ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIX, 2013, f.1

A NOTE ON GENERALIZED ABSOLUTE SUMMABILITY

BY

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Abstract. In this paper, a main theorem on $|A|_k$ summability method has been proved. This theorem also includes two results.

Mathematics Subject Classification 2010: 40F05, 40D25.
Key words: absolute matrix summability, Summability factors

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n), and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots \tag{1.1}$$

The series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} |\overline{\Delta} A_n(s)|^k < \infty, \tag{1.2}$$

where $\overline{\Delta} A_n(s) = A_n(s) - A_{n-1}(s)$. When A is a Riesz matrix, the series $\sum a_n$ is said to be summable $|R, p_n|_k, k \geq 1$, if (see [4]) (1.2) holds. By a Riesz matrix we mean one such that $a_{nv} = \frac{p_v}{P_n}$ for $0 \leq v \leq n$, and $a_{nv} = 0$ for $v > n$, where (p_n) is a sequence of positive real numbers and $P_n = \sum_{v=0}^{n} p_v$ as $n \to \infty$, $(P_{-i} = p_{-i} = 0, i \geq 1)$. Also if we take $a_{nv} = \frac{1}{n}$, i.e. Cesàro matrix, the series $\sum a_n$ is said to be summable $|C, 1|_k, k \geq 1$ (see [1]).

Given any sequences $(x_n), (y_n)$, it is customary to write $y_n = O(x_n)$, if there exist η and N, for every $n > N, |\frac{y_n}{x_n}| \leq \eta$. For any matrix entry $a_{nv}, \Delta_v a_{nv} = a_{nv} - a_{n,v+1}$.

2. Known results

Given a normal matrix $A = (a_{nv})$, we may associate two lower semi-matrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni} \ , n, v = 0, 1, \ldots, \tag{2.1}$$

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}; \ \widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \ n = 1, 2, \ldots \tag{2.2}$$

It may be noted that $\overline{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i = \sum_{i=0}^{n} a_i \sum_{v=i}^{n} a_{nv} = \sum_{i=0}^{n} \overline{a}_{ni} a_i \tag{2.3}$$

$$\overline{\Delta} A_n(s) = \sum_{i=0}^{n} \overline{a}_{ni} a_i - \sum_{i=0}^{n-1} \overline{a}_{n-1,i} a_i = \overline{a}_{nn} a_n + \sum_{i=0}^{n-1} (\overline{a}_{ni} - \overline{a}_{n-1,i}) a_i$$

$$= \widehat{a}_{nn} a_n + \sum_{i=0}^{n-1} \widehat{a}_{ni} a_i = \sum_{i=0}^{n} \widehat{a}_{ni} a_i. \tag{2.4}$$

The aim of this paper is to prove the following main theorem. For this we need the following lemma.

Lemma 2.1 ([2]). *Under the conditions*

$$|\Delta \lambda_n| \leq \beta_n, \tag{2.5}$$

$$\beta_n \to 0, \ n \to \infty, \tag{2.6}$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \tag{2.7}$$

where (X_n) is a positive non-decreasing sequence, we have that

$$n \beta_n X_n = O(1), \ n \to \infty, \tag{2.8}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{2.9}$$

3. The main result

Before we state our main result, we show $A = (a_{nv})$ is said to be of class $\mathfrak{U}$ if the following hold;

A is lower triangular

$$a_{nv} \geq 0, \ n, v = 0, 1, \ldots; \tag{3.1}$$

$$a_{n-1,v} \geq a_{nv} \text{ for } n \geq v + 1; \tag{3.2}$$

$$\overline{a}_{n0} = 1, \ n = 0, 1, \ldots. \tag{3.3}$$

A given by $A_1(x) = x_1$ and $A_n(x) = \frac{x_{n-1}+x_n}{2}$ for $n > 1$ is an example of a matrix of class $\mathfrak{U}$.

Theorem 3.1. *Let* $A \in \mathfrak{U}$ *satisfying*

$$na_{nn} = O(1), \tag{3.4}$$

$$\widehat{a}_{n,v+1} = O(v|\Delta_v \widehat{a}_{nv}|) \tag{3.5}$$

and let there be sequences $(X_n), (\beta_n)$ *and* (λ_n) *such that the conditions taken in the statement of Lemma* 2.1 *be satisfied. If*

$$\sum_{n=1}^{m} a_{nn}|s_n|^k = O(X_m), \ m \to \infty, \tag{3.6}$$

$$|\lambda_n| X_n = O(1), \ n \to \infty, \tag{3.7}$$

then the series $\sum a_n \lambda_n$ *is summable* $|A|_k, k \geq 1$.

Proof. Let (T_n) be the $n-$th term of the $A-$transform of the series $\sum_{i=0}^{n} a_i \lambda_i$. Then, by means of (2.3), we have $T_n = \sum_{v=0}^{n} \overline{a}_{nv} a_v \lambda_v$. Applying Abel's transformation we have that

$$\begin{aligned} \overline{\Delta} T_n &= \sum_{v=0}^{n} \widehat{a}_{nv} a_v \lambda_v = \sum_{v=0}^{n-1} \Delta_v \widehat{a}_{nv} \lambda_v s_v + \sum_{v=0}^{n-1} \widehat{a}_{n,v+1} \Delta \lambda_v s_v + a_{nn} \lambda_n s_n \\ &= T_n(1) + T_n(2) + T_n(3), \text{ say.} \end{aligned}$$

To complete the proof of the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_n(r)|^k < \infty, \text{ for } r = 1, 2, 3. \tag{3.8}$$

Firstly, since

$$\begin{aligned}\Delta_v \widehat{a}_{nv} &= \widehat{a}_{nv} - \widehat{a}_{n,v+1} = \overline{a}_{nv} - \overline{a}_{n-1,v} - \overline{a}_{n,v+1} + \overline{a}_{n-1,v+1} \\ &= a_{nv} - a_{n-1,v} \leq 0. \end{aligned} \tag{3.9}$$

By using (3.2) and (3.3), $\sum_{v=0}^{n-1} |\Delta_v \widehat{a}_{nv}| = \sum_{v=0}^{n-1} (a_{n-1,v} - a_{nv}) = a_{nn}$. Using Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned} I_1 &= \sum_{n=1}^{m+1} n^{k-1} |T_n(1)|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{v=0}^{n-1} |\Delta_v \widehat{a}_{nv}| |\lambda_v| |s_v| \right)^k \\ &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{v=0}^{n-1} |\Delta_v \widehat{a}_{nv}| |\lambda_v|^k |s_v|^k \right) \times \left(\sum_{v=0}^{n-1} |\Delta_v \widehat{a}_{nv}| \right)^{k-1} \\ &= O(1) \sum_{n=1}^{m+1} (n a_{nn})^{k-1} \sum_{v=0}^{n-1} |\Delta_v \widehat{a}_{nv}| |\lambda_v|^k |s_v|^k = O(1) \sum_{v=0}^{m} |\lambda_v|^k |s_v|^k \sum_{n=v+1}^{m+1} |\Delta_v \widehat{a}_{nv}|. \end{aligned}$$

By (3.1) and (3.9),

$$\sum_{n=v+1}^{m+1} |\Delta_v \widehat{a}_{nv}| = \sum_{n=v+1}^{m+1} (a_{n-1,v} - a_{nv}) = a_{vv} - a_{m+1,v} \leq a_{vv}.$$

Thus, we have

$$\begin{aligned} I_1 &= O(1) \sum_{v=0}^{m} |\lambda_v|^{k-1} |\lambda_v| |s_v|^k a_{vv} = O(1) \sum_{v=0}^{m} |\lambda_v| a_{vv} |s_v|^k \\ &= O(1) \sum_{v=0}^{m-1} \Delta|\lambda_v| \sum_{i=1}^{v} a_{ii} |s_i|^k + O(1) |\lambda_m| \sum_{v=1}^{m} a_{vv} |s_v|^k \\ &= O(1) \sum_{v=0}^{m-1} |\Delta \lambda_v| X_v + O(1) |\lambda_m| X_m \\ &= O(1) \sum_{v=0}^{m-1} \beta_v X_v + O(1) |\lambda_m| X_m = O(1),\ m \to \infty \end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 2.1.

Since $n\beta_n = O(1/X_n) = O(1)$ by (2.8), we have that

$$\begin{aligned}
I_2 &= \sum_{n=1}^{m+1} n^{k-1}|T_n(2)|^k \le \sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=0}^{n-1}\widehat{a}_{n,v+1}|\Delta\lambda_v||s_v|\right)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1} v|\Delta_v\widehat{a}_{nv}|\beta_v|s_v|\right)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1}(v\beta_v)^k|\Delta_v\widehat{a}_{nv}||s_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v\widehat{a}_{nv}|\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}(v\beta_v)^k|s_v|^k\sum_{n=v+1}^{m+1}|\Delta_v\widehat{a}_{nv}| \\
&= O(1)\sum_{v=1}^{m}(v\beta_v)^{k-1}(v\beta_v)|s_v|^k a_{vv} = O(1)\sum_{v=1}^{m} v\beta_v|s_v|^k a_{vv}.
\end{aligned}$$

Applying Abel's transformation, we get that

$$\begin{aligned}
I_2 &= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{i=1}^{v} a_{ii}|s_i|^k + O(1)m\beta_m\sum_{v=1}^{m} a_{vv}|s_v|^k \\
&= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)m\beta_m X_m \\
&= O(1),\ m\to\infty
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 2.1.Finally, we have

$$\begin{aligned}
I_3 &= \sum_{n=1}^{m} n^{k-1}|T_n(3)|^k = O(1)\sum_{n=1}^{m} n^{k-1}a_{nn}^k|\lambda_n|^k|s_n|^k \\
&= O(1)\sum_{n=1}^{m} a_{nn}|\lambda_n||s_n|^k = O(1),\ m\to\infty
\end{aligned}$$

as in the proof of I_1.

Thus, we obtain (3.8). This completes the proof of the Theorem. □

4. Corollaries

Setting $a_{nv} = \frac{p_v}{P_n}$ and $a_{nv} = \frac{1}{n}$ in the Theorem 3.1, the following corollaries can be stated.

Corollary 4.1. *Let* (X_n) *be a positive non-decreasing sequence and let there be sequences* (β_n) *and* (λ_n) *such that the conditions* (2.5)-(2.7) *and* (3.7) *are satisfied. If*

$$np_n = O(P_n), \quad n \to \infty, \tag{4.1}$$

$$P_n = O(np_n), \quad n \to \infty, \tag{4.2}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |s_n|^k = O(X_m), \ m \to \infty, \tag{4.3}$$

then the series $\sum a_n \lambda_n$ *is summable* $|R, p_n|_k, k \geq 1$.

Corollary 4.2 ([3])**.** *Let* (X_n) *be a positive non-decreasing sequence and let there be sequences* (β_n) *and* (λ_n) *such that the conditions* (2.5)-(2.7) *and* (3.7) *are satisfied. If*

$$\sum_{n=1}^{m} \frac{1}{n} |s_n|^k = O(X_m), \ m \to \infty, \tag{4.4}$$

then the series $\sum a_n \lambda_n$ *is summable* $|C, 1|_k, k \geq 1$.

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Received: 7.VI.2010
Revised: 24.I.2011

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