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ON LATTICE-THEORETICAL CONSTRUCTION OF MATROIDS

BY

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Abstract. This paper builds up a lattice-theoretical construction for all of matroids under isomorphism with the help of Galois isomorphisms among contexts. It discusses the lattice operations defined on this lattice construction with the direct sum between matroids.

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1. Introduction and preliminaries

Matroid theory was introduced in 1935 by Whitney [1]. From then on, it has been reorganized to naturally use in many fields ([2]). The notion of a Galois connection is derived from a context and has its roots in Galois theory ([3], [4]). In Formal Concept Analysis, this notion is much more to be used ([5], [6]). In data mining, the notion is also discussed ([7]). Concept lattice theory was born in 1982 and is the core of FCA ([5]). It has a large amount algorithms to build up concept lattices ([5], [6], [8-10]).

With the help of Galois isomorphisms among contexts, this article draws up thoroughly the lattice construction of all the matroids under isomorphism. Further, between matroids, it compares the lattice operations provided here with the direct sum operator. We know that the former results

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only discuss some special classes of matroids to draw up their lattice structures. So it is difficult to compare the method here with others, which as far as we know always handle special problems. We regard the method here as valuable in its universality. The application of the lattice construction provided here will be carried out in future work.

In order to highlight our main conclusions, this article omits to write out the reviewing basic contents. The knowledge about Galois connections are referred to [3]; contexts (or say, concept lattices) are ([5], [6]); matroid theory is seen [11]; lattice theory is cf. [6], [11], [12].

We make a standing assumption that all the discussions under consideration are finite.

Let X be a set. $\mathcal{P}(X)$ is the collection of all subsets of X.

Two matroids M_1 and M_2 are isomorphic, in notation, $M_1 \cong M_2$.

Let (O, P, R) be a context with (K, L) as its corresponding Galois connection. $Gal(O, P, R)$ is all the concepts of (O, P, R); the concept lattice of (O, P, R) is still denoted by $Gal(O, P, R)$; the lattice $(\mathcal{B}_O = \{X \subseteq O | LK(X) = X\}, \subseteq)$ is still denoted by $\mathcal{B}_O$ if no confusion.

For dealing with our aim, we firstly present definitions.

Definition 1. (1) If a Galois connection (K, L) could generate a matroid (O, LK), we call (O, P, R) a *Galois context*, where (O, P, R) is the context corresponding to (K, L).

(2) Let (O_j, P_j, R_j) be a context and the corresponding Galois connection be $(K_j, L_j), (j = 1, 2)$. If there is a bijection $\pi : O_1 \to O_2$ satisfying the following condition:

$$\forall X \subseteq O_1, \ \ X = L_1K_1(X) \Leftrightarrow \pi X = L_2K_2(\pi X),$$

then:

(I) (K_1, L_1) is *Galois isomorphic* to (K_2, L_2), in notation, $(K_1, L_1) \cong_G (K_2, L_2)$.

(II) π is called a *Galois isomorphic map* between (K_1, L_1) and (K_2, L_2), and also say that π is a *Galois isomorphic map* between (O_1, P_1, R_1) and (O_2, P_2, R_2).

(III) (O_1, P_1, R_1) is *isomorphic* to (O_2, P_2, R_2), and in notation $(O_1, P_1, R_1) \cong_G (O_2, P_2, R_2)$.

Let (O_j, P_j, R_j) be a context with (K_j, L_j) as its generated Galois connection $(j = 1, 2)$. We could prove the following $(\alpha 1)$ and $(\alpha 2)$:

$(\alpha 1)$ $(O_1, P_1, R_1) \cong (O_2, P_2, R_2) \Rightarrow (O_1, P_1, R_1) \cong_G (O_2, P_2, R_2)$, but not vice versa.

$(\alpha 2)$ If (O_j, P_j, R_j) is a Galois context and M_j is the corresponding matroid $(j = 1, 2)$, then

$$M_1 \cong M_2 \Leftrightarrow (O_1, P_1, R_1) \cong_G (O_2, P_2, R_2) \Leftrightarrow (K_1, L_1) \cong_G (K_2, L_2).$$

Because $(\alpha 1)$ and $(\alpha 2)$ are not very difficult to be proved, we omit the proof here. Actually, we in [13] could find the proofs for $(\alpha 1)$ and $(\alpha 2)$ and their some applications. Certainly, this paper will not be affected by [13]. It is self-contained.

We simply analyze the significance of Definition 1.

(1) Combining [3, Proposition 3] with [11, p.8, Theorem 4], it is not difficult to know that a Galois connection (K, L) will not necessarily generate a matroid, and besides, for a matroid M on O with σ as its closure operator, there is at least a Galois connection (K, L) satisfying $\sigma = LK$. Namely, Definition 1 (1) is reasonable.

(2) For a context (O, P, R), the most valuable and substantive researched part is the construction of its concept lattice $Gal(O, P, R)$, or equivalently its $\mathcal{B}_O$. We care about its content not its mark expression. Hence, Definition 1 (2) has the significance.

(3) [11, p.52, Example] implies that for two matroids $M_1 = (O_1, \sigma_1)$ and $M_2 = (O_2, \sigma_2)$ with $\mathcal{F}_1$ and $\mathcal{F}_2$ as its family of closed sets respectively, even if two lattices $(\mathcal{F}_1, \subseteq)$ and $(\mathcal{F}_2, \subseteq)$ are isomorphic, but there is no bijective map $\pi : O_1 \to O_2$ satisfying $X \in \mathcal{F}_1 \Leftrightarrow \pi X \in \mathcal{F}_2$, then it could not pledge to $M_1 \cong M_2$. Namely it is a necessary supposition in Definition 1 (2) that π is a bijection between O_1 and O_2.

2. Lattice-theoretical structures

Lattice theory has a longer history than that of matroid theory, and has applied into the research on matroids and produced many results ([11, Ch.3 and Ch.17]). Recently, some researchers put their attentions to study on the lattice constructions for some classes of matroids such as all the matroids defined on the same set ([14]) or the class of the matroids owned the same special properties ([15-18]). As the recent research tendency, for the family of all the matroids, in this section, with the assistance of Galois isomorphisms between contexts, we search out its lattice-theoretical construction under isomorphism.

We state the preparative lemmas for the lattice construction of the set of matroids.

Lemma 1. (1) *Let* (O, P, R) *be a context. If* $O \cap P \neq \emptyset$*, then there is a new context* (O, P_N, R_N) *satisfying* $(O, P, R) \cong_G (O, P_N, R_N)$ *and* $O \cap P_N = \emptyset$*.*

(2) *Let* (O_1, P_1, R_1) *and* (O_2, P_2, R_2) *be two contexts. If* $O_1 \cap O_2 \neq \emptyset$*, then there is a new context* (O_{2N}, P_{2N}, R_{2N}) *satisfying* $(O_2, P_2, R_2) \cong_G (O_{2N}, P_{2N}, R_{2N})$ *and* $O_1 \cap O_{2N} = \emptyset$*.*

Proof. (1) It is no harm to suppose $O = \{a_1, a_2, \ldots, a_n, o_1, \ldots, o_t\}$ and $P = \{a_1, a_2, \ldots, a_n, p_1, \ldots, p_s\}$, where $\{o_1, \ldots, o_t\} = O \setminus P$ and $\{p_1, \ldots, p_s\} = P \setminus O$.

Setting $P_N = \{b_1, b_2, \ldots, b_n, p_1, \ldots, p_s\}, b_j \notin O \cup P, (j = 1, 2, \ldots, n)$. $R_N \subseteq O \times P_N$ is defined as follows:

- if $y \in \{p_1, \ldots, p_s\}$, then $(x, y) \in R_N \Leftrightarrow (x, y) \in R$;
- if $y \in \{b_1, b_2, \ldots, b_n\}$, i.e. $y = b_j$, then $(x, y) = (x, b_j) \in R_N \Leftrightarrow (x, a_j) \in R, (j = 1, 2, \ldots, n)$.

It is obviously, $\pi : a_j \mapsto b_j, (j = 1, 2, \ldots, n); p_i \mapsto p_i, (i = 1, 2, \ldots, s)$ is a Galois isomorphic map between (O, P, R) and (O, P_N, R_N) and $O \cap P_N = \emptyset$.

(2) Putting $O_1 = \{e_1, e_2, \ldots, e_m, x_1, \ldots, x_p\}$ and $O_2 = \{e_1, e_2, \ldots, e_m, y_1, \ldots, y_q\}$, where $\{x_1, \ldots, x_p\} = O_1 \setminus O_2$ and $\{y_1, \ldots, y_q\} = O_2 \setminus O_1$.

Posit $O_{2N} = \{d_1, d_2, \ldots, d_m, y_1, \ldots, y_q\}, (d_j \notin O_1 \cup O_2; j = 1, 2, \ldots, m)$. $R_{2N} \subseteq O_{2N} \times P_2$ is defined as follows:

- if $x \in O_2 \setminus O_1$, then $(x, y) \in R_{2N} \Leftrightarrow (x, y) \in R$;
- if $x \in \{d_1, d_2, \ldots, d_m\}$, then $(x, y) = (d_j, y) \in R_{2N} \Leftrightarrow (e_j, y) \in R$ for $j = 1, 2, \ldots, m$.

Evidently, $(O_{2N}, P_2, R_{2N}) \cong_G (O_2, P_2, R_2)$ and $O_{2N} \cap O_1 = \emptyset$.

Before proceeding, we provide some our views which are useful to the main aims.

(v1) Lemma 1 implies that for the contexts (O_1, P_1, R_1) and (O_2, P_2, R_2), we always could find out contexts (O_{1N}, P_{1N}, R_{1N}) and (O_{2N}, P_{2N}, R_{2N}) satisfying

(N1) $O_{1N} \cap O_{2N} = P_{1N} \cap P_{2N} = O_{1N} \cap P_{1N} = O_{2N} \cap P_{2N} = \emptyset$.

(N2) $(O_1, P_1, R_1) \cong_G (O_{1N}, P_{1N}, R_{1N})$ and $(O_2, P_2, R_2) \cong_G (O_{2N}, P_{2N}, R_{2N})$.

(v2) [5], [6] tell us that if the lattice construction of $Gal(O_j, P_j, R_j)$ is found out, then all the concepts $Gal(O_j, P_j, R_j)$ of (O_j, P_j, R_j) are to be born promptly, $(j = 1, 2)$.

(v3) (v1) and (v2) means that if we search $Gal(O_j, P_j, R_j)$ using the properties of matroid theory, we only need to pay attention to $\mathcal{DGM} = \{M_\alpha = (O_\alpha, L_\alpha K_\alpha) | \alpha \in \mathcal{A}\}$ which is the subset of all of matroids defined as:

- a matroid $M = (O, LK) \in \mathcal{DGM}$ if and only if for any $M_2 = (O_2, L_2K_2) \in \mathcal{DGM} \setminus M$, the condition (D1) and (D2) are satisfied, where

(D1): (N1) is satisfied.

(D2): $M \cong M_2$ does not hold.

(v4) Let $M_1 = (O_1, L_1K_1)$ be a matroid generated from the context $(O_1 = \{o_1, o_2, \ldots, o_n\}, P_1 = \{p_1, p_2, \ldots, p_p\}, R_1)$. Let $O_{1\diamond} = \{a_1, a_2, \ldots, a_n\}$, $P_{1\diamond} = \{b_1, b_2, \ldots, b_p\}$, and $R_{1\diamond} \subseteq O_{1\diamond} \times P_{1\diamond}$ be defined as to $(a_i, b_j) \in R_{1\diamond} \Leftrightarrow (o_i, p_j) \in R_1, (i = 1, \ldots, n; j = 1, \ldots, p)$, where

$$a_i \notin \Big(\bigcup_{\alpha \in \mathcal{A} \setminus 1\diamond} (O_\alpha \cup P_\alpha)\Big) \cup (O_1 \cup P_1), (i = 1, \ldots, n);$$

$$b_j \notin \Big(\bigcup_{\alpha \in \mathcal{A} \setminus 1\diamond} (O_\alpha \cup P_\alpha)\Big) \cup (O_1 \cup P_1), (j = 1, 2, \ldots, p) \text{ and } O_{1\diamond} \cap P_{1\diamond} = \emptyset.$$

It is easy to prove that the Galois connection $(K_{1\diamond}, L_{1\diamond})$ produces a matroid $M_{1\diamond} = (O_{1\diamond}, L_{1\diamond}K_{1\diamond})$ satisfying $M_{1\diamond} \cong M_1$ and $M_{1\diamond} \in \mathcal{DGM}$.

(v5) The above (v4) implies that for any matroid M, up to isomorphism, there is one and only one element M_D in $\mathcal{DGM}$ satisfying $M \cong M_D$.

Let $M_j = (O_j, L_jK_j)$ be a matroid $(j = 1, 2)$ and the corresponding contexts be (O_1, P_1, R_1) and (O_2, P_2, R_2) satisfying $(N1)$. Evidently, $(O_1 \cup O_2, P_1 \cup P_2, R_{12})$ is a context, where $R_{12} \subseteq (O_1 \cup O_2) \times (P_1 \cup P_2)$ is defined as: $(x, y) \in R_{12} \Leftrightarrow (x, y) \in R_1$ or $(x, y) \in R_2$.

For this new context, we will prove that it owns the following property.

Lemma 2. *Let* (K_{12}, L_{12}) *be the Galois connection generated from* $(O_1 \cup O_2, P_1 \cup P_2, R_{12})$*. Then* $(O_1 \cup O_2, L_{12}K_{12})$ *is a matroid.*

Proof. In virtue of [11, p.8-9 and 3, Proposition 3], we only need to check that $L_{12}K_{12}$ satisfies (s4): if $y \notin L_{12}K_{12}(Y)$ but $y \in L_{12}K_{12}(Y \cup z)$, then $z \in L_{12}K_{12}(Y \cup y)$.

Suppose $a, b \in O_1 \cup O_2, a \neq b, Y \subseteq O_1 \cup O_2$, and $a \in L_{12}K_{12}(Y \cup b) \setminus L_{12}K_{12}(Y)$. Because the property of (N1), it follows $O_1 \cap O_2 = \emptyset$. It is no harm to suppose $a \in O_1$. In light of definition of Galois conncetion in [3], it gets $L_{12}K_{12}(Y \cup b) = L_{12}(K_{12}(Y \cup b)) = \{x \in O_1 \cup O_2 | \forall y \in K_{12}(Y \cup b), ((x, y) \in R_{12})\}$ and $K_{12}(Y \cup b) = \{y \in P_1 \cup P_2 | \forall x \in Y \cup b, ((x, y) \in R_{12})\}$.

$a \in L_{12}K_{12}(Y \cup b) \setminus L_{12}K_{12}(Y)$ implies $(a, y) \in R_{12}$ for any $y \in K_{12}(Y \cup b)$. Considering $a \in O_1$ and the definition of R_{12}, it has $K_{12}(Y \cup b) \subseteq P_1$. $(b, y) \in R_{12}$ for any $y \in K_{12}(Y \cup b)$ gets $b \in O_1$ and $Y \cap O_2 = \emptyset$. So, $L_{12}K_{12}(Y) = L_1K_1(Y)$ and $L_{12}K_{12}(Y \cup b) = L_1K_1(Y \cup b)$.

By the satisfaction of (s4) for L_1K_1, it leads to $b \in L_1K_1(Y \cup a) = L_{12}K_{12}(Y \cup a)$.

Similarly to prove the case of $a \in O_2$. Hence, $b \in L_{12}K_{12}(Y \cup a)$ is true.

By [5, p.26], for the context $(\emptyset, \emptyset, \emptyset)$, its concept lattice has only one element. In addition, the Galois connection $(K_\emptyset, L_\emptyset)$ corresponding to $(\emptyset, \emptyset, \emptyset)$ produces a matroid $M_\emptyset = (\emptyset, L_\emptyset K_\emptyset)$ with $\{\emptyset\}$ as its only closed set.

Let M_j be a matroid and a corresponding context as (O_j, P_j, R_j), $(j = 1, 2, 3)$. Let (O_j, P_j, R_j) and (O_i, P_i, R_i) satisfy (N1) $(i \neq j;$ $i, j = 1, 2, 3)$ and $O_2 \neq \emptyset$. By Lemma 1, both $M_{12} = (O_1 \cup O_2, L_{12}K_{12})$ and $M_{23} = (O_2 \cup O_3, L_{23}K_{23})$ are matroids. If we continue to obtain a matroid for M_{12} and M_{23} by Lemma 2, it causes a non-meaningful result from $(O_1 \cup O_2) \cap (O_2 \cup O_3) = O_2 \neq \emptyset$. Let $M_1, M_2, M_3 \in \mathcal{DGM}$. It gets $M_{12\diamond} = (O_{12\diamond}, L_{12\diamond}K_{12\diamond})$ satisfying $M_{12\diamond} \in \mathcal{DGM}$ and $M_{12\diamond} \cong M_{12}$. We denote $M_{12\diamond}$ as $M_1 \vee M_2$. For the operation $\vee$, it gets the following result.

Theorem 1. $(\mathcal{DGM}, \vee)$ *is a semilattice with* M_0 *as the least element, where* $M_0 \in \mathcal{DGM}$ *is the matroid which is isomorphic to* $M_\emptyset$*.*

Proof. Routine to check that "$\vee$" satisfies the definition of semilattice ([12]).

Corollary 1. *Let* $M_1, M_2, \ldots, M_n \in \mathcal{DGM}$*. Then* $(\{M_1, M_2, \ldots, M_n\} \cup M_0, \vee)$ *is a complete, also a geometric and distributive lattice with length* n*.*

Proof. By Theorem 1 and [6, p.8, p.12, p.39-40], it is easy to prove that it is a lattice. The other needed results are routine from the definitions in [12, p.6, p.12, p.80] respectively.

[11, p.72-74] causes that for two matroids $M_j = (O_j, L_jK_j), j = (1, 2)$, $M_{12} = (O_1 \cup O_2, L_{12}K_{12})$ yielded out of Lemma 2 is exactly the direct sum $M_1 + M_2$ of M_1 and M_2. It is easy to see when $M_1 = (O_1, \sigma_1), M_2 = (O_2, \sigma_2), \ldots, M_n = (O_n, \sigma_n), M_0 = (O_0, \sigma_0) = M_\emptyset$ are matroids satisfying $O_i \cap O_j = \emptyset, (i \neq j; i, j = 1, 2, \ldots, n)$, $\sigma_0(\emptyset) = \emptyset$ and $n > 2$, it follows that $(\{M_1, M_2, \ldots, M_n\} \cup M_0, +)$ is a poset and $M_i + M_j + M_k$ does not exist if $j = i + 1$ and $k = j + 1$. In other words, $(\{M_1, M_2, \ldots, M_n\} \cup M_0, +)$ is not a lattice. However, Corollary 1 pledges when $M_j \in \mathcal{DGM}$, $(j = 1, \ldots, n), (\{M_1, M_2, \ldots, M_n\} \cup M_0, \vee)$ is a lattice. This analysis means that the operation "$\vee$" defined on $\mathcal{DGM}$ is not "+".

Let (O, P, R) be a context and (K, L) be the corresponding Galois connection. A matroid (P, σ_P) is called a *Galois dual matroid* if it is born from (K, L) satisfying $\sigma_P = KL$. Additionally, the set of closed sets of the matroid (O, LK) is $\mathcal{B}_O$ and the set of closed sets of (P, KL) is $\mathcal{B}_P$. In addition, two lattices $(\mathcal{B}_O, \subseteq)$ and $(\mathcal{B}_P, \subseteq_P)$ are isomorphic where $\subseteq_P$: $X \subseteq_P Y \Leftrightarrow Y \subseteq X$ for any $X, Y \in \mathcal{B}_P$. The approaches to the discussions for Galois dual matroids are the "dual" way comparing with the methods for matroids. About the connections between a matroid and its Galois dual matroid will be studied in future.

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