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ALGORITHM FOR THE CALCULATION OF THE TWO VARIABLES CUBIC SPLINE FUNCTION

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Abstract. When having just one variable, the existence and uniqueness of the interpolation spline function reduces to studying the solutions of an algebrical system of equations. This allows us to find a practical way of calculating the interpolation spline function. Also in the case of two variables spline functions, we can construct a linear system of equations determined by the continuity conditions of the spline function and of its partial derivatives on the edge of each division rectangle. The existence and uniqueness of the solution of the obtained system ensure the existence and uniqueness of the two variables interpolation spline function and offers a practical calculation method.

This can be used to determine approximate global solutions, of some partial differential equations, solutions whose values can be determined at any point of their domain of definition and can provide information on derivatives approximate of solutions. After calculating the two variable cubic spline function, we must assess the rest of the approximation.

Mathematics Subject Classification 2010: 65D07.

Key words: cubic spline function of two variables, tangent plane, the flatness conditions, connection to nodes, interpolation algorithm, partial derivatives.

1. Introduction

Although there is a well-founded theory about the spline functions of two variables, it can not be applied in practice, due to high formulas that appear and for the fact that in these formulas become involved partial derivatives of the function going up to five or six orders, etc. This article proposes a practical method for calculating a spline interpolation function of two variables based on continuity of the spline function and of its partial derivatives on the border of each division rectangle. In order to do so, we set conditions of connection in nodes, smoothness conditions (existence of a single tangent plane in each node) and additional conditions on partial derivatives of order two, maximum three. Out of these conditions we obtain systems of linear equations, systems to be filled with new equations, given by the limit conditions (head of range).

In the literature there are several ways to supplement systems such as: to set conditions where explicitly appear the function derivatives (but not knowing the function we want to interpolate, obviously we do not know these derivatives), we set conditions that on the first two (i.e. the last two intervals of interpolation) to have the same spline function (in this case we break the tridiagonal form of the linear system so that some algorithms for solving them are not applicable), on the first and the last interval the interpolation function to be of degree two, etc. In the article are obtained new conditions on the limit, compatible with the existing ones within range, by a process of transition to limit in the existing conditions and cancelation of all parameters with negative or zero index (which practically does not exist). These are systems with a single solution that ensures the existence and uniqueness of spline interpolation functions of two variables, provides a practical method of calculation and the possibility of developing an algorithm for calculating the spline interpolation function of two variables.

Using in the article the continuity module of a two variable function, we obtain an assessment formula for the rest of the approximation, for a semi-natural two variable cubic spline function (the partial mixed derivatives of order two are zero on half of the domain's boundary).

2. The calculation of the two variables cubic spline function

Given $D = [a,b] \times [c,d]$, $f : D \to \mathbb{R}$, $f \in C(D)$.

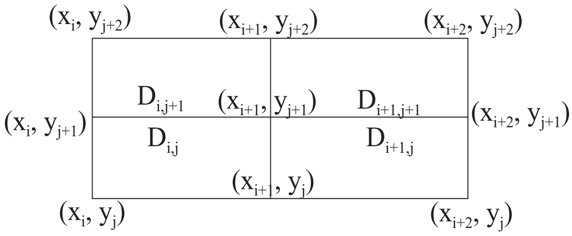
$\Delta_x : a = x_1 < x_2 < \ldots < x_m = b$, division of the interval $[a,b]$, with the pace $h_i^x = x_{i+1} - x_i$, $\forall i = \overline{1, m-1}$.

$\Delta_y : c = y_1 < y_2 < \ldots < y_m = d$, division of the interval $[c,d]$, with the pace $h_j^y = y_{j+1} - y_j$, $\forall j = \overline{1, n-1}$.

$f(x_i, y_j) = z_{ij}$-given, $\forall i = \overline{1,m}, \forall j = \overline{1,n}$;

$D_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $\forall i = \overline{1, m-1}$, $\forall j = \overline{1, n-1}$ ([2]).

We search for $s_{ij} : D_{ij} \to \mathbb{R}$, interpolation cubic spline function $(s(x_i, y_j) =$



$z_{ij}; \forall i = \overline{1,m}, \forall j = \overline{1,n})$ as:

$$
\begin{aligned}
s_{ij}(x,y) &= a_{ij}(x-x_i)^3 + b_{ij}(x-x_i)^2 + c_{ij}(x-x_i) \\
&+ \alpha_{ij}(y-y_j)^3 + \beta_{ij}(y-y_j)^2 \\
&+ \gamma_{ij}(y-y_j) + d_{ij}(x-x_i)^2(y-y_j) \\
&+ \delta_{ij}(x-x_i)(y-y_j)^2 + \varepsilon_{ij}(x-x_i)(y-y_j) + z_{ij}, \\
&\forall x \in [x_i, x_{i+1}], \forall y \in [y_j, y_{j+1}], \forall i = \overline{1, m-1}, \forall j = \overline{1, n-1}.
\end{aligned} \tag{1}
$$

Obviously $s(x_j, y_j) = z_{ij}, \forall i = \overline{1,m}, \forall j = \overline{1,n}$. Out of the joining conditions in the nodes: $s_{ij}(x_{i+1}, y_j) = z_{i+1,j}$, $s_{ij}(x_i, y_{j+1}) = z_{i,j+1}$, we get:

$$a_{ij}(h_i^x)^3 + b_{ij}(h_i^x)^2 + c_{ij}(h_i^x) + z_{ij} = z_{i+1,j}, \ \forall i = \overline{1, m-1}, \forall j = \overline{1,n}, \tag{2}$$

$$\alpha_{ij}(h_j^y)^3 + \beta_{ij}(h_j^y)^2 + \gamma_{ij}(h_j^y) + z_{ij} = z_{i,j+1}, \ \forall i = \overline{1,m}, \forall j = \overline{1, n-1}. \tag{3}$$

If in the case of one variable cubic spline function, the smoothness conditions imposed the existence of an unique tangent in every node, at the two curves, for the two variables spline functions we impose the existence of a single tangent plan at the two surfaces in every node. As the tangent plan has the normal vector: $\overrightarrow{N}(-\frac{\partial s_{ij}}{\partial x}, -\frac{\partial s_{ij}}{\partial y}, 1)$, we set the conditions:

a) $\begin{cases} \frac{\partial s_{ij}}{\partial x}(x_{i+1}, y_j) = \frac{\partial s_{i+1,j}}{\partial x}(x_{i+1}, y_j) \\ \frac{\partial s_{ij}}{\partial y}(x_{i+1}, y_j) = \frac{\partial s_{i+1,j}}{\partial y}(x_{i+1}, y_j) \end{cases}$

b) $\begin{cases} \frac{\partial s_{ij}}{\partial x}(x_i, y_{j+1}) = \frac{\partial s_{i,j+1}}{\partial x}(x_i, y_{j+1}) \\ \frac{\partial s_{ij}}{\partial y}(x_i, y_{j+1}) = \frac{\partial s_{i,j+1}}{\partial y}(x_i, y_{j+1}) \end{cases}$

$\frac{\partial s_{ij}}{\partial x}(x,y) = 3a_{ij}(x-x_i)^2 + 2b_{ij}(x-x_i) + c_{ij} + 2d_{ij}(x-x_i)(y-y_j)^2 + \delta_{ij}(y-y_j)^2 + \varepsilon_{ij}(y-y_j)$

$\frac{\partial s_{ij}}{\partial y}(x,y) = 3\alpha_{ij}(y-y_i)^2 + 2\beta_{ij}(y-y_j) + \gamma_{ij} + d_{ij}(x-x_i)^2 + 2\delta_{ij}(x-$

$x_i)(y-y_j)+\varepsilon_{ij}(x-x_i)$ From a) and b) we obtain:

$$(4)\qquad 3a_{ij}(h_i^x)^2+2b_{ij}(h_i^x)+c_{ij}=c_{i+1,j},\ \forall i=\overline{1,m-1},\forall j=\overline{1,n}$$

$$(5)\qquad d_{ij}(h_i^x)^2+\varepsilon_{ij}(h_i^x)+\gamma_{ij}=\gamma_{i+1,j},\forall i=\overline{1,m-1},\forall j=\overline{1,n}$$

$$(6)\qquad \delta_{ij}(h_j^y)^2+\varepsilon_{ij}(h_j^y)+c_{ij}=c_{i,j+1},\ \forall i=\overline{1,m},\forall j=\overline{1,n-1}$$

$$(7)\qquad 3\alpha_{ij}(h_j^y)^2+2\beta_{ij}(h_j^y)+\gamma_{ij}=\gamma_{i,j+1},\ \forall i=\overline{1,m},\forall j=\overline{1,n-1}.$$

The existence of an unique tangent plan to the two surfaces, in every node, does not provide the uniqueness of the surface. We must set supplementary conditions connected to the partial derivatives of order two:

$$\text{a)}\ \begin{cases}\frac{\partial^2 s_{ij}}{\partial x^2}(x_{i+1},y_j)=\frac{\partial^2 s_{i+1,j}}{\partial x^2}(x_{i+1},y_j)\\ \frac{\partial^2 s_{ij}}{\partial x\partial y}(x_{i+1},y_j)=\frac{\partial s^2_{i+1,j}}{\partial x\partial y}(x_{i+1},y_j)\\ \frac{\partial^2 s_{ij}}{\partial y^2}(x_{i+1},y_j)=\frac{\partial^2 s_{i+1,j}}{\partial y^2}(x_{i+1},y_j)\end{cases}$$

$$\text{b)}\ \begin{cases}\frac{\partial^2 s_{ij}}{\partial x^2}(x_i,y_{j+1})=\frac{\partial^2 s_{i,j+1}}{\partial x^2}(x_i,y_{j+1})\\ \frac{\partial^2 s_{ij}}{\partial x\partial y}(x_i,y_{j+1})=\frac{\partial^2 s_{i,j+1}}{\partial x\partial y}(x_i,y_{j+1})\\ \frac{\partial s^2_{ij}}{\partial y^2}(x_i,y_{j+1})=\frac{\partial^2 s_{i,j+1}}{\partial y^2}(x_i,y_{j+1})\end{cases}$$

$$\frac{\partial^2 s_{ij}}{\partial x^2}=6a_{ij}(x-x_i)+2b_{ij}+2d_{ij}(y-y_j)$$

$$\frac{\partial^2 s_{ij}}{\partial x\partial y}=2d_{ij}(x-x_i)+2\delta_{ij}(y-y_j)+\varepsilon_{ij}$$

$$\frac{\partial^2 s_{ij}}{\partial y^2}=6\alpha_{ij}(y-y_j)+2\beta_{ij}+2\delta_{ij}(y-y_j)$$ From a) and b) we get:

$$(8)\qquad 3a_{ij}(h_i^x)+b_{ij}=b_{i+1,j},\ \forall i=\overline{1,m-1},\forall j=\overline{1,n},$$

$$(9)\qquad 2d_{ij}(h_i^x)+\varepsilon_{ij}=\varepsilon_{i+1,j},\ \forall i=\overline{1,m-1},\forall j=\overline{1,n}$$

$$(10)\qquad \delta_{ij}(h_i^x)+\beta_{ij}=\beta_{i+1,j},\ \forall i=\overline{1,m-1},\forall j=\overline{1,n}$$

$$(11)\qquad d_{ij}(h_j^y)+b_{ij}=b_{i,j+1},\ \forall i=\overline{1,m},\forall j=\overline{1,n-1}$$

$$(12)\qquad 2\delta_{ij}(h_j^y)+\varepsilon_{ij}=\varepsilon_{i,j+1},\ \forall i=\overline{1,m},\forall j=\overline{1,n-1}$$

$$(13)\qquad 3\alpha_{ij}(h_j^y)+\beta_{ij}=\beta_{i,j+1},\ \forall i=\overline{1,m},\forall j=\overline{1,n-1}.$$

From (2), (4) and (8) we obtain the system:

$$(14)\qquad \begin{cases}a_{ij}(h_i^x)^3+b_{ij}(h_i^x)^2+c_{ij}(h_i^x)+z_{ij}=z_{i+1,j}\\ 3a_{ij}(h_i^x)^2+2b_{ij}(h_i^x)+c_{ij}=c_{i+1,j}\\ 3a_{ij}(h_i^x)+b_{ij}=b_{i+1,j},\ \forall i=\overline{1,m-1},\forall j=\overline{1,n}.\end{cases}$$

Successively eliminating between (2) and (4) a_{ij} and b_{ij} we get:

$$b_{ij} = 3\frac{z_{i+1,j} - z_{ij}}{(h_i^x)^3} - \frac{c_{i+1,j} + 2c_{ij}}{(h_i^x)}, \forall i = \overline{1, m-1}, \forall j = \overline{1, n} \tag{15}$$

$$a_{ij} = -2\frac{z_{i+1,j} - z_{ij}}{(h_i^x)^3} + \frac{c_{i+1,j} + c_{ij}}{(h_i^x)^2},\ \forall i = \overline{1, m-1}, \forall j = \overline{1, n}. \tag{16}$$

Introducing (15) and (16) in (8) we obtain:

$$\begin{aligned} & h_{i+1}^x c_{ij} + 2[(h_i^x) + (h_{i+1}^x)]c_{i+1,j} + (h_i^x)c_{i+2,j} \\ & = 3[(h_i^x)\frac{z_{i+2,j} - z_{i+1,j}}{(h_{i+1}^x)} + (h_{i+1}^x)\frac{z_{i+1,j} - z_{ij}}{(h_i^x)}], \\ & \forall i = \overline{1, m-2}, \forall j = \overline{1, n}. \end{aligned} \tag{17}$$

So that the system (17) to have an unique solution we still need "2n" equations (conditions). Going to the limit in (17) with $i \to 0, i \to m-1$ and cancelling all parameters with zero or negative index, we get:

$$\begin{cases} 2c_{1,j} + c_{2j} = 3\frac{z_{2j} - z_{1j}}{(h_1^x)} \\ c_{m-1,j} + 2c_m^j = 3\frac{z_{mj} - z_{m-1,j}}{(h_{m-1}^x)} \end{cases}, \forall j = \overline{1, n}. \tag{18}$$

In conclusion, we calculate c_{ij} from (17) and (18), and then a_{ij} and b_{ij} out of (16) and (15). Analogically, out of (3), (7) and (13) we obtain:

$$\beta_{ij} = 3\frac{z_{i,j+1} - z_{ij}}{(h_j^y)^3} - \frac{\gamma_{i,j+1} + 2\gamma_{ij}}{(h_j^y)}, \forall i = \overline{1, m}, \forall j = \overline{1, n-1} \tag{19}$$

$$\alpha_{ij} = -2\frac{z_{i,j+1} - z_{ij}}{(h_j^y)^3} + \frac{\gamma_{i,j+1} + \gamma_{ij}}{(h_j^y)^2},\ \ \forall i = \overline{1, m}, \forall j = \overline{1, n-1} \tag{20}$$

(21)

$$\begin{aligned} & (h_{j+1}^y)\gamma_{ij} + 2[(h_j^y) + (h_{j+1}^y)]\gamma_{i,j+1} + (h_j^y)\gamma_{ij+2} \\ & = 3[(h_j^y)\frac{z_{i,j+2} - z_{i,j+1}}{(h_{j+1}^y)} + (h_{j+1}^y)\frac{z_{i,j+1} - z_{i,j}}{(h_j^y)}], \forall i = \overline{1, m}, \forall j = \overline{1, n-2}. \end{aligned}$$

In order that (21) to have an unique solution, we go to the limit with and cancel all the parametres with zero or negative index. We obtain the following equations:

$$\begin{cases} 2\gamma_{i,1} + \gamma_{i2} = 3\frac{z_{i2} - z_{i1}}{(h_1^y)} \\ \gamma_{i,n-1} + 2\gamma_{in} = 3\frac{z_{in} - z_{i,n-1}}{(h_{n-1}^y)} \end{cases}, \forall i = \overline{1, m}. \tag{22}$$

Out of (21) and (22) we determine γ_{ij}, then from (20) and (19) α_{ij} and β_{ij}. Further on, from (11) we determine:

$$d_{ij} = \frac{b_{i,j+1} - b_{i,j}}{h_j^y}, \ \forall i = \overline{1,m}, \forall j = \overline{1,n-1} \tag{23}$$

from (10) we find out

$$\delta_{ij} = \frac{\beta_{i+1,j} - \beta_{ij}}{h_i^x}, \ \forall i = \overline{1,m-1}, \forall j = \overline{1,n}, \tag{24}$$

out of (5) or (6) we find:

$$\varepsilon_{ij} = \frac{\gamma_{i+1,j} - \gamma_{ij} - d_{ij}(h_i^x)^2}{(h_i^x)}, \ \forall i = \overline{1,m-1}, \forall j = \overline{1,n} \tag{25}$$

$$\varepsilon_{ij} = \frac{c_{i,j+1} - c_{ij} - \delta_{ij}(h_j^y)^2}{(h_j^y)}, \ \forall i = \overline{1,m}, \forall j = \overline{1,n-1} \ \ ([1]). \tag{26}$$

Special case

Although they are useful from a practical point of view, the above presented facts have some disadvantages such as: the great calculation number (we practically reach to solving a system of $9mn$ equations with $9mn$ unknowns, system which is decomposed in three systems of $3mn$ equations with $3mn$ unknowns), it does not allow the passing to a three variables interpolation spline function, etc.

Considering in (1) $d_{ij} = \delta_{ij} = \varepsilon_{ij} = 0$ (that is dropping the terms in x^2y, xy^2, xy), we get:

$$\begin{aligned} s_{ij}(x,y) &= a_{ij}(x-x_i)^3 + b_{ij}(x-x_i)^2 + c_{ij}(x-x_i) + \alpha_{ij}(y-y_j)^3 \\ &\quad + \beta_{ij}(y-y_j)^2 + \gamma_{ij}(y-y_j), \\ &\quad \forall x \in [x_i, x_{i+1}], \forall y \in [y_j, y_{j+1}], \forall i = \overline{1,m-1}, \forall j = \overline{1,n-1} \end{aligned} \tag{1'}$$

meaning the sum of two spline functions with one variable: one compared to x, the other compared to y.

In order to obtain an assessment of the rest of the approximation of the function f, with the two variable cubic spline function given by (1), we express first the coefficients of the function s_{ij} with the help of its partial derivatives of order two.

First let us decompose s_{ij} into three components:

$$(27)\qquad \begin{cases} f_{ij}(x) = a_{ij}(x-x_i)^3 + b_{ij}(x-x_i)^2 + c_{ij}(x-x_i) \\ g_{ij}(y) = \alpha_{ij}(y-y_j)^3 + \beta_{ij}(y-y_j)^2 + \gamma_{ij}(y-y_j) \\ h_{ij}(x,y) = d_{ij}(x-x_i)^2(y-y_j) + \delta_{ij}(x-x_i)(y-y_1)^2 \\ \qquad\qquad +\varepsilon_{ij}(x-x_i)(y-y_j) + z_{ij}. \end{cases}$$

We denote

$$(28)\qquad \frac{\partial^2 s_{ij}}{\partial x^2}(x_i,y_j) = E_{ij},\ \frac{\partial^2 s_{ij}}{\partial x \partial y}(x_i,y_j) = F_{ij};\ \frac{\partial^2 s_{ij}}{\partial y^2}(x_i,y_j) = G_{ij}.$$

As $\frac{\partial^2 s_{ij}}{\partial x^2} = 6a_{ij}(x-x_i)+2b_{ij}+2d_{ij}(y-y_j)$ and $\frac{\partial^2 s_{ij}}{\partial x^2}(x_i,y_j) = E_{ij} \Rightarrow 2b_{ij} = E_{ij} \Rightarrow b_{ij} = \frac{1}{2}E_{ij}$,

$\frac{\partial^2 s_{ij}}{\partial x^2}(x_{i+1},y_j) = 6a_{ij}h_i^x + 2b_{ij} \Rightarrow E_{i+1,j} = 6a_{ij}h_i^x + 2b_{ij} \Rightarrow a_{ij} = \frac{E_{i+1,j}-E_{ij}}{6h_i^x}$

$s_{ij}(x_{i+1},y_j) = z_{i+1,j} \Rightarrow a_{ij}(h_i^x)^3 + b_{ij}(h_i^x)^2 + c_{ij}h_i^x + z_{ij} = z_{i+1,j} \Rightarrow c_{ij} = \frac{z_{i+1,j}-z_{ij}}{h_i^x} - \frac{E_{i+1,j}+2E_{ij}}{6}h_i^x$

$$\begin{aligned} f_{ij}(x) &= \frac{E_{i+1,j}-E_{ij}}{6h_i^x}(x-x_i)^3 + \frac{1}{2}E_{ij}(x-x_i)^2 \\ &+ [\frac{z_{i+1,j}-z_{ij}}{6h_i^x} - \frac{E_{i+1,j}+2E_{ij}}{6}h_i^x](x-x_i) \\ &= E_{i+1,j}\frac{(x-x_i)^3}{6h_i^x} - \frac{E_{ij}}{6}[h_i^x - (x_{i+1}-x)^3] \\ &+ \frac{E_{ij}}{2}[h_i^x - (x_{i+1}-x)^2] + z_{i+1,j}\frac{x-x_i}{h_i^x} \\ &- z_{ij}\frac{h_i^x-(x_{i+1}-x)}{h_i^x} - \frac{E_{i+1,j}}{6}h_i^x(x-x_i) - \frac{E_{ij}}{3}h_i^x[h_i^x-(x_{i+1}-x)] \\ &= E_{i+1,j}\frac{(x-x_i)^3}{6h_i^x} - \frac{E_{ij}}{6}(h_i^x)^2 + \frac{E_{ij}}{2}h_i^x(x_{i+1}-x) - \frac{E_{ij}}{2}(x_{i+1}-x)^2 \\ &+ \frac{E_{ij}}{6}\frac{(x_{i+1}-x)^3}{h_i^x} + \frac{E_{ij}}{2}(h_i^x)^2 - E_{ij}h_i^x(x_{i+1}-x) \\ &+ \frac{E_{ij}}{2}(x_{i+1}-x)^2 + z_{i+1,j}\frac{x-x_i}{h_i^x} - z_{ij} + z_{ij}\frac{x_{i+1}-x}{h_i^x} \end{aligned}$$

$$
\begin{aligned}
&-\frac{E_{i+1,j}}{6}h_i^x(x-x_i)-\frac{E_{ij}}{3}(h_i^x)^2+\frac{1}{2}E_{ij}h_i^x(x_{i+1}-x)\\
&=E_{i+1,j}\frac{x-x_i}{6h_i^x}[(x-x_i)^2-(h_i^x)^2]+E_{ij}\frac{x_{i+1}-x}{6h_i^x}[(x_{i+1}-x)^2-(h_i^x)^2]\\
&+z_{ij}\frac{x_{i+1}-x}{h_i^x}+z_{i+1,j}\frac{x-x_i}{h_i^x}-z_{ij}+\frac{z_{i+1,j}+z_{ij}}{2}-\frac{z_{i+1,j}+z_{ij}}{2}.
\end{aligned}
$$

But

$$
\begin{aligned}
&z_{ij}\frac{x_{i+1}-x}{h_i^x}+z_{i+1,j}\frac{x-x_i}{h_i^x}-\frac{z_{i+1,j}+z_{ij}}{2}\\
&=\frac{x_{i+1}z_{ij}-x_iz_{i+1,j}+x_iz_{ij}-x_{i+1}z_{i+1,j}-2x(z_{ij}-z_{i+1,j})}{2h_i^x}\\
&=(z_{ij}-z_{i+1,j})\frac{x_{i+1}+x_i-2x}{2h_i^x}.
\end{aligned}
$$

Consequently, we have:

$$
\begin{aligned}
f_{ij}(x)=&E_{i+1,j}\frac{x-x_i}{6h_i^x}[(x-x_i)^2-(h_i^x)^2]\\
&+E_{ij}\frac{x_{i+1}-x}{6h_i^x}[(x_{i+1}-x)^2-(h_i^x)^2]\\
&+(z_{ij}-z_{i+1,j})\frac{x_{i+1}+x_i-2x}{2h_i^x}+\frac{z_{i+1,j}+z_{ij}}{2}\\
&-f(x,y)+f(x,y)-z_{ij}.
\end{aligned}
\tag{29}
$$

We write $\overline{\Delta}_x=\max\{h_i^x|i=\overline{1,m-1}\}$, $\underline{\Delta}_x=\min\{h_i^x|i=\overline{1,m-1}\}$, $\alpha=\frac{\overline{\Delta}_x}{\underline{\Delta}_x}$.

$\overline{\Delta}_y=\max\{h_j^y|j=\overline{1,n-1}\}$, $\underline{\Delta}_y=\min\{h_j^y|j=\overline{1,n-1}\}$, $\beta=\frac{\overline{\Delta}_y}{\underline{\Delta}_y}$, $\overline{\Delta}=\overline{\Delta}_x\times\overline{\Delta}_y$

$\omega(f,h)=\sup\{|f(x+t_1,y+t_2)-f(x,y)|\,|x,x+t_1\in[a,b],y,y+t_2\in[c,d],|t_1|<h_1,|t_2|<h_2\}$

- the continuity module of $f(h=(h_1,h_2))$,

$$
\begin{aligned}
|z_{ij}-f(x,y)|<\omega(f,\overline{\Delta})\Rightarrow&\left|\frac{z_{i+1,j}+z_{ij}}{2}-f(x,y)\right|\\
&\le\frac{1}{2}(|z_{i+1,j}-f(x,y)|+|z_{ij}-f(x,y)|)<\omega(f,\overline{\Delta}).
\end{aligned}
$$

So:

$$|\frac{z_{i+1,j}+z_{ij}}{2}-f(x,y)|<\omega(f,\overline{\Delta});|f(x,y)-z_{ij}|<\omega(f,\overline{\Delta}). \tag{30}$$

Given $\alpha(x)=\frac{x_{i+1}+x_i-2x}{2h_i^x}\Rightarrow\alpha'(x)=-\frac{1}{h_i^x}<0\Rightarrow\alpha(x_{i+1})\le\alpha(x)\le\alpha(x_i)\Rightarrow -\frac{1}{2}\le\alpha(x)\le\frac{1}{2}\Rightarrow|\alpha(x)|\le\frac{1}{2}$,

$$|(z_{ij}-z_{i+1,j})\frac{x_{i+1}+x_i-2x}{2h_i^x}|\le\frac{1}{2}|z_{ij}-z_{i+1,j}|<\frac{1}{2}\omega(f,\overline{\Delta}). \tag{31}$$

Given $\beta(x)=\frac{x-x_i}{6h_i^x}[(x-x_i)^2-(h_i^x)^2]\Rightarrow\beta'(x)=\frac{1}{6h_i^x}[3(x-x_i)^2-(h_i^x)^2]$,

$\beta'(x)=0\Rightarrow x=x_i+\frac{h_i^x}{\sqrt{3}};\beta(x_i)=0,\beta(x_{i+1})=0,\beta(x_i+\frac{h_i^x}{\sqrt{3}})=-\frac{(h_i^x)^2}{9\sqrt{3}}\Rightarrow |\beta(x)|\le\frac{1}{9\sqrt{3}}(h_i^x)^2$,

$$\left|\frac{x-x_i}{6h_i^x}[(x-x_i)^2-(h_i^x)^2]\right|\le\frac{1}{9\sqrt{3}}(h_i^x)^2. \tag{32}$$

Similar:

$$\left|\frac{x_{i+1}-x}{6h_i^x}[(x_{i+1}-x)^2-(h_i^x)^2]\right|\le\frac{1}{9\sqrt{3}}(h_i^x)^2. \tag{33}$$

We must find now an upper bound for $(h_i^x)^2(|E_{i+1,j}|+|E_{ij}|)$. From the condition $\frac{\partial s_{ij}}{\partial x}(x_{i+1},y_j)=\frac{\partial s_{i+1,j}}{\partial x}(x_{i+1},y_j)$, we obtain $E_{i+1,j}\frac{h_i^x}{2}+\frac{1}{h_i^x}z_{i+1,j}-E_{i+1,j}\frac{h_i^x}{6}-\frac{1}{h_i^x}z_{ij}+E_{ij}\frac{h_i^x}{6}=-E_{i+1,j}\frac{h_{i+1}^x}{2}+\frac{1}{h_{i+1}^x}z_{i+2,j}-E_{i+2,j}\frac{h_{i+1}^x}{6}-\frac{1}{h_{i+1}^x}z_{i+1,j}+E_{i+1,j}\frac{h_{i+1}^x}{6}\Rightarrow 6(\frac{z_{i+2,j}-z_{i+1,j}}{h_{i+1}^x}-\frac{z_{i+1,j}-z_{ij}}{h_i^x})=2E_{i+1,j}(h_i^x+h_{i+1}^x)+E_{ij}h_i^x+E_{i+2,j}h_i^x\Rightarrow$

$$\begin{aligned}&h_i^xE_{ij}+2(h_i^x+h_{i+1}^x)E_{i+1,j}+E_{i+2,j}h_{i+1}^x\\&=6(\frac{z_{i+1,j}-z_{i+1,j}}{h_{i+1}^x}-\frac{z_{i+1,j}-z_{ij}}{h_i^x}).\end{aligned} \tag{34}$$

We write:

$$P_{i+1}^x=\frac{h_i^x}{h_i^x+h_{i+1}^x};L_{i+1}^x=\frac{h_{i+1}^x}{h_i^x+h_{i+1}^x};(P_{i+1}^x+L_{i+1}^x=1) \tag{35}$$

$$D_{i+1}^x=\frac{6}{h_i^x+h_{i+1}^x}(\frac{z_{i+2,j}-z_{i+1,j}}{h_{i+1}^x}-\frac{z_{i+1,j}-z_{ij}}{h_i^x}). \tag{36}$$

For $i \to 0$ and $h_0^x = 0 \Rightarrow P_1^x = 0,\ L_1^x = 1,\ D_1^x = 6\frac{z_{2j}-z_{1j}}{(h_1^x)^2}$.

$i \to m-1$ and $h_m^x = 0 \Rightarrow P_m^x = 1,\ L_m^x = 0,\ D_m^x = -6\frac{z_{mj}-z_{m-1,j}}{(h_{m-1}^x)^2}$. With these, the relations (34) become:

$$P_{i+1}^x E_{ij} + 2E_{i+1,j} + L_{i+1}^x E_{i+2,j} = D_{i+1}^x. \tag{37}$$

Being given $A^x = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ P_2^x & 2 & L_2^x & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & P_{m-1}^x & 2 & L_{m-1}^x \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 \end{pmatrix}$,
$E_j = (E_{1j}, E_{2j}, \dots, E_{mj})^t$, $D^x = (D_1^x, D_2^x, \dots, D_m^x)^t \Rightarrow$

$$A^x E_j = D^x \Rightarrow A^x E_j (h_i^x)^2 = D^x (h_i^x)^2 \tag{38}$$

$$\begin{aligned} |(h_i^x)^2 D_{i+1}^x| &= (h_i^x)^2 \frac{6}{h_i^x + h_{i+1}^x} \left| \frac{z_{i+2,j} - z_{i+1,j}}{h_{i+1}^z} - \frac{z_{i+1,j} - z_{ij}}{h_i^x} \right| \\ &= \frac{6(h_i^x)^2}{h_i^x + h_{i+1}^x} \frac{|(z_{i+2,j} - z_{i+1,j})h_i^x - (z_{i+1,j} - z_{ij})h_{i+1}^x|}{h_i^x h_{i+1}^x} \\ &\le 6 \frac{h_i^x \overline{\Delta}_x}{h_{i+1}^x (h_i^x + h_{i+1}^x)} (|z_{i+2,j} - z_{i+1,j}| + |z_{i+1,j} - z_{ij}|) \\ &\le 6\alpha \frac{\overline{\Delta}_x}{2\underline{\Delta}_x} (|z_{i+2,j} - z_{i+1,j}| + |z_{i1,j} - z_{ij}|) \le 6\alpha^2 \omega(f, \overline{\Delta}) \\ &\Rightarrow (h_i^x)^2 (|E_{i+1,j}| + |E_{ij}|) \le 2\|A_x^{-1}\| \cdot \|(h_i^x)^2 E_j\|, \end{aligned}$$

where $\|A\| = \max\{\sum_{j=1}^n |a_{ij}|;\ i = \overline{1,n}\}$.

As A^x is a dominant diagonal matrix ($|a_{ii}| > \sum_{j=1, j\neq 1}^n |a_{ij}|, \forall i = \overline{1,n}$), it is invertible and

$$\begin{aligned} &\|A_x^{-1}\| \le \max\{(|a_{ii}| - \sum_{j=1}^n |a_{ij}|)^{-1} | i = \overline{1,n}\} = 1 \\ &\Rightarrow (h_i^x)^2 (|E_{i+1,j}|) \le 2[6\alpha^2 \omega(f, \overline{\Delta}) + (\overline{\Delta}_x)^2 (|D_1^x| + |D_m^x|)]. \end{aligned}$$

But $(\overline{\Delta}_x)^2(|D_1^x| + |D_m^x|) = 6(\overline{\Delta}_x)^2(\frac{|z_{2j}-z_{1j}|}{(h_1^x)^2} + \frac{|z_{mj}-z_{m-1,j}|}{(h_m^x)^2}) \le 6(\frac{\overline{\Delta}_x}{\underline{\Delta}_x})^2 2\omega(f, \overline{\Delta}) = |2\alpha^2 \omega(f, \overline{\Delta})| \Rightarrow (h_i^x)^2(|E_{i+1,j}| + |E_{ij}|) \le 36\alpha^2 \omega(f, \overline{\Delta})$.

In conclusion, we have:

$$|f_{ij}(x)| \leq \frac{5}{2}\omega(f,\overline{\Delta}) + \frac{4}{\sqrt{3}}\beta^2\omega(f,\overline{\Delta}). \tag{39}$$

Similarly we obtain:

$$|g_{ij}(y)| \leq \frac{5}{2}\omega(f,\overline{\Delta}) + \frac{4}{\sqrt{3}}\beta^2\omega(d,\overline{\Delta}). \tag{40}$$

From (27) $\Rightarrow \frac{\partial h_{ij}}{\partial x} = 2d_{ij}(x-x_i)(y-y_j) + \delta_{ij}(y-y_j)^2 + \varepsilon_{ij}(y-y_j)$

$\frac{\partial h_{ij}}{\partial y} = d_{ij}(x-x_i)^2 + 2\delta_{ij}(x-x_i)(y-y_i) + \varepsilon_{ij}(x-x_i)$

$\frac{\partial^2 h_{ij}}{\partial x^2} = 2d_{ij}(y-y_j)$; $\frac{\partial^2 h_{ij}}{\partial y^2} = 2\delta_{ij}(x-x_i)$; $\frac{\partial^2 h_{ij}}{\partial x \partial y} = 2d_{ij}(x-x_i) + 2\delta_{ij}(y-y_j) + \varepsilon_{ij}$

$\frac{\partial^2 s_{ij}}{\partial x \partial y}(x_i, y_j) = F_{ij} \Rightarrow \varepsilon_{ij} = F_{ij}$

$\frac{\partial^2 s_{ij}}{\partial x \partial y}(x_{i+1}, y_j) = 2d_{ij}h_i^x + \varepsilon_{ij} = F_{i+1,j} \Rightarrow d_{ij} = \frac{F_{i+1,j}-F_{ij}}{2h_i^x}$

$\frac{\partial^2 s_{ij}}{\partial x \partial y}(x_i, y_{j+1}) = 2\delta_{ij}h_j^y + \varepsilon_{ij} = F_{i,j+1} \Rightarrow \delta_{ij} = \frac{F_{i,j+1}-F_{ij}}{2h_j^y}$. So we have:

$$h_{ij}(x,y) = \frac{F_{i+1,j}-F_{ij}}{2h_i^x}(x-x_i)^2(y-y_j) + \frac{F_{i,j+1}-F_{ij}}{2h_j^y}(x-x_i)(y-y_j)^2 + F_{ij}(x-x_i)(y-y_j). \tag{41}$$

We notice that s_{ij} and all its partial derivatives of order one or two are zero in all points (x_i, y_j). So we must set supplementary conditions on the mixed partial derivatives of order three $\frac{\partial^3 s_{ij}}{\partial x^2 \partial y} = \frac{F_{i1,j}-F_{ij}}{2h_i^x}$; $\frac{F_{i,j+1}-F_{ij}}{2h_j^y}$. From the condition $\frac{\partial^3 s_{ij}}{\partial x^2 \partial y}(x_i, y_j) = \frac{\partial^3 s_{i+1,j}}{\partial x \partial y}(x_{i+1}, y_j)$, we get

$$\frac{F_{i+1,j}-F_{ij}}{h_i^x} = \frac{F_{i+2,j}-F_{i+1,j}}{h_{i+1}^x}, \forall i = \overline{1, m-2}. \tag{42}$$

If we set the conditions $F_{mj} = 0$, $\forall j = \overline{1,n}$, $F_{ni} = 0, \forall i = \overline{1,m}$ then we obtain a semi-natural two variable cubic spline function (the partial mixed derivatives of order two are zero on half of D's boundary). Consecutively replacing in (42) i with $i-1, i-2, \ldots, 2, 1$ we get:

$$\begin{aligned}
&\frac{F_{i+1,j}-F_{ij}}{h_i^x} = \frac{F_{2,j}-F_{1,j}}{h_1^x} \Rightarrow |\frac{F_{i+1,j}-F_{ij}}{2h_i^x}(x-x_i)^2(y-y_j)| \\
&= \frac{|F_{2j}|}{2h_i^x}(x-x_i)^2(y-y_j) \qquad (43)\\
&\leq \frac{1}{\underline{\Delta}_x}|F_{2j}|(\overline{\Delta}x)^2\overline{\Delta}_y = \frac{\alpha}{2}|F_{2j}| \cdot \overline{\Delta}_x\overline{\Delta}_y.
\end{aligned}$$

We write $S = \max\{|F_{2j}| \,|\, j = \overline{1,n}\}$, we obtain:

$$|\frac{F_{i+1,j}-F_{ij}}{2h_i^x}(x-x_i)^2(y-y_j)| \leq \frac{\alpha S}{2}\overline{\Delta}_x\overline{\Delta}_y \qquad (44)$$

Similarly, we get:

$$|\frac{F_{i,j+1}-F_{ij}}{2h_j^x}(x-x_i)(y-y_j)^2| \leq \frac{\beta T}{2}\overline{\Delta}_x\overline{\Delta}_y, \qquad (45)$$

where $T = \max\{|F_{i2}|;\, i = \overline{1,m}\}$. From (43) $\Rightarrow |F_{i+1,j} - F_{ij}| = \frac{h_i^x}{h_1^x}|F_{2j}|$. But $|F_{i+1,j}| - |F_{ij}| \leq |F_{i+1,j} - F_{ij}| = \frac{h_i^x}{h_1^x}|F_{2j}|$. Calculating at $i = 1, 2, \ldots, i-1$ and summing up, we obtain $|F_{i+1,j}| - |F_{1j}| \leq \frac{h_1^x + h_2^x + \ldots + h_{i-1}^x}{h_1^x}|F_{2j}|$. But $h_1^x + h_2^x + \ldots + h_{i-1}^x = x_i - a \leq b - a$. So $|F_{ij}| \leq \frac{b-a}{h_1^x}|F_{2j}| \Rightarrow |F_{ij}(x-x_i)(y-y_j)| \leq \frac{b-a}{\underline{\Delta}_x}|F_{2j}|\overline{\Delta}_x\overline{\Delta}_y$, that is:

$$|F_{ij}(x-x_i)(y-y_j)| \leq \alpha(b-a)S\overline{\Delta}_y. \qquad (46)$$

From (27), (44), (45) we obtain:

$$\begin{aligned}
|h_{ij}^{(x,y)} - f(x,y)| &\leq \frac{\alpha S}{2}\overline{\Delta}_x\overline{\Delta}_y + \frac{\beta T}{2}\overline{\Delta}_x\overline{\Delta}_y + \alpha S(b-a)\overline{\Delta}_y + |z_{ij} - f(x,y)| \\
&\leq \frac{\alpha S + \beta T}{2}\overline{\Delta}_x\overline{\Delta}_y + \alpha S(b-a)\overline{\Delta}_y + \omega(f, \overline{\Delta}). \qquad (47)
\end{aligned}$$

From (27), (39), (40) we get

$$\begin{aligned}
|s_{ij}^{(x,y)} - f(x,y)| \leq{}& \frac{5}{2}\omega(f,\overline{\Delta}) + \frac{4}{\sqrt{3}}\alpha^2\omega(f,\overline{\Delta}) + \frac{5}{2}\omega(f,\overline{\Delta}) + \frac{4}{\sqrt{3}}\beta^2\omega(f,\overline{\Delta}) \\
&+ \frac{\alpha S + \beta T}{2}\overline{\Delta}_x\overline{\Delta}_y + \alpha S(b-a)\overline{\Delta}_y + \omega(f,\overline{\Delta}),
\end{aligned}$$

that is:

$$|s_{ij}^{(x,y)} - f(x,y)| \le 6\omega(f,\overline{\Delta}) + \frac{4}{\sqrt{3}}\omega(f,\overline{\Delta})(\alpha^2+\beta^2) + \frac{\alpha S + \beta T}{2}\overline{\Delta}_x\overline{\Delta}_y + \alpha S(b-a)\overline{\Delta}_y. \tag{48}$$

If the divisions Δ_x and Δ_y are uniform, then $\overline{\Delta}_x = \frac{b-a}{m}$, $\overline{\Delta}_y = \frac{d-c}{n}$, $\alpha = \beta = 1$ from where we get:

$$|s_{ij}^{(x,y)} - f(x,y)| \le \omega(f,\overline{\Delta})(6+\frac{4}{\sqrt{3}}) + (b-a)(d-c)(\frac{S+T}{2mn}+\frac{S}{n}). \tag{49}$$

3. Conclusion

For the practical use of spline interpolation functions we have to find effective ways to analytically represent them. Out of the connection in nodes, smoothness conditions, conditions of continuity of partial derivatives of order two or three, plus the limit conditions (head of range) we obtain compatible systems out of which are determined the coefficients of spline function of two variables. This ensures the existence and uniqueness of it and allow the development of an algorithm for actually calculating the spline function of two variables. In order to obtain the assessment formula of the rest, in the range we used convex linear combinations, a fact that ensures the stability of the algorithm.

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