ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIX, 2013, f.1

ON LATTICE-ORDERED REES MATRIX Γ-SEMIGROUPS

BY

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Abstract. The purpose of this paper is to introduce and give some properties of l-Rees matrix Γ-semigroups. Generalizing the results given by GUOWEI and PING, concerning the congruences and lattice of congruences on regular Rees matrix Γ-semigroups, the structure theorem of l-congruences lattice of $l-\Gamma$-semigroup $M = \mu^o(G : I, \wedge, \Gamma_e)$ is given, from which it follows that this l-congruences lattice is distributive.

Mathematics Subject Classification 2010: 06F99, 06F05, 20M10, 20M20.
Key words: Γ-semigroup, Rees matrix Γ-semigroup, $l-\Gamma$-semigroup, l-Rees matrix Γ-semigroup, l-group, l-congruence, γ-idempotent.

1. Introduction and preliminaries

In [16], SETH generalized the notion of Rees matrix semigroup and Rees's theorem for Γ-semigroup. Some properties concerning Rees (cf. [2,3,11,12], [14,17]) matrix Γ-semigroups have been already considered in [7]. In [6], an explicit description of the congruences and lattice of congruences on regular Rees matrix Γ-semigroups is obtained. The concept of the $l-\Gamma$-semigroup is introduced in [8]. The purpose of this paper is to introduce and to reveal some properties of l-Rees matrix Γ-semigroups. Generalizing the results given in [6], the structure theorem of l-congruences lattice of $l-\Gamma$-semigroup $M = \mu^o(G : I, \wedge, \Gamma_e)$ is given, from which it follows that this l-congruences lattice is distributive.

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

In 1986, SEN and SAHA [15] defined Γ-semigroup as a generalization of semigroup and ternary semigroup as follows:

Definition 1.1. Let M and Γ be two nonempty sets. Denote by the letters of the English alphabet the elements of M and with the letters of the Greek alphabet the elements of Γ. Then M is called a Γ-semigroup if there exists a mapping $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \longmapsto a\gamma b$ satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$.

Example 1.2. Let M be a semigroup and Γ be any nonempty set. Define a mapping $M \times M \times M \to M$ by $a\gamma b = ab$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then M is a Γ-semigroup.

Example 1.3. Let M be a set of all negative rational numbers. Obviously M is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p \text{ is prime}\}$. Let $a, b, c \in M$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence M is a Γ-semigroup.

Example 1.4. Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then M is a Γ-semigroup under the multiplication over complex numbers while M is not a semigroup under complex number multiplication.

These examples shows that every semigroup is a Γ-semigroup and Γ-semigroups are a generalization of semigroups.

A nonemtpy subset K of a Γ-semigroup M is called a Γ-*subsemigroup* of M if, for all $a, b \in K$ and $\gamma \in \Gamma$, $a\gamma b \in K$. An element a of a Γ-semigroup M is called an γ-*idempotent* if exists $\gamma \in \Gamma$, $a\gamma a = a$ ([15]).

Example 1.5. Let $M = [0, 1]$ and $\Gamma = \{\frac{1}{n} | n \text{ is a positive integer}\}$. Then M is a Γ-semigroup under usual multiplication. Let $K = [0, 1/2]$. We have that K is a nonemtpy subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a sub-Γ-semigroup of M.

Let M be a Γ-semigroup and x be a fixed element of Γ. We define $a \circ b$ in M by $a \circ b = axb, \forall a, b \in M$. The authors [15] have shown that M is a semigroup and denoted this semigroup by M_x. They have shown that if M_x is a group for some $x \in \Gamma$, then M_x is a group for all $x \in \Gamma$. A Γ-semigroup M is called a Γ-*group* if M_x is a group for some (hence for all) $x \in \Gamma$.

An element a of an Γ-semigroup M is called *left zero element* (resp. *right zero element*) of M if $a\gamma b = a$ (resp. $b\gamma a = a$), for all $b \in M$ and $\gamma \in \Gamma$. It will be called *a zero element* if it is a left and right zero element and it is denoted by 0.

An equivalence relation ρ on a Γ-semigroup M is called a *congruence* if for all $c \in M$ and for all $\gamma \in \Gamma$, $(a, b) \in \rho$ implies $(a\gamma c, b\gamma c) \in \rho$ and $(c\gamma a, c\gamma b) \in \rho$.

Examples of Γ-semigroups can be seen in [9, 10, 15, 16].

Let G be a group, $I, \wedge$ be two index sets and Γ be the collection of some $\wedge \times I$ matrices over $G^o = G \cup \{0\}$, the group with zero. Let μ^o be the set of all elements $(a)_{i\lambda}$ where $i \in I, \lambda \in \wedge$ and $(a)_{i\lambda}$ is the $I \times \wedge$ matrix over G^o having a in the i-th row and λ-th column, its remaining entries being zero. The expression $(0)_{i\lambda}$ will be used to denote $I \times \wedge$ zero matrix. For any $(a)_{i\lambda}, (b)_{j\mu}, (c)_{k\nu} \in \mu^o$ and $\alpha = (p_{\lambda i}), \beta = (q_{\lambda i}) \in \Gamma$ we define $(a)_{i\lambda}\alpha(b)_{j\mu} = (ap_{\lambda j}b)_{i\mu}$. Then it is easy verified that $[(a)_{i\lambda}\alpha(b)_{j\mu}]\beta(c)_{k\nu} = (a)_{i\lambda}\alpha[(b)_{j\mu}\beta(c)_{k\nu}]$. Thus μ^o is a Γ-semigroup.

We shall call Γ the sandwich matrix set and μ^o Rees $I \times \wedge$ matrix Γ-semigroup over G^o with sandwich matrix set Γ and denote it by $\mu^o(G : I, \wedge, \Gamma)$. Sandwich matrix set Γ is called *regular* ([16]), if for each row $i \in I$ there exists a matrix $\alpha = (p_{\mu i}) \in \Gamma$ and for each column $\lambda \in \wedge$ there exists a matrix $\beta = (q_{\lambda i}) \in \Gamma$ such that $\alpha = (p_{\mu i})$ has at least one non-zero entry in i-th row, $\beta = (q_{\mu j})$ has at least one non-zero entry in λ-th column.

A *po-Γ-semigroup* is an ordered set M at the same time Γ-semigroup such that for all $c \in M$ and for all $\gamma \in \Gamma$, $a \leq b \Rightarrow a\gamma c \leq b\gamma c, c\gamma a \leq c\gamma b$.

Let M be a lattice under $\vee$ and $\wedge$ and at the same time a $po - \Gamma$-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$, $a\gamma(b \vee c) = a\gamma b \vee a\gamma c, (a \vee b)\gamma c = a\gamma c \vee b\gamma c$ and $a\gamma(b \wedge c) = a\gamma b \wedge a\gamma c, (a \wedge b)\gamma c = a\gamma c \wedge b\gamma c$. Then M is called a lattice-ordered Γ-semigroup (: $l - \Gamma - semigroup$) ([8]).

The usual order relation $\leq$ on M is defined in the following way $a \leq b \Leftrightarrow a \vee b = b$. Then we can show that for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Examples of lattice ordered Γ-semigroups can be found in [8].

Example 1.6 ([8])**.** Let $(X, \leq)$ and $(Y, \leq)$ be two finite chains. Let M be the set of all isotone mappings from X into Y and Γ be the set of all isotone mappings from Y into X. Let $f, g \in M$ and $\alpha \in \Gamma$. Then M is a Γ-semigroup with respect to the usual mapping composition $f\alpha g$ of f, α and g. For $f, g \in M$, the mappings $f \vee g$ and $f \wedge g$ are defined by letting, for each $a \in X$, $(f \vee g)(a) = \max\{f(a), g(a)\}, (f \wedge g)(a) = \min\{f(a), g(a)\}$ (the maximum and minimum are considered with respect to the order $\leq$ in X and Y). The greatest element e is the mapping that sends every $a \in X$ to the greatest element of finite chains $(Y, \leq)$. Then M is an $le-\Gamma$-semigroup.

Example 1.7 ([8])**.** Let M be a $po-\Gamma$-semigroup. Let M_1 be the set of all ideals of M. Then $(M_1, \subseteq, \cap, \cup)$ is an $le-\Gamma$-semigroup.

Example 1.8 ([8])**.** Let M be a $po-\Gamma$-semigroup. Let M_1 and Γ_1 be the sets of all subsets of M and Γ, respectively. Then M_1 is a $po-\Gamma_1$-semigroup with respect the mapping

$$A\Lambda B = \begin{cases} (A](\Lambda](B] = (A\Lambda B], & \text{if } A, B \in M_1 \backslash \{\emptyset\}, \Lambda \in \Gamma_1 \backslash \{\emptyset\} \\ \emptyset, & \text{if } A = \emptyset \text{ or } B = \emptyset \end{cases}$$

and $(M_1, \subseteq, \cap, \cup)$ is an $le-\Gamma_1$-semigroup.

A subset T of a $po-\Gamma$-semigroup M is said to be *convex*, if for all $x, z \in T$ and $y \in M$, $x < y < z$ implies $y \in T$.

If a Rees matrix Γ-semigroup $M = \mu^o(G : I, \wedge, \Gamma)$ is an $l-\Gamma$-semigroup, it is called an l-Rees matrix Γ-semigroup.

An *l-congruence* ρ on an $l-\Gamma$-semigroup M is a congruence on the Γ-semigroup structure and the lattice structure.

2. Main results

Proposition 2.1. Let M be a Rees matrix Γ-semigroup.

1. If for any $a, b, c \in G$, $\alpha = (p_{\lambda i}) \in \Gamma$, $i, j \in I, \lambda, \mu \in \wedge$, $(a)_{i\lambda}\alpha(b)_{j\mu} = (c)_{i\lambda}\alpha(b)_{j\mu}$, then $a = c$, that is $(a)_{i\lambda} = (c)_{i\lambda}$.
2. If the sandwitch matrix Γ satisfies that its entries are all equal to e, the identity element of G, then M has cancellation law.

Proof. (1) We have that $(a)_{i\lambda}\alpha(b)_{j\mu} = (c)_{i\lambda}\alpha(b)_{j\mu}$ implies $(ap_{\lambda j}b)_{i\mu} = (cp_{\lambda j}b)_{i\mu}$ and so $ap_{\lambda j}b = cp_{\lambda j}b$. Therefore $a = c$.

(2) If for any $a, b, c \in G$, $\alpha = (p_{\lambda i}) \in \Gamma$, $i, j, k \in I, \lambda, \mu, \delta \in \wedge$, $(a)_{i\lambda}\alpha(b)_{j\mu} = (c)_{k\delta}\alpha(b)_{j\mu}$, then by hypothesis $(ab)_{i\mu} = (cb)_{k\mu}$, and so $i = k, c = a$, that is $(a)_{i\lambda} = (c)_{k\delta}$. □

In sequel, we denote by α_e the sandwich matrix $\alpha \in \Gamma$ whose entries are all equal to e, the identity element of G. We denote $\Gamma_e = \{\alpha_e\}$.

Proposition 2.2. Let $M = \mu^o(G : I, \wedge, \Gamma)$. Then any α-idempotent of M, where $\alpha = (p_{\lambda i}) \in \Gamma$, is $(p_{\lambda i}^{-1})_{i\lambda}$ $(\forall i \in I, \lambda \in \wedge)$.

Proof. Let $(a)_{i\lambda}$ ($\forall i \in I, \lambda \in \wedge$) be an α-idempotent of M, where $\alpha \in \Gamma$. Then we have $(a)_{i\lambda}\alpha(a)_{i\lambda} = (ap_{\lambda i}a)_{i\lambda} = (a)_{i\lambda}$, and so $ap_{\lambda i}a = a$. Thus $a = p_{\lambda i}^{-1}$. □

Remark 1. It is clear that there is one such α-idempotent for each pair i, λ where $i \in I, \lambda \in \wedge$ such that $p_{\lambda i} \neq 0$. Therefore, for all $\alpha \in \Gamma$, the number of all α-idempotents in $M = \mu^o(G : I, \wedge, \Gamma)$ is $|I| \times |\wedge|$.

Theorem 2.3. *Let $M = \mu^o(G : I, \wedge, \Gamma)$ be an $l - \Gamma$-semigroup. Then the set $F_\lambda = \{(p_{\lambda i}^{-1})_{i\lambda}, \forall i \in I\}$ of all α-idempotents of M ($\forall \alpha = (p_{\lambda i}) \in \Gamma$) in the λ-th column is a left zero convex $l - \Gamma$-subsemigroup of M.*

Proof. 1. We show that F_λ is a sublattice of M. For any $(p_{\lambda i}^{-1})_{i\lambda}, (p_{\lambda j}^{-1})_{j\lambda} \in F_\lambda$, there exist $k \in I, \delta \in \wedge$ such that $(p_{\lambda i}^{-1})_{i\lambda} \vee (p_{\lambda j}^{-1})_{j\lambda} = (c)_{k\delta}$. Then for $\alpha \in \Gamma$, we have $(p_{\lambda i}^{-1})_{i\lambda}\alpha(p_{\lambda i}^{-1})_{i\lambda} \vee (p_{\lambda i}^{-1})_{i\lambda}\alpha(p_{\lambda j}^{-1})_{j\lambda} = (p_{\lambda i}^{-1})_{i\lambda}\alpha(c)_{k\delta}$ and so $(p_{\lambda i}^{-1})_{i\lambda} \vee (p_{\lambda i}^{-1}p_{\lambda j}p_{\lambda j}^{-1})_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda} = (p_{\lambda i}^{-1}p_{\lambda k}c)_{i\delta}$. Hence $\lambda = \delta$ and $c = p_{\lambda k}^{-1}$, that is there exists $k \in I$ such that $(p_{\lambda i}^{-1})_{i\lambda} \vee (p_{\lambda j}^{-1})_{j\lambda} = (p_{\lambda k}^{-1})_{k\lambda}$ In the same way, we can show the result for the case " $\wedge$ ".

2. F_λ is left zero. Indeed for any $i, j \in I$ and $\alpha \in \Gamma$ we have

$$(p_{\lambda i}^{-1})_{i\lambda}\alpha(p_{\lambda j}^{-1})_{j\lambda} = (p_{\lambda i}^{-1}p_{\lambda j}p_{\lambda j}^{-1})_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda}.$$

3. F_λ is convex. Indeed: let $(p_{\lambda i}^{-1})_{i\lambda} \leq (c)_{k\delta} \leq (p_{\lambda j}^{-1})_{j\lambda}$. Then by step 2 we have $(p_{\lambda i}^{-1})_{i\lambda} \leq (p_{\lambda i}^{-1})_{i\lambda}\alpha(c)_{k\delta} = (p_{\lambda i}^{-1}p_{\lambda k}c)_{i\delta} \leq (p_{\lambda i}^{-1})_{i\lambda}\alpha(p_{\lambda j}^{-1})_{j\lambda} = (p_{\lambda i}^{-1})_{i\lambda}$. Hence $\delta = \lambda$ and $c = p_{\lambda k}^{-1}$. Therefore $(c)_{k\delta} \in F_\lambda$. □

Dually we have

Theorem 2.4. *Let $M = \mu^o(G : I, \wedge, \Gamma)$ be an $l - \Gamma$-semigroup. Then the set $F_\lambda = \{(p_{\lambda i}^{-1})_{i\lambda}, \forall \lambda \in \wedge\}$ of all α-idempotents of M ($\forall \alpha = (p_{\lambda i}) \in \Gamma$) in the i-th row is a right zero convex $l - \Gamma$-subsemigroup of M.*

Theorem 2.5. *Let $M = \mu^o(G : I, \wedge, \Gamma)$ be an $l - \Gamma$-semigroup and Γ satisfies that every matrix of it is such that all elements of its λ_0-th row and i_0-th column are equal to $b(\forall b \in G)$. Then*

(1) $\forall i, j \in I, \exists k \in I$ *such that* $\forall a \in G, \forall \delta \in \wedge, (a)_{i\delta} \vee (a)_{j\delta} = (a)_{k\delta},$

(2) $\forall \delta, \rho \in \wedge, \exists \lambda \in \wedge$ *such that* $\forall a \in G, \forall i \in I, (a)_{i\delta} \vee (a)_{i\rho} = (a)_{i\lambda}.$

Proof. We will prove only (1). The proof of (2) is similar. We show first that $(e)_{i\lambda_0} \vee (e)_{j\lambda_0} = (e)_{k\lambda_0}$.

Indeed: The following identities hold true:

1. $(a)_{i\mu} = (ab^{-1}p_{\lambda j}^{-1})_{i\lambda}\alpha(b)_{j\mu}$, for all $\lambda \in \Lambda, \alpha \in \Gamma$.

2. $(a)_{i\lambda} = (b)_{i\mu}\alpha(p_{\mu j}^{-1}b^{-1}a)_{j\lambda}$, for all $\mu, \alpha \in \Gamma$.

Suppose that $(a)_{i\mu} \leq (a)_{j\mu}$. Then $(a)_{i\mu} \vee (a)_{j\mu} = (a)_{j\mu}$. Using (1), we have $(a)_{i\mu} = (p_{\lambda j}^{-1})_{i\lambda}\alpha(a)_{j\mu}, (a)_{j\mu} = (p_{\lambda j}^{-1})_{j\lambda}\alpha(a)_{j\mu}$, and $(a)_{i\mu} \vee (a)_{j\mu} = [(p_{\lambda j}^{-1})_{i\lambda} \vee (p_{\lambda j}^{-1})_{j\lambda}]\alpha(a)_{j\mu} = (a)_{j\mu}$. The element between the brackets is necessarily of the form $(c)_{j\delta}$ and depends only of i and j. So, we have $(c)_{j\delta}\alpha(a)_{j\mu} = (a)_{j\mu}$, and $cp_{\delta j}a = a$. Consequently, $c = p_{\delta j}^{-1}$ and $(p_{\lambda j}^{-1})_{i\lambda} \vee (p_{\lambda j}^{-1})_{j\lambda} = (p_{\delta j}^{-1})_{j\delta}$. Therefore, we have that that there exists $k \in I$ such that $(e)_{i\lambda_0} \vee (e)_{j\lambda_0} = (c)_{k\lambda_0}$. Then for $\alpha \in \Gamma$ we have $(e)_{i\lambda_0}\alpha(e)_{i\lambda_0} \vee (e)_{i\lambda_0}\alpha(e)_{j\lambda_0} = (p_{\lambda_0 i})_{i\lambda_0} \vee (p_{\lambda_0 j})_{i\lambda_0} = (e)_{i\lambda_0}\alpha(c)_{k\lambda_0} = (p_{\lambda_0 k}c)_{i\lambda_0}$. Thus $(b)_{i\lambda_0} = (bc)_{i\lambda_0}$ since $p_{\lambda_0 i} = p_{\lambda_0 j} = p_{\lambda_0 k} = b$. Hence $c = e$.

In general, $(a)_{i\delta} \vee (a)_{j\delta} = [(e)_{i\lambda_0} \vee (e)_{j\lambda_0}]\alpha(b^{-1}a)_{i\delta} = (e)_{k\lambda_0}\alpha(b^{-1}a)_{i\delta} = (p_{\lambda_0 i}b^{-1}a)_{i\delta} = (a)_{k\delta}$. □

Remark 2. The results of Theorem 2.5 remain true if we replace $\vee$ with $\wedge$.

Proposition 2.6. Let $M = \mu^o(G : I, \wedge, \Gamma)$ be an $l-\Gamma$-semigroup and Γ satisfies that every matrix of it is such that all elements of its λ_0-th row and i_0-th column are equal to e the identity element of G. Then I and $\wedge$ are distributive lattice ordered sets.

Proof. Similar to the proof of Proposition 6 ([13]), we obtain it. □

By the Proposition 2.6 and the last paragraph of [13] we have:

Theorem 2.7. *Let $M = \mu^o(G : I, \wedge, \Gamma_e)$. Then M is an $l-\Gamma$-semigroup if and only if G is an l-group and I, $\wedge$ are lattices.*

The following lemma holds true:

Lemma 2.8 ([6, Lemma 2.1])**.** *Let $M = \mu^o(G : I, \wedge, \Gamma)$, Γ be regular and ρ be a congruence of M. Then $N_\rho = \{a \in G : (a)_{11}\rho(e)_{11}\}$ is a normal subgroup of G.*

Theorem 2.9. *Let $M = \mu^o(G : I, \wedge, \Gamma_e)$ be an $l-\Gamma$-semigroup. Then N_ρ is an l-ideal of the l-group G for any l-congruence ρ of M.*

Proof. By Lemma 2.8 we only need to verify that N_ρ is a convex sublattice of G. If $a,b \in N_\rho$, then $\forall c \in G$ and $a \leq c \leq b, (a)_{11} \leq (c)_{11} \leq (b)_{11}$, and thus $(a)_{11}\rho \leq (c)_{11}\rho \leq (b)_{11}\rho$. Therefore, $(c)_{11}\rho(e)_{11}$, i.e., $c \in N_\rho, N_\rho$ is also a sublattice of G since $(a)_{11} \vee (b)_{11} = (a \vee b)_{11}\rho(e)_{11} = (e)_{11} \vee (e)_{11}$, $(a)_{11} \wedge (b)_{11} = (a \wedge b)_{11}\rho(e)_{11}$.

Let $M = \mu^o(G : I, \wedge, \Gamma)$. We define an equivalence relation ϵ_I on I by the rule that

$$(i,j) \in \epsilon_I \text{ if } \{\lambda \in \wedge : p_{\lambda i} = 0\} = \{\lambda \in \wedge : p_{\lambda j} = 0\} \text{ for any } \alpha = (p_{\lambda i}) \in \Gamma$$

and an equivalence relation $\epsilon_\wedge$ on $\wedge$ by the analogues rule that

$$(\lambda, \mu) \in \epsilon_\wedge \text{ if } \{i \in I : p_{\lambda i} = 0\} = \{i \in I : p_{\lambda i} = 0\} \text{ for any } \alpha = (p_{\lambda j}) \in \Gamma.$$

If ρ is a proper congruence on M, we define a relation ρ_I on I by the rule that $(i,j) \in \rho_I$ if $(i,j) \in \epsilon_I$ and if $(p_{\lambda i}^{-1})_{i\lambda}\rho(p_{\lambda j}^{-1})_{j\lambda}$ for any $\alpha = (p_{\lambda i}) \in \Gamma$ and λ in $\wedge$ such that $p_{\lambda i}$ (and hence also $p_{\lambda j}$) is non-zero. □

In [6] it is proved that ρ_I and $\rho_\wedge$ are equivalence relations on I and $\wedge$ respectively. For the case of Γ_e the equivalence relations ρ_I and $\rho_\wedge$ are defined as follows:

$$\forall i,j \in I, (i,j) \in \rho_I \text{ if } (e)_{i\lambda} = (e)_{j\lambda}, \forall \lambda \in \wedge,$$
$$\forall \lambda, \mu \in \wedge, (\lambda, \mu) \in \rho_\wedge \text{ if } (e)_{i\lambda} = (e)_{i\mu}, \forall i \in I.$$

Lemma 2.10. *Let $M = \mu^o(G : I, \wedge, \Gamma_e)$ be an $l-\Gamma$-semigroup and ρ an l-congruence of M. Then ρ_I and $\rho_\wedge$ are l-congruences of lattice I and $\wedge$ respectively.*

Proof. For any $k \in I, (i,j) \in \rho_I$ we have that $(e)_{i\lambda} = (e)_{j\lambda}$, and thus for all $\lambda \in \wedge$, $(e)_{i\lambda} \vee (e)_{k\lambda} = (e)_{i\vee k,\lambda} = (e)_{j\vee k,\lambda}$. Therefore $(i \vee k, j \vee k) \in \rho_I$. Dually, we can prove that $(i \wedge k, j \wedge k) \in \rho_I$. □

In analogues way we can show that $\rho_\wedge$ is a l-congruence of $\wedge$.

In [5] it is proved the following lemma:

Lemma 2.11. *Let N be an l-ideal of an l-group G. If $ab \in N^{-1}$ for any $a, b \in G$, then $\left|(a \vee c)(b \vee c)^{-1}\right| \leq \left|ab^{-1}\right|, \left|(a \wedge c)(b \wedge c)^{-1}\right| \leq \left|ab^{-1}\right|$, for any $c \in G$.*

Lemma 2.12. *Let $M = \mu^o(G : I, \wedge, \Gamma_e)$ be an $l-\Gamma$-semigroup. Then for any l-ideal N of G and any l-congruence ψ, φ of lattices I and $\wedge$ there exists an l-congruence ρ of G such that $N = N_\rho, \psi = \rho_I, \varphi = \rho_\wedge$.*

Proof. For any $(a)_{i\lambda}, (b)_{j\mu} \in M$ we define the relation ρ as follows:

$$(a)_{i\lambda}\rho(b)_{j\lambda} \text{ if } ab^{-1} \in N, (i,j) \in \psi, (\lambda,\mu) \in \phi.$$

It is clear that ρ is an equivalence relation of M. Since $(ca)(cb^{-1}) \in N$, $(k,k) \in \psi, (\lambda,\mu) \in \phi$ for $(a)_{i\lambda}\rho(b)_{j\lambda}$ and any $(c)_{k\delta} \in M$, so that for $\alpha_e \in \Gamma_e$, $(c)_{k\delta}\alpha_e(a)_{i\lambda}\rho(c)_{k\delta}\alpha_e(b)_{j\mu}$, that is, ρ is left compatible with the operation on M. In the same way we can deduce that ρ is right compatible.

Since $(a)_{i\lambda}\rho(b)_{j\mu}$, one has $ab^{-1} \in N$, and hence $\left|ab^{-1}\right| \in N$. For any $c \in G$, by Lemma 2.11 we have $\left|(a\vee c)(b\vee c)^{-1}\right| \leq \left|ab^{-1}\right| \in N$, $\left|(a\wedge c)(b\wedge c)^{-1}\right| \leq \left|ab^{-1}\right| \in N$. Consequently, $(a\vee c)(b\vee c)^{-1}, (a\wedge c)(b\wedge c)^{-1} \in N$.

Moreover, $(i,j) \in \psi, (\lambda,\mu) \in \phi$, and hence for any $k \in I, \delta \in \wedge$ we have $(i\vee k, j\vee k) \in \psi, (i\wedge k, j\wedge k) \in \psi$, $(\lambda\vee\delta, \mu\vee\delta) \in \phi$, $(\lambda\wedge\delta, \mu\wedge\delta) \in \phi$. Therefore, $(a)_{i\lambda}\vee(c)_{k\delta}\rho(b)_{j\mu}\vee(c)_{k\delta}$, $(a)_{i\lambda}\wedge(c)_{k\delta}\rho(b)_{j\mu}\wedge(c)_{k\delta}$. So, ρ is an l-congruence of M. It is easily to verify that $N_\rho = N, \psi = \rho_I, \phi = \rho_\wedge$.

The following lemma is proved in spirit of Lemma 2.5 ([6]) in case of l-Rees matrix Γ-semigroup.

Lemma 2.13. *Let ρ and σ be l-congruences of an l-Rees matrix Γ-semigroup $M = \mu^o(G : I, \wedge, \Gamma_e)$. If $N_\rho = N, \rho_I = \sigma_I, \rho_\lambda = \sigma_\lambda$ then $\rho = \sigma$.*

Proof. If $(a)_{i\lambda}\rho(b)_{j\mu}$, then $(i,j) \in \rho_I = \sigma_I, (\lambda,\mu) \in \rho_\wedge = \sigma_\wedge$ and $ab^{-1} \in N_\rho = N_\sigma$, so that $(ab^{-1})_{11}\rho(e)_{11}$. Therefore, $(e)_{i1}\alpha_e(ab^{-1})_{11}\alpha_e(e)_{1\lambda} = (ab^{-1}_{i\lambda}\sigma(e)_{i1}\alpha_e(e)_{11}\alpha_e(e)_{i\lambda} = (e)_{i\lambda}$, $(e)_{i\lambda}\sigma(e)_{j\lambda}\sigma(e)_{i\mu}(\forall j \in I, \mu \in \wedge)$, that is

(1) $$(a)_{i\lambda}\sigma(b)_{i\lambda}\sigma(b)_{j\lambda},$$

(2) $$(e)_{i\lambda}\sigma(e)_{i\mu}.$$

By (1)×(2), we have $(a)_{i\lambda}\sigma(b)_{j\mu}$ so that $\rho \subseteq \sigma$.

In analogues way we can prove that $\sigma \subseteq \rho$. □

In spirit of Theorem 2.1 and 2.2 ([6]) and summarizing the above results we get the following theorem

Theorem 2.14. *Let $M = \mu^o(G : I, \wedge, \Gamma_e)$ be an $l-\Gamma$-semigroup. The the mapping $\Phi : \rho \mapsto (N_\rho, \rho_I, \rho_\wedge)$ is an ordered-preserving bijection from the set of proper l-congruences on M onto the set of triples, where N_ρ is an l-ideal of an l-group G, ρ_I and $\rho_\wedge$ are l-congruences of the lattices I and $\wedge$ respectively.*

We use the above correspondence between the proper l-congruences on M and the triples as above to derive information on the nature of the lattice of l-congruences of the $l-\Gamma$-semigroups. If $(N_\rho, \rho_I, \rho_\wedge)$ is a triples of M as above we shall write the l-congruence $(N_\rho, \rho_I, \rho_\wedge)\Phi^{-1}$ corresponding to the triple as $[N_\rho, \rho_I, \rho_\wedge]$ (with square brackets). Thus $\rho = [N_\rho, \rho_I, \rho_\wedge]$.

Lemma 2.15. *Let ρ and σ be l-congruences of an l-Rees matrix Γ-semigroup $M = \mu^o(G : I, \wedge, \Gamma_e)$. Then* $\rho \wedge \sigma = [N_\rho \cap N_\sigma, \rho_I \cap \sigma_I, \rho_\wedge \cap \sigma_\wedge]$, $\rho \vee \sigma = [N_\rho N_\sigma, \rho_I \vee \sigma_I, \rho_\wedge \vee \sigma_\wedge]$.

Proof. The proof is analogues to the proof of Lemma 3.1 in [6], and we omit it here. □

Remark 3. The intersection and union of l-congruences of an $l-\Gamma$-semigroup M are same as the cases of Γ-semigroup about congruences on it. The union is given by $a(\vee_{i\in I}\rho_i)b$ if $a \equiv a_1(\rho_1), a_1 \equiv a_2(\rho_2), ..., a_n \equiv b(\rho_n)$, for some $a_1, a_2, ..., a_n \in M, \{1, 2, ..., n\} \in I$.

By Lemma 2.15, the mapping Φ in Theorem 2.14 is an l-morphism from the lattice of l-congruences of M onto the triples lattice whose three components are lattices with respect to the intersection and the union as above respectively. It is known that the lattice of l-ideals of an l-group and the l-congruences of a lattice are distributive [1]. Therefore, we have the following theorem:

Theorem 2.16. *The lattice of l-congruences of any l-Rees matrix Γ-semigroup $M = \mu^o(G : I, \wedge, \Gamma_e)$ is distributive.*

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Received: 30.III.2010

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