = \{0\}$ and every sequence $(B_n)_{n \in \mathbb{N}} \subset \mathcal{C}$, with $\lim_{n \rightarrow \infty} |\mu(B_n)| = 0$, we have $\lim_{n \rightarrow \infty} |\mu(A \cup B_n)| = 0$.

X) *asymptotic null-additive* if for every sequences $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{C}$, with $\lim_{n \rightarrow \infty} |\mu(A_n)| = \lim_{n \rightarrow \infty} |\mu(B_n)| = 0$, we have $\lim_{n \rightarrow \infty} |\mu(A_n \cup B_n)| = 0$.

All over the paper, unless stated otherwise, $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ is supposed to be a *fuzzy (i.e., monotone) set multifunction*, with $\mu(\emptyset) = \{0\}$.

Remark 2.2. We observe that the above definitions I)-X) generalize to the fuzzy set-valued measure case the corresponding notions from the classical fuzzy measure case (see [1], [13], [16], [22]). An example in this sense is the following:

One may consider *the fuzzy set multifunction* $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(\mathbb{R}_+)$, defined for every $A \in \mathcal{C}$ by $\mu(A) = [0, m(A)]$, which is *induced by a fuzzy set function* $m : \mathcal{C} \rightarrow \mathbb{R}_+$. We immediately observe that μ satisfies one of the notions I)-X) if and only if the same does m , due to the inequalities: $h([0, a], [0, b]) = |a - b|$ and $|[0, a]| = a$, for every $a, b \in \mathbb{R}_+$.

Proposition 2.3. I) *i) If μ is asymptotic null-additive, then it is single asymptotic null-additive.*

ii) If μ is single asymptotic null-additive, then it is null-null-additive.

II) *Suppose \mathcal{C} is a δ -ring.*

i) If μ is null-null-additive and decreasing convergent, then it is asymptotic null-additive.

ii) If μ is null-null-additive and decreasing convergent, then it is single asymptotic null-additive.

Proof. I) The statements are straightforward by the definitions.

II) i) Let be $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{C}$, with $\lim_{n \rightarrow \infty} |\mu(A_n)| = \lim_{n \rightarrow \infty} |\mu(B_n)| = 0$.

We denote $A = \bigcap_{n=1}^{\infty} A_n$ and $B = \bigcap_{n=1}^{\infty} B_n$. Without loss of generality, suppose $A_n \searrow A$ and $B_n \searrow B$. Because μ is decreasing convergent, $\lim_{n \rightarrow \infty} h(\mu(A_n), \mu(A)) = 0$, so,

$$0 \leq |\mu(A)| \leq \lim_{n \rightarrow \infty} |\mu(A_n)| + \lim_{n \rightarrow \infty} h(\mu(A_n), \mu(A)) = 0,$$

which implies that $\mu(A) = \{0\}$. Analogously, $\mu(B) = \{0\}$, so $\mu(A \cup B) = \{0\}$. Again, by the decreasing convergence of μ , we have $\lim_{n \rightarrow \infty} h(\mu(A_n \cup B_n), \mu(A \cup B)) = 0$. Consequently,

$$0 \leq \lim_{n \rightarrow \infty} |\mu(A_n \cup B_n)| \leq \lim_{n \rightarrow \infty} h(\mu(A_n \cup B_n), \mu(A \cup B)) + |\mu(A \cup B)| = 0,$$

hence, μ is asymptotic null-additive.

ii) The statement immediately follows by I i) and II i). \square

Corollary 2.4. *Suppose \mathcal{C} is a δ -ring. If μ is decreasing convergent, then i) μ is asymptotic null-additive \Leftrightarrow ii) μ is single asymptotic null-additive \Leftrightarrow iii) μ is null-null-additive.*

One can easily check the following results:

Remark 2.5. i) If μ is uniformly autocontinuous, then it is autocontinuous from above.

ii) If μ is autocontinuous from above, then μ is asymptotic null-additive, so, by Proposition 2.3 I), it is also null-null-additive.

iii) If $\mu : \mathcal{C} \rightarrow \mathcal{P}_{bf}(X)$ is autocontinuous from above, then it is null-additive.

iv) a) If $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ is decreasing convergent, then it is order continuous.

b) If \mathcal{C} is a σ -ring and μ is order continuous, then μ is exhaustive.

Definition 2.6 ([9]). We say that $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ has the *pseudometric generating property* (briefly, *PGP*) if for every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ so that for every $A, B \in \mathcal{C}$, with $|\mu(A)| < \delta$ and $|\mu(B)| < \delta$, we have $|\mu(A \cup B)| < \varepsilon$.

Remark 2.7 ([9]). i) If μ is uniformly autocontinuous, then it has PGP.

ii) If \mathcal{C} is a σ -ring and μ is fuzzy, increasing convergent and autocontinuous from above, then μ has PGP.

By the definitions, in like manner as in the proof of Proposition 2.3 II) i), one can easily verify:

Remark 2.8. i) If \mathcal{C} is a δ -ring and μ is null-additive and decreasing convergent, then it is autocontinuous from above, so it is also asymptotic null-additive.

ii) a) Any autocontinuous from above order continuous set multifunction is increasing convergent and decreasing convergent.

b) Any decreasing convergent set multifunction is order continuous. Consequently, if μ is fuzzy and autocontinuous from above, then μ is (S)-fuzzy if and only if it is order continuous.

iii) If μ has PGP, then μ is asymptotic null-additive.

By Remark 2.8 i) and Remark 2.5 iii), we get:

Remark 2.9. If \mathcal{C} is a δ -ring and $\mu : \mathcal{C} \rightarrow \mathcal{P}_{bf}(X)$ is decreasing convergent, then μ is null-additive if and only if it is autocontinuous from above.

In what follows in this section, let \mathcal{C} be a σ -ring. We recall from [26] the following:

Definition 2.10. A double sequence $\{A_{m,n}\}_{(m,n) \in \mathbb{N}^2} \subset \mathcal{C}$ is called a μ -regulator if it satisfies the following two conditions:

(R₁) $A_{m,n} \supset A_{m,n'}$, whenever $m, n, n' \in \mathbb{N}$ and $n \leq n'$;

(R'₂) $\mu(\bigcap_{n=1}^{\infty} A_{m,n}) = \{0\}$, for any $m \in \mathbb{N}$.

Definition 2.11. μ is said to fulfil condition (E') if for any $\varepsilon > 0$ and any μ -regulator $\{A_{m,n}\}_{(m,n) \in \mathbb{N}^2} \subset \mathcal{C}$, there exists an increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ of naturals such that $|\mu(\bigcup_{i=1}^{\infty} A_{i,n_i})| < \varepsilon$.

Proposition 2.12 ([26]). *Let μ be (S) -fuzzy. Then μ fulfils condition (E') if and only if it is null-null-additive.*

By Remark 2.5 ii), Remark 2.7 ii) and Proposition 2.12, we have:

Corollary 2.13. *i) If μ is (S) -fuzzy and autocontinuous from above, then it fulfils (E') .*

ii) Moreover, if \mathcal{C} is a σ -ring and μ is (S) -fuzzy and autocontinuous from above, then μ fulfils (E') and has PGP.

By Proposition 2.12 and Corollary 2.4, we have:

Corollary 2.14. *Suppose \mathcal{C} is a δ -ring and μ is (S) -fuzzy. Then i) μ is null-null-additive \Leftrightarrow ii) μ fulfils (E') \Leftrightarrow iii) μ is asymptotic null-additive \Leftrightarrow iv) μ is single asymptotic null-additive.*

Many of the results we obtained in this section generalize to the set-valued case previous results of KAWABE [16], JIANG and SUZUKI [13], PAP [22] concerning corresponding asymptotic structural properties of fuzzy set functions.

3. Abstract regularity for monotone set multifunctions

In this section, we shall introduce different types of abstract regularity in the fuzzy set-valued measure case and we shall establish various relationships among them. These results will find their direct applicability in concrete regularities that will be studied in Section 4.

Some of the results concerning abstract regularity we shall obtain in this section generalize other previous results we obtained in [4-6] and [8].

Let T be a Hausdorff space, X a real normed space, \mathcal{C} a ring of subsets of T , $A \in \mathcal{C}$ an arbitrary set, $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ a fuzzy (i.e., monotone) set multifunction, with $\mu(\emptyset) = \{0\}$ and $\mathcal{M}, \mathcal{N} \subset \mathcal{P}(T)$ two arbitrary nonvoid families of subsets of T .

For the consistency of the following notions, one may place itself in one of the following situations (but not only, as we shall see, for instance, in Theorem 3.10 or Example 3.12):

(i) T is a Hausdorff space, \mathcal{C} is the Borel σ -algebra $\tilde{\mathcal{B}}$ generated by the open sets of T , $\mathcal{M} = \mathcal{F}$, the family of closed subsets of T and $\mathcal{N} = \mathcal{D}$, the family of open subsets of T or $\mathcal{M} = \mathcal{K}$, the family of compact subsets of T and $\mathcal{N} = \mathcal{D}$;

(It is known that $\tilde{\mathcal{B}} = \mathcal{A}(\mathcal{F}) = \mathcal{A}(\mathcal{D})$, and, particularly, if $T = \mathbb{R}^n$, then $\tilde{\mathcal{B}} = \mathcal{A}(\mathcal{K})$, where, by $\mathcal{A}(\mathcal{L})$, we mean the σ -algebra generated by a nonvoid family \mathcal{L} of subsets of T .)

(ii) T is, particularly, a locally compact Hausdorff space, \mathcal{C} is \mathcal{B}_0 (respectively, \mathcal{B}'_0) - the Baire δ -ring (respectively, σ -ring) generated by compact sets, which are G_δ (i.e., countable intersections of open sets) or \mathcal{C} is \mathcal{B} (respectively, \mathcal{B}') - the Borel δ -ring (respectively, σ -ring) generated by the compact sets of T , $\mathcal{M} = \mathcal{K}$ and $\mathcal{N} = \mathcal{D}$.

Note that $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{B}_0 \subset \mathcal{B}'_0$, $\mathcal{B}'_0 \subset \mathcal{B}'$ and $\mathcal{B} \subset \mathcal{B}'$.

According to the usage of \mathcal{M} and \mathcal{N} , \mathcal{F} and \mathcal{D} or \mathcal{K} and \mathcal{D} , it will be understood that we place ourselves, respectively, in the general situation, situation (i)/(ii) or situation (ii) (we remark that in situation (i) we may have \mathcal{F} and \mathcal{D} or \mathcal{K} and \mathcal{D} and in situation (ii) we may have \mathcal{K} and \mathcal{D}).

We shall particularly study situation (i) when T is a locally compact Hausdorff space.

Since $e(N, M) = h(M, N)$, for every $M, N \in \mathcal{P}_f(X)$, with $M \subseteq N$, we give the following:

Definition 3.1. I) A is said to be:

- (i) $R_{\mathcal{M},\mathcal{N}}$ -regular if for every $\varepsilon > 0$, there are $M \in \mathcal{M} \cap \mathcal{C}$, $M \subset A$ and $N \in \mathcal{N} \cap \mathcal{C}$, $N \supset A$ so that $e(\mu(N), \mu(M)) < \varepsilon$.
- (ii) $R_{\mathcal{M}}$ -regular if for every $\varepsilon > 0$, there exists $M \in \mathcal{M} \cap \mathcal{C}$, $M \subset A$ so that $e(\mu(A), \mu(M)) < \varepsilon$.
- (iii) $R_{\mathcal{N}}$ -regular if for every $\varepsilon > 0$, there exists $N \in \mathcal{N} \cap \mathcal{C}$, $N \supset A$ such that $e(\mu(N), \mu(A)) < \varepsilon$.
- (iv) $R'_{\mathcal{M},\mathcal{N}}$ -regular if for every $\varepsilon > 0$, there are $M \in \mathcal{M} \cap \mathcal{C}$, $M \subset A$ and $N \in \mathcal{N} \cap \mathcal{C}$, $A \subset N$ so that $|\mu(N \setminus M)| < \varepsilon$.
- (v) $R'_{\mathcal{M}}$ -regular if for every $\varepsilon > 0$, there is $M \in \mathcal{M} \cap \mathcal{C}$, $M \subset A$ so that $|\mu(A \setminus M)| < \varepsilon$.
- (vi) $R'_{\mathcal{N}}$ -regular if for every $\varepsilon > 0$, there is $N \in \mathcal{N} \cap \mathcal{C}$, $A \subset N$ such that $|\mu(N \setminus A)| < \varepsilon$.

II) μ is said to be:

- i) $R_{\mathcal{M},\mathcal{N}}$ -regular (respectively, $R_{\mathcal{M}}$ -regular, $R_{\mathcal{N}}$ -regular) if every set $A \in \mathcal{C}$ is $R_{\mathcal{M},\mathcal{N}}$ -regular (respectively, $R_{\mathcal{M}}$ -regular, $R_{\mathcal{N}}$ -regular).
- ii) $R'_{\mathcal{M},\mathcal{N}}$ -regular (respectively, $R'_{\mathcal{M}}$ -regular, $R'_{\mathcal{N}}$ -regular) if every set $A \in \mathcal{C}$ is $R'_{\mathcal{M},\mathcal{N}}$ -regular (respectively, $R'_{\mathcal{M}}$ -regular, $R'_{\mathcal{N}}$ -regular).

Remark 3.2. I) In situation (ii), our definitions generalize the well-known notions we introduced and studied in [4-9].

In situation (i), by the aid of the fuzzy induced set multifunction, we get the generalization to the set-valued case of the well-known notions of regularity from the classical fuzzy measures theory.

II) Every $M \in \mathcal{M} \cap \mathcal{C}$ is $R_{\mathcal{M}}$ -regular and $R'_{\mathcal{M}}$ -regular and every $N \in \mathcal{N} \cap \mathcal{C}$ is $R_{\mathcal{N}}$ -regular and $R'_{\mathcal{N}}$ -regular.

III) i) μ is $R'_{\mathcal{M},\mathcal{N}}$ -regular if and only if for every $A \in \mathcal{C}$ there are a sequence $(M_n) \subset \mathcal{M} \cap \mathcal{C}$ and a sequence $(N_n) \subset \mathcal{N} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $M_n \subset A \subset N_n$ and $\lim_{n \rightarrow \infty} |\mu(N_n \setminus M_n)| = 0$.

ii) μ is $R'_{\mathcal{M}}$ -regular if and only if for every $A \in \mathcal{C}$ there is a sequence $(M_n) \subset \mathcal{M} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $M_n \subset A$ and $\lim_{n \rightarrow \infty} |\mu(A \setminus M_n)| = 0$.

- iii) μ is $R'_{\mathcal{N}}$ -regular if and only if for every $A \in \mathcal{C}$ there is a sequence $(N_n) \subset \mathcal{N} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $A \subset N_n$ and $\lim_{n \rightarrow \infty} |\mu(N_n \setminus A)| = 0$.
 - iv) μ is $R_{\mathcal{M}, \mathcal{N}}$ -regular if and only if for every $A \in \mathcal{C}$ there are a sequence $(M_n) \subset \mathcal{M} \cap \mathcal{C}$ and a sequence $(N_n) \subset \mathcal{N} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $M_n \subset A \subset N_n$ and $\lim_{n \rightarrow \infty} e(\mu(N_n), \mu(M_n)) = 0$.
 - v) μ is $R_{\mathcal{M}}$ -regular if and only if for every $A \in \mathcal{C}$ there is a sequence $(M_n) \subset \mathcal{M} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $M_n \subset A$ and $\lim_{n \rightarrow \infty} e(\mu(A), \mu(M_n)) = 0$.
 - vi) μ is $R_{\mathcal{N}}$ -regular if and only if for every $A \in \mathcal{C}$ there is a sequence $(N_n) \subset \mathcal{N} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $A \subset N_n$ and $\lim_{n \rightarrow \infty} e(\mu(N_n), \mu(A)) = 0$.
- IV) (M_n) may be chosen increasing and (N_n) may be chosen decreasing.
- V) If $\mathcal{M} = \mathcal{C}$ or if $\mathcal{N} = \mathcal{C}$, then μ is $R'_{\mathcal{C}}$ -regular and $R_{\mathcal{C}}$ -regular.
- VI) i) If $\mathcal{M}_1 \subset \mathcal{M}_2$ and μ is $R'_{\mathcal{M}_1}$ -regular ($R_{\mathcal{M}_1}$ -regular, respectively), then μ is $R'_{\mathcal{M}_2}$ -regular ($R_{\mathcal{M}_2}$ -regular, respectively).
- ii) If $\mathcal{N}_1 \subset \mathcal{N}_2$ and μ is $R'_{\mathcal{N}_1}$ -regular ($R_{\mathcal{N}_1}$ -regular, respectively), then μ is $R'_{\mathcal{N}_2}$ -regular ($R_{\mathcal{N}_2}$ -regular, respectively).
- iii) If $\mathcal{M}_1 \subset \mathcal{M}_2, \mathcal{N}_1 \subset \mathcal{N}_2$ and μ is $R'_{\mathcal{M}_1, \mathcal{N}_1}$ -regular ($R_{\mathcal{M}_1, \mathcal{N}_1}$ -regular, respectively), then μ is $R'_{\mathcal{M}_2, \mathcal{N}_2}$ -regular ($R_{\mathcal{M}_2, \mathcal{N}_2}$ -regular, respectively).

In what follows, we establish different relationships among these types of abstract regularity, which generalize some other previous results we established in [6] and [8].

Theorem 3.3. *i) μ is $R_{\mathcal{M}, \mathcal{N}}$ -regular if and only if it is $R_{\mathcal{M}}$ -regular and $R_{\mathcal{N}}$ -regular.*

- ii) If μ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular, then it is $R'_{\mathcal{M}}$ -regular and $R'_{\mathcal{N}}$ -regular.*
- iii) If μ is asymptotic null-additive, then μ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular if and only if it is $R'_{\mathcal{M}}$ -regular and $R'_{\mathcal{N}}$ -regular.*
- iv) If \mathcal{C} is a δ -ring and μ is decreasing convergent and null-null-additive, then μ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular if and only if it is $R'_{\mathcal{M}}$ -regular and $R'_{\mathcal{N}}$ -regular.*

Proof. The statements i) and ii) can be proved analogously as in [8].

iii) The *Only if part* follows by ii).

For *The if part*, let $A \in \mathcal{C}$ be arbitrarily. Since A is $R'_{\mathcal{M}}$ -regular, there is $(M_n) \subset \mathcal{M} \cap \mathcal{C}$ so that $M_n \subset A$ and $\lim_{n \rightarrow \infty} |\mu(A \setminus M_n)| = 0$. Also, because A is $R'_{\mathcal{N}}$ -regular, there is $(N_n) \subset \mathcal{D} \cap \mathcal{C}$ so that $A \subset N_n$ and $\lim_{n \rightarrow \infty} |\mu(N_n \setminus A)| = 0$. Then $\lim_{n \rightarrow \infty} |\mu(N_n \setminus M_n)| = \lim_{n \rightarrow \infty} |\mu((N_n \setminus A) \cup (A \setminus M_n))| = 0$, hence A is $R'_{\mathcal{M}, \mathcal{N}}$ -regular.

iv) The statement is straightforward according to iii) and Proposition 2.3 II i). \square

Corollary 3.4. (i) *A set $M \in \mathcal{M} \cap \mathcal{C}$ is $R_{\mathcal{M}, \mathcal{N}}$ -regular if and only if it is $R_{\mathcal{N}}$ -regular and a set $N \in \mathcal{N} \cap \mathcal{C}$ is $R_{\mathcal{M}, \mathcal{N}}$ -regular if and only if it is $R_{\mathcal{M}}$ -regular.*

(ii) *If μ is asymptotic null-additive, then a set $M \in \mathcal{M} \cap \mathcal{C}$ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular if and only if it is $R'_{\mathcal{N}}$ -regular and a set $N \in \mathcal{N} \cap \mathcal{C}$ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular if and only if it is $R'_{\mathcal{M}}$ -regular.*

Theorem 3.5. *Suppose μ is autocontinuous from above.*

i) *If μ is $R'_{\mathcal{M}}$ -regular, then μ is $R_{\mathcal{M}}$ -regular.*

ii) *If μ is $R'_{\mathcal{N}}$ -regular, then μ is $R_{\mathcal{N}}$ -regular.*

iii) *If μ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular, then μ is $R_{\mathcal{M}, \mathcal{N}}$ -regular.*

Proof. i) and ii) are immediate because μ is autocontinuous from above.

iii) The statement is straightforward by i), ii), Theorem 3.3 ii) and i) and also by Remark 2.5 ii). \square

Proposition 3.6. *Suppose \mathcal{C} is an algebra, \mathcal{M} is arbitrarily, $\mathcal{N} = \{A \in \mathcal{C}; cA \in \mathcal{M}\}$ and μ is asymptotic null-additive and $R'_{\mathcal{M}}$ -regular (or, respectively, $R'_{\mathcal{N}}$ -regular). Then μ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular.*

Proof. Let us consider arbitrary $A \in \mathcal{C}$.

If μ is $R'_{\mathcal{M}}$ -regular, there exists an increasing sequence of sets $(M_n) \subset \mathcal{M} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $M_n \subset A$ and $\lim_{n \rightarrow \infty} |\mu(A \setminus M_n)| = 0$.

Analogously, for $cA \in \mathcal{C}$, there exists an increasing sequence of sets $(P_n) \subset \mathcal{M} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $P_n \subset cA$ and $\lim_{n \rightarrow \infty} |\mu(cA \setminus P_n)| = 0$.

If we denote $N_n = cP_n$, for every $n \in \mathbb{N}$, then $(N_n) \subset \mathcal{N} \cap \mathcal{C}$, $A \subset N_n$ and $\lim_{n \rightarrow \infty} |\mu(N_n \setminus A)| = 0$.

Since μ is asymptotic null-additive, $\lim_{n \rightarrow \infty} |\mu(N_n \setminus M_n)| = 0$, so, finally, μ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular.

Similar argues in case when μ is $R'_{\mathcal{N}}$ -regular. \square

By Theorem 3.3 and Theorem 3.6, we have:

Corollary 3.7. *Suppose \mathcal{C} is an algebra, \mathcal{M} is arbitrarily, $\mathcal{N} = \{A \in \mathcal{C}; cA \in \mathcal{M}\}$ and μ is asymptotic null-additive. Then i) μ is $R'_{\mathcal{M}}$ -regular \Leftrightarrow ii) μ is $R'_{\mathcal{N}}$ -regular \Leftrightarrow iii) μ is $R'_{\mathcal{M}, \mathcal{N}}$ -regular.*

Theorem 3.8. *Suppose $\mu_1, \mu_2 : \mathcal{C} \rightarrow \mathcal{P}_{bf}(X)$ are fuzzy and let \mathcal{M} be closed for finite unions and \mathcal{N} be closed for finite intersections. Then:*

i) *If μ_1, μ_2 are $R_{\mathcal{M}}$ -regular, then $\mu_1 = \mu_2$ on $\mathcal{M} \cap \mathcal{C}$ if and only if $\mu_1 = \mu_2$ on \mathcal{C} .*

ii) *If μ_1, μ_2 are $R_{\mathcal{N}}$ -regular, then $\mu_1 = \mu_2$ on $\mathcal{N} \cap \mathcal{C}$ if and only if $\mu_1 = \mu_2$ on \mathcal{C} .*

Proof. Let $A \in \mathcal{C}$ be an arbitrary set.

i) The *If part* is straightforward.

The *Only if part*: Suppose μ_1 and μ_2 are $R_{\mathcal{M}}$ -regular and $\mu_1 = \mu_2$ on $\mathcal{M} \cap \mathcal{C}$. Because μ_1 is $R_{\mathcal{M}}$ -regular, there is a sequence $(M_n^1) \subset \mathcal{M} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $M_n^1 \subset A$ and $\lim_{n \rightarrow \infty} e(\mu_1(A), \mu_1(M_n^1)) = 0$. Analogously, for μ_2 , there is an increasing sequence $(M_n^2) \subset \mathcal{M} \cap \mathcal{C}$ so that for every $n \in \mathbb{N}$, $M_n^2 \subset A$ and $\lim_{n \rightarrow \infty} e(\mu_2(A), \mu_2(M_n^2)) = 0$. For every $n \in \mathbb{N}$, we denote $M_n = M_n^1 \cup M_n^2 \in \mathcal{M} \cap \mathcal{C}$. Obviously, $\lim_{n \rightarrow \infty} h(\mu_1(A), \mu_1(M_n)) = 0$ and $\lim_{n \rightarrow \infty} h(\mu_2(A), \mu_2(M_n)) = 0$. Because for every $n \in \mathbb{N}$, $\mu_1(M_n) = \mu_2(M_n)$, then $h(\mu_1(A), \mu_2(A)) = 0$, so, because $\mu_1, \mu_2 : \mathcal{C} \rightarrow \mathcal{P}_{bf}(X)$, we get $\mu_1(A) = \mu_2(A)$.

ii) One can obtain the result using similar argues as in i). \square

In what follows, let \mathcal{C} be a ring of subsets of T and $\mu_1, \mu_2 : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ two fuzzy set multifunctions, with $\mu_1(\emptyset) = \mu_2(\emptyset) = \{0\}$.

We introduce now the fuzzy set multifunction $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$, with $\mu(\emptyset) = \{0\}$, defined for every $A \in \mathcal{C}$ by $\mu(A) = \mu_1(A) \dot{+} \mu_2(A)$ (the so called *Minkowski sum of the fuzzy set multifunctions*).

In the sequel, we observe that μ preserves the continuity properties which both μ_1 and μ_2 have:

Theorem 3.9. *i) If both $\mu_1, \mu_2 : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ are order continuous (increasing convergent, decreasing convergent, (S)-fuzzy or exhaustive, respectively), then the same is μ ;*

ii) If μ_1 is $R'_{\mathcal{M}_1}$ -regular ($R_{\mathcal{M}_1}$ -regular, respectively) and μ_2 is $R'_{\mathcal{M}_2}$ -regular ($R_{\mathcal{M}_2}$ -regular, respectively), where \mathcal{M}_1 and \mathcal{M}_2 are two arbitrary families of subsets of T , then μ is $R'_{\mathcal{M}_1 \cup \mathcal{M}_2}$ -regular ($R_{\mathcal{M}_1 \cup \mathcal{M}_2}$ -regular, respectively), where $\mathcal{M}_1 \cup \mathcal{M}_2 = \{M_1 \cup M_2; M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2\}$.

Particularly, if μ_1 and μ_2 are $R'_{\mathcal{M}}$ -regular ($R_{\mathcal{M}}$ -regular, respectively), then the same is μ , where \mathcal{M} is an arbitrary, closed for finite unions, family of subsets of T .

iii) If μ_1 is $R'_{\mathcal{N}_1}$ -regular ($R_{\mathcal{N}_1}$ -regular, respectively) and μ_2 is $R'_{\mathcal{N}_2}$ -regular ($R_{\mathcal{N}_2}$ -regular, respectively), where \mathcal{N}_1 and \mathcal{N}_2 are two arbitrary families of subsets of T , then μ is $R'_{\mathcal{N}_1 \cap \mathcal{N}_2}$ -regular ($R_{\mathcal{N}_1 \cap \mathcal{N}_2}$ -regular, respectively), where $\mathcal{N}_1 \cap \mathcal{N}_2 = \{N_1 \cap N_2; N_1 \in \mathcal{N}_1, N_2 \in \mathcal{N}_2\}$.

Particularly, if μ_1 and μ_2 are $R'_{\mathcal{N}}$ -regular ($R_{\mathcal{N}}$ -regular, respectively), then the same is μ , where \mathcal{N} is an arbitrary, closed for finite intersections, family of subsets of T .

Proof. i) One may easily verify the statements using the inequalities:

$$(1) \quad |M \dot{+} N| \leq |M| + |N|, \text{ for every } M, N \in \mathcal{P}_f(X);$$

$$(2) \quad h(M \dot{+} P, N \dot{+} P) \leq h(M, N),$$

for every $M, N, P \in \mathcal{P}_f(X)$, which implies that for every $n \in \mathbb{N}$, (3)

$$(3) \quad \begin{aligned} & h(\mu_1(A_n) \dot{+} \mu_2(A_n), \mu_1(A) \dot{+} \mu_2(A)) \\ & \leq h(\mu_1(A_n) \dot{+} \mu_2(A_n), \mu_1(A_n) \dot{+} \mu_2(A)) \\ & + h(\mu_1(A_n) \dot{+} \mu_2(A), \mu_1(A) \dot{+} \mu_2(A)) \\ & \leq h(\mu_1(A_n), \mu_1(A)) + h(\mu_2(A_n), \mu_2(A)), \end{aligned}$$

for every $(A_n)_n \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$ or $A_n \searrow A \in \mathcal{C}$.

ii) and iii) are simple consequences of the definitions, also taking into account that, generally,

$$h(M_1 \dot{+} M_2, N_1 \dot{+} N_2) \leq h(M_1, N_1) + h(M_2, N_2),$$

for every $M_1, M_2, N_1, N_2 \in \mathcal{P}_f(X)$. □

In what follows, we consider:

$\mathcal{C}_\delta = \{A \subset T; \exists (A_n)_n \subset \mathcal{C} \text{ so that } A_n \searrow A\}$ and $\mathcal{C}_\sigma = \{A \subset T; \exists (A_n)_n \subset \mathcal{C} \text{ so that } A_n \nearrow A\}$.

We easily observe that $\mathcal{C} \subset \mathcal{C}_\delta, \mathcal{C} \subset \mathcal{C}_\sigma, (\mathcal{C}_\delta)_\delta = \mathcal{C}_\delta$ and $(\mathcal{C}_\sigma)_\sigma = \mathcal{C}_\sigma$. By $\sigma(\mathcal{C})$ we denote the σ -ring generated by \mathcal{C} .

Obviously, $\mathcal{C}_\delta \subset \sigma(\mathcal{C})$ and $\mathcal{C}_\sigma \subset \sigma(\mathcal{C})$.

As we shall see in the following, regularity does not necessarily mean approximation by compact/open sets.

Theorem 3.10. *Suppose $\mu : \sigma(\mathcal{C}) \rightarrow \mathcal{P}_f(X)$ is decreasing convergent and has PGP and (E') . Then μ is $R'_{\mathcal{C}_\delta, \mathcal{C}_\sigma}$ -regular.*

Proof. Since μ has PGP, then, also, for every $\varepsilon > 0$, there is $\tilde{\delta}(\varepsilon) > 0$ so that for every $A, B, C \in \mathcal{C}$, with $|\mu(A)| < \tilde{\delta}$, $|\mu(B)| < \tilde{\delta}$ and $|\mu(C)| < \tilde{\delta}$, we have $|\mu(A \cup B \cup C)| < \varepsilon$.

Let $\mathcal{M} = \{A \in \sigma(\mathcal{C}); A \text{ is } R'_{\mathcal{C}_\delta, \mathcal{C}_\sigma}\text{-regular}\}$. Obviously, $\mathcal{C} \subset \mathcal{M}$.

We prove that \mathcal{M} is a σ -ring, and this will imply that $\mathcal{M} = \sigma(\mathcal{C})$, so, μ is $R'_{\mathcal{C}_\delta, \mathcal{C}_\sigma}$ -regular.

For this, let be $A_1, A_2 \in \mathcal{M}$. There are two sequences $(M_n^1), (M_n^2) \subset \mathcal{C}_\delta$ and two sequences $(N_n^1), (N_n^2) \subset \mathcal{C}_\sigma$ so that for every $n \in \mathbb{N}$, $M_n^1 \subset A_1 \subset N_n^1, M_n^2 \subset A_2 \subset N_n^2$, $\lim_{n \rightarrow \infty} |\mu(N_n^1 \setminus M_n^1)| = 0$ and $\lim_{n \rightarrow \infty} |\mu(N_n^2 \setminus M_n^2)| = 0$. Since μ has PGP, by Remark 2.8 iii), it is asymptotic null-additive, so $\lim_{n \rightarrow \infty} |\mu((N_n^1 \setminus M_n^1) \cup \mu(N_n^2 \setminus M_n^2))| = 0$.

Because for every $n \in \mathbb{N}$, $N_n^1 \setminus M_n^2 \in \mathcal{C}_\sigma$, $M_n^1 \setminus N_n^2 \in \mathcal{C}_\delta$, $M_n^1 \setminus N_n^2 \subset A_1 \setminus A_2 \subset N_n^1 \setminus M_n^2$ and

$$|\mu((N_n^1 \setminus M_n^2) \setminus (M_n^1 \setminus N_n^2))| \leq |\mu((N_n^1 \setminus M_n^1) \cup \mu(N_n^2 \setminus M_n^2))|,$$

then $A_1 \setminus A_2 \in \mathcal{M}$.

Let be $(A_n)_n \subset \mathcal{M}, A_n \nearrow A$. We prove that $A \in \mathcal{M}$.

Because μ is decreasing convergent, by Remark 2.5 iv) a), it is order continuous, so, we have $\lim_{n \rightarrow \infty} |\mu(A \setminus A_n)| = 0$. Consequently, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $|\mu(A \setminus A_{n_0})| < \tilde{\delta}$.

Since $A_{n_0} \in \mathcal{M}$, then A_{n_0} is $R'_{\mathcal{C}_\delta, \mathcal{C}_\sigma}$ -regular, so, by Theorem 3.3 ii), it is also $R'_{\mathcal{C}_\delta}$ -regular, which implies that there is $(M_{n_0, m})_m \subset \mathcal{C}_\delta$, so that for every $m \in \mathbb{N}$, $M_{n_0, m} \subset A_{n_0}$ and $\lim_{m \rightarrow \infty} |\mu(A_{n_0} \setminus M_{n_0, m})| = 0$. Consequently, there is $m_0(\varepsilon) \in \mathbb{N}$ such that $|\mu(A_{n_0} \setminus M_{n_0, m_0})| < \tilde{\delta}$.

Analogously, according to Theorem 3.3 ii), because $R'_{\mathcal{C}_\delta, \mathcal{C}_\sigma}$ -regularity implies $R'_{\mathcal{C}_\sigma}$ -regularity, then, for every $n \in \mathbb{N}$, with $n \geq n_0(\varepsilon)$, for A_n there

is a decreasing sequence $(N_{n,m})_m \subset \mathcal{C}_\sigma$ so that for every $m \in \mathbb{N}$, $A_n \subset N_{n,m}$ and $\lim_{m \rightarrow \infty} |\mu(N_{n,m} \setminus A_n)| = 0$.

We observe that for every $n \in \mathbb{N}$, with $n \geq n_0(\varepsilon)$, $N_{n,m} \setminus A_n \xrightarrow{m \rightarrow \infty} \bigcap_{m=1}^{\infty} (N_{n,m} \setminus A_n)$ and, since $\lim_{m \rightarrow \infty} |\mu(N_{n,m} \setminus A_n)| = 0$, by the decreasing convergence property of μ , we get that for every $n \in \mathbb{N}$, with $n \geq n_0(\varepsilon)$, we have $\mu(\bigcap_{m=1}^{\infty} (N_{n,m} \setminus A_n)) = \{0\}$.

Therefore, since μ has (E'), then for $\tilde{\delta}$, there is an increasing sequence of naturals $(m_i)_{i \in \mathbb{N}}$ so that $|\mu(\bigcup_{i=1}^{\infty} (N_{n_0+i, m_i} \setminus A_{n_0+i}))| < \tilde{\delta}$.

Let us consider $N = \bigcup_{i=1}^{\infty} N_{n_0+i, m_i}$. Because $(\mathcal{C}_\sigma)_\sigma = \mathcal{C}_\sigma$, then $N \in \mathcal{C}_\sigma$. Also,

$$A = A_{n_0} \cup \left(\bigcup_{i=1}^{\infty} A_{n_0+i} \right) \subset N_{n_0+1, m_1} \cup \left(\bigcup_{i=1}^{\infty} N_{n_0+i, m_i} \right) = \bigcup_{i=1}^{\infty} N_{n_0+i, m_i} = N.$$

Because

$$\begin{aligned} N \setminus A &= \left(\bigcup_{i=1}^{\infty} N_{n_0+i, m_i} \right) \setminus \left(\bigcup_{i=1}^{\infty} A_i \right) \subset \left(\bigcup_{i=1}^{\infty} N_{n_0+i, m_i} \right) \setminus \left(\bigcup_{i=1}^{\infty} A_{n_0+i} \right) \\ &\subset \bigcup_{i=1}^{\infty} (N_{n_0+i, m_i} \setminus A_{n_0+i}), \end{aligned}$$

we get that $|\mu(N \setminus A)| < \tilde{\delta}$.

Since $N \setminus M_{n_0, m_0} = (N \setminus A) \cup (A \setminus A_{n_0}) \cup (A_{n_0} \setminus M_{n_0, m_0})$, we finally have $|\mu(N \setminus M_{n_0, m_0})| < \varepsilon$, where $M_{n_0, m_0} \in \mathcal{C}_\delta$ and $N \in \mathcal{C}_\sigma$, with $M_{n_0, m_0} \subset A \subset N$. This says $A \in \mathcal{M}$. \square

By Theorem 3.10, Remark 2.8 ii) and Corollary 2.14, we easily get:

Corollary 3.11. *If $\mu : \sigma(\mathcal{C}) \rightarrow \mathcal{P}_f(X)$ is order continuous and auto-continuous from above, then μ is $R'_{\mathcal{C}_\delta, \mathcal{C}_\sigma}$ -regular.*

Example 3.12. i) If \mathcal{C} is a δ -ring and if $\mu : \sigma(\mathcal{C}) \rightarrow \mathcal{P}_f(X)$ is order continuous, then μ is $R'_\mathcal{C}$ -regular.

Indeed, if μ is order continuous and $A \in \sigma(\mathcal{C})$ is arbitrarily, there is an increasing sequence of sets $(A_n)_n \subset \mathcal{C}$ so that $A_n \nearrow A$. Because μ is order continuous, for every $\varepsilon > 0$, there is $A_{n_0} \in \mathcal{C}$ so that $|\mu(A \setminus A_{n_0})| < \varepsilon$, i.e., μ is $R'_\mathcal{C}$ -regular.

ii) If \mathcal{C} is a δ -ring, \mathcal{N} is an arbitrary nonvoid family of subsets of T , $(A_n)_n \subset \mathcal{C} \cap \mathcal{N}$, $A_n \searrow A$ and $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ is decreasing convergent, then A is $R'_\mathcal{N}$ -regular.

iii) If \mathcal{C} is a σ -ring, \mathcal{M} is an arbitrary nonvoid family of subsets of T , $(A_n)_n \subset \mathcal{C} \cap \mathcal{M}$, $A_n \nearrow A$ and $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ is increasing convergent, then A is $R'_{\mathcal{M}}$ -regular.

Same argues as before for the proofs.

4. Regularity properties in concrete situations

In this section, for the case when:

T is a Hausdorff space, \mathcal{C} is an algebra, for instance, the Borel σ -algebra $\tilde{\mathcal{B}}$ generated by the open sets of T , $\mathcal{M} = \mathcal{F}$, the family of closed subsets of T or $\mathcal{M} = \mathcal{K}$, the family of compact subsets of T and $\mathcal{N} = \mathcal{D}$, the family of open subsets of T or, if T is, particularly, a locally compact Hausdorff space, \mathcal{C} is a ring, for instance, \mathcal{B}_0 (respectively, \mathcal{B}'_0) or \mathcal{B} (respectively, \mathcal{B}') (but not only), $\mathcal{M} = \mathcal{K}$ and $\mathcal{N} = \mathcal{D}$, we shall present important consequences of the results established in Section 3, pointing out various relationships existing among concrete regularities.

As we shall see, our results generalize other previous results obtained by GAVRILUȚ [4-9], KAWABE [16], WU and WU [29], LI and YASUDA [18], NARUKAWA [20] and NARUKAWA ET. AL [21].

By Corollary 3.7, we have:

Corollary 4.1. *If T is a Hausdorff space and $\mu : \tilde{\mathcal{B}} \rightarrow \mathcal{P}_f(X)$ is asymptotic null-additive, then μ is $R'_{\mathcal{F}}$ -regular $\Leftrightarrow \mu$ is $R'_{\mathcal{D}}$ -regular $\Leftrightarrow \mu$ is $R'_{\mathcal{F},\mathcal{D}}$ -regular.*

Moreover, if T is compact, then a) μ is $R'_{\mathcal{F}}$ -regular \Leftrightarrow b) μ is $R'_{\mathcal{K}}$ -regular \Leftrightarrow c) μ is $R'_{\mathcal{D}}$ -regular \Leftrightarrow d) μ is $R'_{\mathcal{F},\mathcal{D}}$ -regular \Leftrightarrow e) μ is $R'_{\mathcal{K},\mathcal{D}}$ -regular.

Corollary 4.2. *Suppose T is a compact space and $\mu : \mathcal{C} \rightarrow \mathcal{P}_f(X)$ is asymptotic null-additive.*

- i) *If \mathcal{C} is \mathcal{B} or \mathcal{B}' , then μ is $R'_{\mathcal{K}}$ -regular if and only if it is $R'_{\mathcal{D}}$ -regular if and only if it is $R'_{\mathcal{K},\mathcal{D}}$ -regular.*
- ii) *If, moreover, T is metrisable and \mathcal{C} is \mathcal{B}_0 or \mathcal{B}'_0 , then μ is $R'_{\mathcal{K}}$ -regular if and only if it is $R'_{\mathcal{D}}$ -regular if and only if it is $R'_{\mathcal{K},\mathcal{D}}$ -regular.*

Also, by Theorem 3.8, we get:

Corollary 4.3. *Suppose T is a Hausdorff space and $\mu_1, \mu_2 : \tilde{\mathcal{B}} \rightarrow \mathcal{P}_{bf}(X)$ are fuzzy.*

- a) If μ_1, μ_2 are $R_{\mathcal{F}}$ -regular, then $\mu_1 = \mu_2$ on $\mathcal{F} \cap \mathcal{C}$ if and only if $\mu_1 = \mu_2$ on \mathcal{C} ;
- b) If μ_1, μ_2 are $R_{\mathcal{D}}$ -regular, then $\mu_1 = \mu_2$ on $\mathcal{D} \cap \mathcal{C}$ if and only if $\mu_1 = \mu_2$ on \mathcal{C} ;
- c) If μ_1, μ_2 are $R_{\mathcal{K}}$ -regular, then $\mu_1 = \mu_2$ on $\mathcal{K} \cap \mathcal{C}$ if and only if $\mu_1 = \mu_2$ on \mathcal{C} .

By Corollary 4.3 and Theorem 3.3 i), we get:

Corollary 4.4. *Suppose T is a Hausdorff space and $\mu_1, \mu_2 : \tilde{\mathcal{B}} \rightarrow \mathcal{P}_{bf}(X)$ are fuzzy and $R_{\mathcal{K}, \mathcal{D}}$ -regular. Then i) $\mu_1 = \mu_2$ on $\mathcal{C} \Leftrightarrow$ ii) $\mu_1 = \mu_2$ on $\mathcal{F} \cap \mathcal{C} \Leftrightarrow$ iii) $\mu_1 = \mu_2$ on $\mathcal{K} \cap \mathcal{C} \Leftrightarrow$ iv) $\mu_1 = \mu_2$ on $\mathcal{D} \cap \mathcal{C}$.*

By Theorem 3.8, we also get:

Corollary 4.5. *Suppose T is a locally compact Hausdorff space and \mathcal{C} is, for instance, $\mathcal{B}, \mathcal{B}_0, \mathcal{B}'$ or \mathcal{B}'_0 .*

- a) If $\mu_1, \mu_2 : \mathcal{C} \rightarrow \mathcal{P}_{bf}(X)$ are fuzzy and $R_{\mathcal{K}}$ -regular, then $\mu_1 = \mu_2$ on $\mathcal{K} \cap \mathcal{C}$ if and only if $\mu_1 = \mu_2$ on \mathcal{C} ;
- b) If $\mu_1, \mu_2 : \mathcal{C} \rightarrow \mathcal{P}_{bf}(X)$ are fuzzy and $R_{\mathcal{D}}$ -regular, then $\mu_1 = \mu_2$ on $\mathcal{D} \cap \mathcal{C}$ if and only if $\mu_1 = \mu_2$ on \mathcal{C} .

By Corollary 4.5 and Theorem 3.3 i), we have:

Corollary 4.6. *Suppose T is a locally compact Hausdorff space, \mathcal{C} is, for instance, $\mathcal{B}, \mathcal{B}_0, \mathcal{B}'$ or \mathcal{B}'_0 and $\mu_1, \mu_2 : \mathcal{C} \rightarrow \mathcal{P}_{bf}(X)$ are fuzzy and $R_{\mathcal{K}, \mathcal{D}}$ -regular. Then i) $\mu_1 = \mu_2$ on $\mathcal{C} \Leftrightarrow$ ii) $\mu_1 = \mu_2$ on $\mathcal{K} \cap \mathcal{C} \Leftrightarrow$ iii) $\mu_1 = \mu_2$ on $\mathcal{D} \cap \mathcal{C}$.*

In the following, we obtain a result which is stronger than Corollary 4.1:

Proposition 4.7. *Suppose T is a Hausdorff space.*

If $\mu : \tilde{\mathcal{B}} \rightarrow \mathcal{P}_f(X)$ is asymptotic null-additive, then μ is $R'_{\mathcal{K}, \mathcal{D}}$ -regular if and only if it is $R'_{\mathcal{F}, \mathcal{D}}$ -regular and T is $R'_{\mathcal{K}}$ -regular.

Proof. The *Only if part* is straightforward.

The *If part*: Let $A \in \tilde{\mathcal{B}}$ be arbitrarily. Because μ is $R'_{\mathcal{F}, \mathcal{D}}$ -regular, there is an increasing sequence $(F_n) \subset \mathcal{F} \cap \mathcal{C}$ and a decreasing sequence $(D_n) \subset \mathcal{D} \cap \mathcal{C}$ so that $F_n \subset A \subset D_n$ and $\lim_{n \rightarrow \infty} |\mu(D_n \setminus F_n)| = 0$.

Since T is $R'_{\mathcal{K}}$ -regular, there is an increasing sequence $(K_n) \subset \mathcal{K} \cap \mathcal{C}$ so that $\lim_{n \rightarrow \infty} |\mu(T \setminus K_n)| = 0$.

If for any $n \in \mathbb{N}$ we denote $L_n = F_n \cap K_n$, then (L_n) is an increasing sequence of compact sets and $\{0\} \subset \mu(D_n \setminus L_n) \subset \mu((T \setminus K_n) \cup (D_n \setminus F_n))$.

By the asymptotic null-additivity, since $\lim_{n \rightarrow \infty} |\mu((T \setminus K_n) \cup (D_n \setminus F_n))| = 0$, then $\lim_{n \rightarrow \infty} |\mu(D_n \setminus L_n)| = 0$, which says that μ is $R'_{\mathcal{K}, \mathcal{D}}$ -regular. \square

Theorem 4.8 ([8]). *Suppose T is a locally compact Hausdorff space. If \mathcal{C} is the ring (or the δ -ring) generated by the compact sets or by the compact, G_δ sets, then μ is $R'_{\mathcal{K}}$ -regular if and only if μ is $R'_{\mathcal{D}}$ -regular.*

By Theorem 4.8 and Theorem 3.3 iii) we have:

Corollary 4.9. *Suppose T is a locally compact Hausdorff space. If \mathcal{C} is the ring (or the δ -ring) generated by the compact sets or by the compact, G_δ sets and if μ is asymptotic null-additive, then i) μ is $R'_{\mathcal{K}}$ -regular \Leftrightarrow ii) μ is $R'_{\mathcal{D}}$ -regular \Leftrightarrow iii) μ is $R'_{\mathcal{K}, \mathcal{D}}$ -regular.*

5. Concluding remarks

In this paper, abstract regularity is studied in the fuzzy set-valued case, with direct applications in some concrete situations. We shall apply these results in future research, in order to obtain set-valued Alexandroff and Lusin type theorems.

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