

NEW CATEGORICAL APPROACHES TO MOVABILITY IN SHAPE THEORY

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Abstract. In the present paper we define some notions of movability for functors and natural transformations which permit to characterize the movability of inverse systems considered as functors, and the movability of the morphisms of inverse systems considered as natural transformations.

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1. Introduction

The notion of movability for metric compacta was introduced by BORSUK [1] as an important shape invariant. The movable spaces are a generalization of spaces having the shape of ANR's. The movability assumption allows a series of important results in algebraic topology (like the Whitehead and Hurewicz theorems) to remain valid when the homotopy pro-groups are replaced by the corresponding shape groups. The term "movability" comes from the geometric interpretation of the definition in the compact case: if X is a compactum lying in a space $M \in AR$, one says that X is movable if for every neighborhood U of X in M there exists a neighborhood $V \subset U$ of X such that for every neighborhood $W \subset U$ of X there is a homotopy $H : V \times [0, 1] \rightarrow U$ such that $H(x, 0) = x$ and $H(x, 1) \in W$ for every $x \in V$. One shows that the choice of $M \in AR$ is irrelevant [1]. After the notion of movability had been expressed in terms of ANR-systems, for arbitrary topological spaces, [5], [6], it became clear that one could define it

in arbitrary pro-categories. The definitions of a movable object in an arbitrary pro-category and of uniform movability were given by MOSZYŃSKA [8]. Uniform movability is important in the study of mono- and epi-morphisms in pro-categories and in the study of the shape of pointed spaces. In the book of MARDESIC and SEGAL [6] all these approaches and applications of various types of movability are discussed.

Some categorical approaches to movability in shape theory were given by GEVORGYAN [3], POP [11], GEVORGYAN and POP [4], and PAŞA [9]. In all these papers the authors consider a notion of movability (simple, uniform or strong) for an abstract category which is then applied to a comma category of an object in order to obtain the corresponding notions of movability for an object in a shape theory. In the present paper we define some notions of movability for functors and natural transformations (functor morphisms) which permit to characterize the movability of inverse systems considered as functors, and the movability of the morphisms of inverse systems (in the sense of [12]) considered as functor morphisms.

2. Movable, uniformly and strongly movable functors

Definition 2.1. Let a covariant functor $F : \mathcal{K} \rightarrow \mathcal{K}'$ be given. Then we say that:

(i) F is a movable functor if for any object $X \in \mathcal{K}$ there exist an object $M_X^F \in \mathcal{K}$ and a morphism $m_X^F \in \mathcal{K}(M_X^F, X)$, such that any morphism $p \in \mathcal{K}(Y, X)$ admits a morphism $u^F(p) \in \mathcal{K}'(F(M_X^F), F(Y))$ satisfying the relation $F(m_X^F) = F(p) \circ u^F(p)$,

$$\begin{array}{ccc} F(M_X^F) & \xrightarrow{F(m_X^F)} & F(X) \\ & \searrow u^F(p) & \nearrow F(p) \\ & & F(Y) \end{array}$$

(ii) F is a uniformly movable functor if it is movable and moreover, with the notations from (i), the following condition is satisfied: for three morphisms $p \in \mathcal{K}(Y, X)$, $q \in \mathcal{K}(Z, X)$ and $r \in \mathcal{K}(Z, Y)$ in the relation $p \circ r = q$, we have $u^F(p) = F(r) \circ u^F(q)$,

$$\begin{array}{ccc} Z & \xrightarrow{r} & Y \\ & \searrow q & \nearrow p \\ & & X \end{array}$$

\Rightarrow

$$\begin{array}{ccc}
 F(Z) & \xrightarrow{F(r)} & F(Y) \\
 & \swarrow u^F(q) & \nearrow u^F(p) \\
 & F(M_X^F) &
 \end{array}$$

(iii) F is a strongly movable functor if it is movable and moreover, with the notations from (i), for the morphism $u^F(p)$ we have $u^F(p) \circ F(\beta) = F(\alpha)$, for two morphisms $\beta : Z \rightarrow M_X^F$ and $\alpha : Z \rightarrow Y$, (and $F(m_X^F) = F(p) \circ u^F(p)$).

$$\begin{array}{ccccc}
 & & F(M_X^F) & & \\
 & \nearrow F(\beta) & \downarrow u^F(p) & \searrow F(m_X^F) & \\
 F(Z) & & & & F(X) \\
 & \searrow F(\alpha) & & \nearrow F(p) & \\
 & & F(Y) & &
 \end{array}$$

Remark 2.2. In [10] a notion of movability for a functor was introduced but that definition is not adequate for our purpose, so that we use another definitions, as above.

Example 2.3. A category \mathcal{K} is movable (uniformly movable, strongly movable) in the sense of the papers [3] (resp. [4], and [9]) if and only if the identity functor is movable (resp. uniformly movable or strongly movable) in the sense of the above definition.

Example 2.4. If a category \mathcal{K} is movable (uniformly movable or strongly movable) then any functor $F : \mathcal{K} \rightarrow \mathcal{K}'$ is a movable (resp. uniformly movable or strongly movable) functor, for an arbitrary category \mathcal{K}' .

Example 2.5. Consider $Sh_{(\mathcal{T}, \mathcal{P})}$ a shape theory, with \mathcal{P} a dense subcategory of the category \mathcal{T} (see [6], p. 25), and let $X \in \mathcal{T}$. Denote by $X_{\mathcal{P}}$ the comma category of X in \mathcal{T} over \mathcal{P} . Suppose that the sets $\mathcal{T}(X, P)$ are pointed sets, with the base points u_P^0 , such that the correspondences $P \rightarrow u_P^0, f \in \mathcal{P}(P, P') \rightarrow f \in X_{\mathcal{P}}(u_P^0, u_{P'}^0)$ define a covariant functor $F : \mathcal{P} \rightarrow X_{\mathcal{P}}$. Then, if X is a movable (uniformly movable, strongly movable) the functor F is movable (uniformly movable, strongly movable). This follows from the fact that the movability properties of the object X

are equivalent with the corresponding movability properties for the comma category $X_{\mathcal{P}}$ (cf. [3], [4], [9]) and this easily implies the corresponding movability properties in sense of Definition 2.1. In fact this implies the corresponding movability properties for the category \mathcal{P} and then we apply Example 2.4. This situation can appear in the case of some categories of pointed topological spaces or of groups, etc.

Let $\mathbf{X} = (X_{\lambda}, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in the category \mathcal{C} . We can consider this inverse system as a covariant functor (see [2], p. 23, 39). More precisely, we first consider the category \mathcal{K} defined by the directed preordered set (Λ, \leq) , which has all elements of Λ as objects, and for two such elements $\lambda, \lambda' \in \Lambda$, we put $\mathcal{K}(\lambda, \lambda') = \{(\lambda, \lambda')\}$, a set with a single element, if $\lambda \leq \lambda'$, and $\mathcal{K}(\lambda, \lambda') = \emptyset$ otherwise. Then, if \mathcal{K}^0 is the dual category of \mathcal{K} , we can define a functor $F_{\mathbf{X}} : \mathcal{K}^0 \rightarrow \mathcal{C}$, with $F_{\mathbf{X}}(\lambda) = X_{\lambda}$, for any $\lambda \in \Lambda$, and $F_{\mathbf{X}}((\lambda, \lambda')) = p_{\lambda\lambda'} : X_{\lambda'} \rightarrow X_{\lambda}$, for $\lambda \leq \lambda'$.

Theorem 2.6. *An inverse system $\mathbf{X} = (X_{\lambda}, p_{\lambda\lambda'}, \Lambda)$ in a category \mathcal{C} is movable (strongly movable, uniformly movable) if and only if the covariant functor $F_{\mathbf{X}} : \mathcal{K}^0 \rightarrow \mathcal{C}$ is movable (strongly movable, uniformly movable) in the sense of Definition 2.1.*

Proof. It is immediate that the usual conditions of movability or strongly movability for the inverse system \mathbf{X} , as in the book [6] (p. 159 and p. 226 resp.), are precisely the conditions of movability and strong movability respectively for the functor $F_{\mathbf{X}}$. As regarding the condition of uniform movability, this can also be translated from the inverse system \mathbf{X} to the functor $F_{\mathbf{X}}$ as follows. Suppose that $F_{\mathbf{X}}$ is a uniformly movable functor and let us consider an index $\lambda \in \Lambda$. Then there is an index $\lambda' \in \Lambda$, ($\lambda' = M_{\lambda}^{F_{\mathbf{X}}}$), with $\lambda' \geq \lambda$ (in order to define $m_{\lambda}^{F_{\mathbf{X}}} = (\lambda, \lambda')$), for which we consider $p_{\lambda\lambda'} : X_{\lambda'} \rightarrow X_{\lambda}$. Now we consider an arbitrary index $\nu \in \Lambda$, and we define a morphism $r_{\nu} : X_{\lambda'} \rightarrow X_{\nu}$ by $r_{\nu} = p_{\nu\mu} \circ u^{F_{\mathbf{X}}}(\lambda, \mu) : X_{\lambda'} \xrightarrow{u^{F_{\mathbf{X}}}(\lambda, \mu)} X_{\mu} \xrightarrow{p_{\nu\mu}} X_{\nu}$, for an index $\mu \in \Lambda$, with $\mu \geq \lambda', \mu \geq \nu$. Then we can see that the set of these morphisms $\{r_{\nu} | \nu \in \Lambda\}$, in the category \mathcal{C} , defines a morphism of inverse systems $\mathbf{r} : X_{\lambda'} \rightarrow \mathbf{X}$. For this, at first we observe that if $\mu \geq \lambda', \mu \geq \nu$ and $\mu' \geq \mu$, then by the condition (ii) of uniform movability we have $p_{\nu\mu} \circ u^{F_{\mathbf{X}}}(\lambda, \mu) = p_{\nu\mu} \circ p_{\mu\mu'} \circ u^{F_{\mathbf{X}}}(\lambda, \mu') = p_{\nu\mu'} \circ u^{F_{\mathbf{X}}}(\lambda, \mu')$. Then, if $\nu, \nu' \in \Lambda$ are in the relation $\nu' \geq \nu$, we can consider an index $\bar{\mu} \in \Lambda$ satisfying the conditions $\bar{\mu} \geq \lambda', \bar{\mu} \geq \nu, \bar{\mu} \geq \nu'$, and we can take

$r_\nu = p_{\nu\bar{\mu}} \circ u^{F\mathbf{X}}(\lambda, \bar{\mu})$ and $r_{\nu'} = p_{\nu'\bar{\mu}} \circ u^{F\mathbf{X}}(\lambda, \bar{\mu})$. This implies the equality $p_{\lambda\nu'} \circ r_{\nu'} = p_{\lambda\nu'} \circ p_{\nu'\bar{\mu}} \circ u^{F\mathbf{X}}(\lambda, \bar{\mu}) = p_{\lambda\bar{\mu}} \circ u^{F\mathbf{X}}(\lambda, \bar{\mu}) = r_\nu$. Therefore $\mathbf{r} = (r_\nu) : X_{\lambda'} \rightarrow \mathbf{X}$ is a morphism of inverse systems. Afterwards, since for $\nu = \lambda$ we can take $\mu = \lambda'$ and because $u^{F\mathbf{X}}(\lambda, \lambda') = 1_{X_{\lambda'}}$, we deduce that $r_\lambda = p_{\lambda\lambda'}$. Then this implies that, if $\mathbf{p}_\lambda : \mathbf{X} \rightarrow X_\lambda$ is the morphism of *pro*- \mathcal{C} given by the identity morphism 1_{X_λ} , we have $\mathbf{p}_\lambda \circ \mathbf{r} = p_{\lambda\lambda'}$.

$$\begin{array}{ccc} X_{\lambda'} & \xrightarrow{p_{\lambda\lambda'}} & X_\lambda \\ & \searrow \mathbf{r} & \nearrow \mathbf{p}_\lambda \\ & \mathbf{X} & \end{array}$$

Thus the condition of uniform movability for the inverse system \mathbf{X} is verified. Conversely, if \mathbf{X} is uniformly movable, in order to verify that the functor $F_{\mathbf{X}}$ is uniformly movable, for the index $\lambda \in \Lambda$ assign λ' as for \mathbf{X} , and if $\nu \geq \lambda$, we take $u^{F\mathbf{X}}((\lambda, \nu)) = r_\nu$. Then, since $p_{\lambda\nu} \circ r_\nu = r_\lambda$ and $r_\lambda = p_{\lambda\lambda'}$, we deduce that $p_{\lambda\nu} \circ r_\nu = p_{\lambda\lambda'}$. Moreover, with this notation if $\nu' \geq \nu \geq \lambda'$, then we have $p_{\nu\nu'} \circ u^{F\mathbf{X}}((\lambda, \nu')) = u^{F\mathbf{X}}((\lambda, \nu))$. These facts show that $F_{\mathbf{X}}$ is uniformly movable. \square

Example 2.7. If $F : \mathcal{K} \rightarrow \mathcal{K}'$ is a constant functor, i.e., there is an object $X'_0 \in \mathcal{K}'$ such that $F(X) = X'_0$, for any object $X \in \mathcal{K}$, and $F(f) = 1_{X'_0}$, for any morphism f in \mathcal{K} , then F is a uniformly and strongly movable functor.

Example 2.8. Let $F : \mathcal{K} \rightarrow \mathcal{K}'$ be a movable (uniformly movable or strongly movable) functor. Then for any covariant functor $G : \mathcal{K}' \rightarrow \mathcal{K}''$, the composition functor $G \circ F : \mathcal{K} \rightarrow \mathcal{K}''$ is a movable (resp. uniformly movable or strongly movable) functor. This is a generalization of Example 2.4. Particularly, if $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is a movable (uniformly movable, strongly movable) inverse system in a category \mathcal{C} , then for an arbitrary covariant functor $G : \mathcal{C} \rightarrow \mathcal{D}$, the inverse system $G(\mathbf{X}) = (G(X_\lambda), G(p_{\lambda\lambda'}), \Lambda)$ has the same movability properties. This result can be used as a necessary condition of movability (uniform or strong movability). For example, if $(X, *)$ is a pointed space, and if for a HPol_* -expansion $\mathbf{p} = (p_\lambda) : (X, *) \rightarrow (\mathbf{X}, *) = ((X_\lambda, *), p_{\lambda\lambda'}, \Lambda)$, the inverse system of pointed sets $\pi_0(\mathbf{X}, *) = (\pi_0(X_\lambda, *), (p_{\lambda\lambda'})_0, \Lambda)$, or one of the inverse system of groups $\pi_k(\mathbf{X}, *) = (\pi_k(X_\lambda, *), (p_{\lambda\lambda'})_k, \Lambda)$, $k \geq 1$, is not movable (uniformly movable, strongly movable), then $(X, *)$ is not a movable (uniformly movable, strongly movable) pointed space.

Example 2.9. We recall that a groupoid is a small category in which every morphism is an isomorphism. For \mathcal{K}' a groupoid, every covariant functor $F : \mathcal{K} \rightarrow \mathcal{K}'$ is movable, uniformly and strongly movable. Indeed, for an object $X \in \mathcal{K}$ and some morphisms $p \in \mathcal{K}(Z, X), q \in \mathcal{K}(Z, X)$, with $q = p \circ r$, we take $M_X^F = X, m_X^F = 1_X, u^F(p) = F(p)^{-1}, u^F(q) = F(q)^{-1}$, and then we have $F(p) \circ u^F(p) = F(m_X^F), u^F(p) = F(r) \circ u^F(q)$, and for $\beta = 1_X, \alpha = p, u^F(p) \circ F(\beta) = F(\alpha)$.

Definition 2.10. Let $F : \mathcal{F} \rightarrow \mathcal{C}$ and $G : \mathcal{G} \rightarrow \mathcal{C}$ be two covariant functors with the same codomain. A generalized natural transformation from the functor F to the functor G consists of a covariant functor $\Phi : \mathcal{G} \rightarrow \mathcal{F}$ and of a natural transformation $\varphi : F\Phi \rightarrow G$.

Example 2.11. If $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu,\mu'}, M)$ are two inverse systems, then a generalized natural transformation from the functor $F_{\mathbf{X}}$ to the functor $F_{\mathbf{Y}}$ is a morphism $(f_\mu, \phi) : \mathbf{X} \rightarrow \mathbf{Y}$ of inverse systems in the usual sense, but with $\phi : (M, \leq) \rightarrow (\Lambda, \leq)$ an increasing function, and such that if $\mu \leq \mu'$ in M , then

$$(2.1) \quad f_\mu \circ p_{\phi(\mu)\phi(\mu')} = q_{\mu\mu'} \circ f_{\mu'}.$$

We will call such a morphism of inverse systems a "natural" morphism of inverse systems. Conversely, if (f_μ, ϕ) is a "natural" morphism of inverse systems from \mathbf{X} to \mathbf{Y} , for example if (M, \leq) is a cofinite directed set (see [6], Ch. I, §1.2, Lemma 2), then this can be considered as a generalized natural transformation from $F_{\mathbf{X}}$ to the functor $F_{\mathbf{Y}}$.

Remark 2.12. By Theorem 3 from [6], Ch. I, §1.3, the hypothesis of "natural" morphisms is not too restrictive.

Remark 2.13. If $(\Phi, \varphi) : F \rightarrow G$ and $(\Psi, \psi) : G \rightarrow H$ are two generalized natural transformations, then there exists the composition $(\Phi\Psi, \chi)$, where for an object Z from the domain \mathcal{H} of H , $\chi_Z = \psi_Z \circ \varphi_{\Psi(Z)}$.

Theorem 2.14. Let $F : \mathcal{F} \rightarrow \mathcal{C}, G : \mathcal{G} \rightarrow \mathcal{C}$ be functors with the same codomain, such that F dominates G by the generalized natural transformations. Then if F is a movable (uniformly movable, strongly movable) functor, so is G .

Proof. Suppose that the generalized natural transformations $(\Phi, \varphi) : F \rightarrow G$ and $(\Psi, \psi) : G \rightarrow F$ satisfy the domination conditions $\Psi \circ \Phi =$

$1_{\mathcal{G}}$, and, for any $X \in \mathcal{G}$, $\varphi_X \psi_{\Phi(X)} = 1_{G(X)}$. Suppose that F is movable. For an object $X \in \mathcal{G}$, consider $\Phi(X) \in \mathcal{F}$ and for this, the object $M_{\Phi(X)}^F$ and the morphism $m_{\Phi(X)}^F : M_{\Phi(X)}^F \rightarrow \Phi(X)$. Then we take $M_X^G := \Psi(M_{\Phi(X)}^F)$ and $m_X^G := \Psi(m_{\Phi(X)}^F) : M_X^G \rightarrow X$. Then if $p \in \mathcal{G}(Y, X)$, we have $\Phi(p) \in \mathcal{F}(\Phi(Y), \Phi(X))$ and we consider $u^F(\Phi(p)) : F(M_{\Phi(X)}^F) \rightarrow F(\Phi(Y))$ in \mathcal{C} , with $F(m_{\Phi(X)}^F) = F(\Phi(p))u^F(\Phi(p))$. Now we can define $u^G(p) : G(\Psi(M_{\Phi(X)}^F)) \rightarrow G(Y)$ by putting $u^G(p) = \varphi_Y \circ u^F(\Phi(p)) \circ \psi_{M_{\Phi(X)}^F}$.

$$\begin{array}{ccccc}
 & & F(M_{\Phi(X)}^F) & & \\
 & \swarrow u^F(\Phi(p)) & \uparrow & \searrow F(m_{\Phi(X)}^F) & \\
 F(\Phi(Y)) & \xrightarrow{F(\Phi(p))} & F(\Phi(X)) & & \\
 \downarrow \varphi_Y & & \downarrow \Psi_{M_{\Phi(X)}^F} & & \downarrow \varphi_X \\
 & & G(\Psi(M_{\Phi(X)}^F)) & & \\
 & \swarrow u^G(p) & & \searrow G(\Psi(m_{\Phi(X)}^F)) & \\
 G(Y) & \xrightarrow{G(p)} & G(Y) & &
 \end{array}$$

For this morphism we have $G(p) \circ u^G(p) = G(p) \circ \varphi_Y \circ u^F(\Phi(p)) \circ \psi_{M_{\Phi(X)}^F} = \varphi_X \circ F(\Phi(p)) \circ u^F(\Phi(p)) \circ \psi_{M_{\Phi(X)}^F} = \varphi_X \circ F(m_{\Phi(X)}^F) \circ \psi_{M_{\Phi(X)}^F} = \varphi_X \circ \Psi_{\Phi(X)} \circ G(\Psi(m_{\Phi(X)}^F)) = G(\Psi(m_{\Phi(X)}^F)) = G(\Psi(m_{\Phi(X)}^F))$. This shows that G is a movable functor.

Suppose now that F is uniformly movable and that the morphisms $p \in \mathcal{G}(Y, X)$, $q \in \mathcal{G}(Z, X)$ and $r \in \mathcal{G}(Z, Y)$ satisfy the relation $p \circ r = q$. This implies the relation $\Phi(p) \circ \Phi(r) = \Phi(q)$, and by hypothesis we can write $u^F(\Phi(p)) = F(\Phi(r)) \circ u^F(\Phi(q))$. Then by the above notations we have $u^G(p) = \varphi_Y \circ u^F(\Phi(p)) \circ \psi_{M_{\Phi(X)}^F}$, $u^G(q) = \varphi_Z \circ u^F(\Phi(q)) \circ \psi_{M_{\Phi(X)}^F}$. Therefore, $G(r) \circ u^G(q) = G(r) \circ \varphi_Z \circ u^F(\Phi(q)) \circ \psi_{M_{\Phi(X)}^F} = \varphi_Y \circ F(\Phi(r)) \circ u^F(\Phi(q)) \circ \psi_{M_{\Phi(X)}^F} = \varphi_Y \circ u^F(\Phi(p)) \circ \psi_{M_{\Phi(X)}^F} = u^G(p)$, and this shows that G is uniformly movable.

Finally suppose that F is strongly movable. For $X, Y \in \mathcal{G}$ and $p \in \mathcal{G}(Y, X)$ as in the first part of the proof, we consider $u^G(p)$ and for this

we have $G(p) \circ u^G(p) = G(m_X^G)$. Then by hypothesis there exist an object $Z' \in \mathcal{F}$ and some morphisms $\alpha' : Z' \rightarrow \Phi(Y)$, $\beta' : Z' \rightarrow M_{\Phi(X)}^F$ in \mathcal{F} , with $F(\alpha') = u^F(\Phi(p)) \circ F(\beta')$. We take $Z = \Psi(Z')$, $\alpha = \Psi(\alpha') : Z \rightarrow Y$ and $\beta = \Psi(\beta') : Z \rightarrow \Psi(M_{\Phi(X)}^F) = M_X^G$. With these notation, we have $u^G(p) \circ G(\beta) = \varphi_Y \circ u^F(\Phi(p)) \circ \psi_{M_{\Phi(X)}^F} \circ G(\Psi(\beta')) = \varphi_Y \circ u^F(\Phi(p)) \circ F(\beta') \circ \psi_{Z'} = \varphi_Y \circ F(\alpha') \circ \psi_{Z'} = \varphi_Y \circ \psi_{\Phi(Y)} \circ G(\Psi(\alpha')) = G(\Psi(\alpha')) = G(\alpha)$. So, G is a strongly movable functor. \square

Corollary 2.15. *Let \mathbf{X} and \mathbf{Y} be inverse systems in a category \mathcal{C} . If \mathbf{Y} is dominated by \mathbf{X} in the category $\text{inv} - \mathcal{C}$ by "natural" morphisms, and \mathbf{X} is a movable (uniformly movable, strongly movable) system, then so is \mathbf{Y} .*

Remark 2.16. It is well known that the result of Corollary 2.15 is more general: if $\mathbf{Y} \leq \mathbf{X}$ in $\text{pro} - \mathcal{C}$, and if \mathbf{X} is movable (uniformly, strongly movable) then so is \mathbf{Y} (cf. [6], Ch. II, §6.1, Theorem 2).

Corollary 2.17. *A functor $F : \mathcal{F} \rightarrow \mathcal{C}$ equivalent by a generalized isomorphism of functors with a movable (uniformly movable, strongly movable) functor $G : \mathcal{G} \rightarrow \mathcal{C}$, has itself the same movability property.*

Corollary 2.18. *If a functor $F : \mathcal{F} \rightarrow \mathcal{C}$ is dominated (by the generalized morphisms of functors) by a constant functor $G : \mathcal{G} \rightarrow \mathcal{C}$, then F is uniformly movable and strongly movable.*

Definition 2.19. Let $F : \mathcal{K} \rightarrow \mathcal{K}'$ be a covariant functor and \mathcal{K}'_0 a subcategory of \mathcal{K}' . We say that F is \mathcal{K}'_0 -movable if for any object $X \in \mathcal{K}$ there exists an object $M_X^{F, \mathcal{K}'_0} \in \mathcal{K}$ and a morphism $m_X^{F, \mathcal{K}'_0} \in \mathcal{K}(M_X^{F, \mathcal{K}'_0}, X)$ such that for any morphism $p \in \mathcal{K}(Y, X)$ and any object $X'_0 \in \mathcal{K}'_0$ and any morphism $h \in \mathcal{K}'(X'_0, F(M_X^{F, \mathcal{K}'_0}))$, there exists a morphism $u^{F, \mathcal{K}'_0}(h, p) \in \mathcal{K}'(X'_0, F(Y))$ satisfying the relation $F(m_X^{F, \mathcal{K}'_0}) \circ h = F(p) \circ u^{F, \mathcal{K}'_0}(h, p)$,

$$\begin{array}{ccc}
 & F(M_X^{F, \mathcal{K}'_0}) & \\
 h \nearrow & & \searrow F(m_X^{F, \mathcal{K}'_0}) \\
 X'_0 & & F(X) \\
 u^{F, \mathcal{K}'_0}(h, p) \searrow & & \nearrow F(p) \\
 & F(Y) &
 \end{array}$$

If in addition, for three morphisms $p \in K(Y, X)$, $q \in K(Z, X)$ and $r \in$

$K(Z, Y)$ satisfying the relation $p \circ r = q$, and an arbitrary morphism $h \in \mathcal{K}'(X'_0, F(M_X^{F, \mathcal{K}'_0}))$ we have $u^{F, \mathcal{K}'_0}(h, p) = F(r) \circ u^{F, \mathcal{K}'_0}(h, q)$, we say that the functor F is \mathcal{K}'_0 -uniformly movable.

Remark 2.20. Clearly, the movability of the functor F implies \mathcal{K}'_0 -movability of F , for every subcategory \mathcal{K}'_0 of the category \mathcal{K} , since we can take $M_X^{F, \mathcal{K}'_0} = M_X^F$, $m_X^{F, \mathcal{K}'_0} = m_X^F$, $u^{F, \mathcal{K}'_0}(p) = u^F(p) \circ h$. If $\mathcal{K}'_0 = \mathcal{K}'$ then, conversely, \mathcal{K}'_0 -movability of the functor F implies the movability of F . Indeed, we take $M_X^F = M_X^{F, \mathcal{K}'}$, $m_X^F = m_X^{F, \mathcal{K}'}$, and for $p \in \mathcal{K}(Y, X)$, $u^F(p) = u^{F, \mathcal{K}'}(1_{M_X^{F, \mathcal{K}'}}(p))$. The statement also holds for the uniform movability.

Example 2.21. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system in a category \mathcal{C} and let \mathcal{C}_0 be a subcategory of \mathcal{C} . The inverse system \mathbf{X} is \mathcal{C}_0 -movable (\mathcal{C}_0 -uniformly movable) in the sense of [6], p.164-165, if and only if the functor $F_{\mathbf{X}} : \mathcal{K}^0 \rightarrow \mathcal{C}$ (see Theorem 2.6) is \mathcal{C}_0 -movable (\mathcal{C}_0 -uniformly movable) in the sense of Definition 2.19. Particularly an inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in the category \mathbf{HPol} is n -movable in the sense of [6], p. 198, if and only if the functor $F_{\mathbf{X}} : \mathcal{K}^0 \rightarrow \mathbf{HPol}$ is \mathbf{HPol}_n -movable.

3. Movable natural transformations

Taking into consideration that the notions of movability as defined above for a functor lead to the usual properties of movability for inverse systems (see Theorem 2.6, Examples 2.11 and 2.21, Corollary 2.15), in order to introduce some notions of movability for morphisms of inverse systems (in an inv-category or a pro-category), it is natural to define some appropriate notions of movability for natural transformation between functors.

Definition 3.1. Let $F, G : \mathcal{K} \rightarrow \mathcal{K}'$ be covariant functors and let $\phi : F \rightarrow G$ be a natural transformation. Then we say that:

(i) ϕ is a movable natural transformation if for every object $X \in \mathcal{K}$ there exists an object $M_X^\phi \in \mathcal{K}$ and a morphism $m_X^\phi \in \mathcal{K}(M_X^\phi, X)$ such that any morphism $p \in \mathcal{K}(Y, X)$ admits a morphism $u^\phi(p) \in \mathcal{K}'(F(M_X^\phi), G(Y))$ such that $G(m_X^\phi) \circ \phi_{M_X^\phi} = G(p) \circ u^\phi(p)$, i.e., the following diagram commutes

$$\begin{array}{ccc}
 G(M_X^\phi) & \xrightarrow{G(m_X^\phi)} & G(X) \\
 \phi_{M_X^\phi} \uparrow & & \uparrow G(p) \\
 F(M_X^\phi) & \xrightarrow{u^\phi(p)} & G(Y)
 \end{array}$$

(ii) ϕ is a strongly movable natural transformation if it is movable and moreover, with the notations from (i), for the morphism $u^\phi(p)$ we have $u^\phi(p) \circ F(\beta) = \phi_Y \circ F(\alpha)$, for two morphisms $\beta : Z \rightarrow M_X^\phi$ and $\alpha : Z \rightarrow Y$.

(iii) ϕ is a uniformly movable natural transformation if it is movable and moreover, with the notations from (i), the following condition is satisfied: for three morphisms $p \in \mathcal{K}(Y, X)$, $q \in \mathcal{K}(Z, X)$ and $r \in \mathcal{K}(Z, Y)$ in the relation $p \circ r = q$, we have $u^\phi(p) = G(r) \circ u^\phi(q)$.

Example 3.2. Let $F, G : \mathcal{K} \rightarrow \mathcal{K}'$ be covariant functors and let $\phi : F \rightarrow G$ be a natural transformation. If G is a (uniformly, strongly) movable functor, then ϕ is a (uniformly, strongly) movable natural transformation: we can take $M_X^\phi = M_X^G$, $m_X^\phi = m_X^G$ and $u^\phi(p) = u^G(p) \circ \phi_{M_X^\phi}$. Particularly, if G is a constant functor, then any natural transformation $\phi : F \rightarrow G$ is uniformly and strongly movable.

First we recall from [12] the following definitions.

Definition 3.3 ([12]). Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ be inverse systems (in a category \mathcal{C}) over the same directed set (Λ, \leq) , and $(f_\lambda) : \mathbf{X} \rightarrow \mathbf{Y}$ be a level morphism in the category $inv - \mathcal{C}$. Then we say that:

(a) (f_λ) is a movable level morphism if every $\lambda \in \Lambda$ admits a $\lambda' \geq \lambda$ (called a *movability index of λ relative to (f_λ)*) such that each $\lambda'' \geq \lambda$ admits a morphism $r : X_{\lambda'} \rightarrow Y_{\lambda''}$ of \mathcal{C} which satisfies

$$(3.1) \quad q_{\lambda\lambda'} \circ f_{\lambda'} = q_{\lambda\lambda''} \circ r,$$

i.e., it makes the following diagram commutative

$$\begin{array}{ccc} Y_{\lambda'} & \xrightarrow{q_{\lambda\lambda'}} & Y_\lambda \\ f_{\lambda'} \uparrow & & \uparrow q_{\lambda\lambda''} \\ X_{\lambda'} & \xrightarrow{r} & Y_{\lambda''} \end{array}$$

(b) (f_λ) is a strongly movable level morphism if every $\lambda \in \Lambda$ admits a $\lambda' \geq \lambda$ (called a *strong movability index of λ relative to (f_λ)*) such that for every $\lambda'' \geq \lambda$ there exist a $\lambda^* \geq \lambda', \lambda''$ and a morphism $r : X_{\lambda'} \rightarrow Y_{\lambda''}$ satisfying

$$(3.2) \quad q_{\lambda\lambda'} \circ f_{\lambda'} = q_{\lambda\lambda''} \circ r,$$

$$(3.3) \quad r \circ p_{\lambda'\lambda^*} = q_{\lambda'\lambda^*} \circ f_{\lambda^*}.$$

(c) (f_λ) is uniformly movable if every $\lambda \in \Lambda$ admits a $\lambda' \in \Lambda, \lambda' \geq \lambda$ (a *uniform movability index of λ relative to (f_λ)*) such that there is a morphism of inverse systems $\mathbf{r} : X_{\lambda'} \rightarrow \mathbf{Y}$ satisfying

$$(3.4) \quad q_{\lambda\lambda'} \circ f_{\lambda'} = \mathbf{q}_\lambda \circ \mathbf{r}$$

i.e., making the following diagram commutative

$$\begin{array}{ccc} Y_{\lambda'} & \xrightarrow{q_{\lambda\lambda'}} & Y_\lambda \\ f_{\lambda'} \uparrow & & \uparrow \mathbf{q}_\lambda \\ X_{\lambda'} & \xrightarrow{\mathbf{r}} & \mathbf{Y} \end{array}$$

where $\mathbf{q}_\lambda : \mathbf{Y} \rightarrow Y_\lambda$ is the morphism of *inv* - \mathcal{C} given by $1_{Y_\lambda} : Y_\lambda \rightarrow Y_\lambda$, i.e., \mathbf{q}_λ is the restriction of \mathbf{Y} to Y_λ .

Theorem 3.4. *Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ be two inverse systems in a category \mathcal{C} indexed by the same set of indices, (Λ, \leq) , and let $(f_\lambda) : \mathbf{X} \rightarrow \mathbf{Y}$ be level morphism of inverse systems. Consider the associated functors $F_{\mathbf{X}}, F_{\mathbf{Y}} : \mathcal{K}^0 \rightarrow \mathcal{C}$ (see Theorem 2.6) and the natural transformation $\phi_{(f_\lambda)} : F_{\mathbf{X}} \rightarrow F_{\mathbf{Y}}$ defined by (f_λ) (see Example 2.11). Then the morphism (f_λ) is movable (uniformly movable, strongly movable) in the sense of Definition 3.3 if and only if $\phi_{(f_\lambda)}$ is movable (uniformly movable, strongly movable) in the sense of Definition 3.1.*

Proof. The cases of simple movability and strongly movability are obtained by simple transcription of conditions (i)-(a) and (ii)-(b) in the two definitions. For the case of uniformly movability we give details.

Suppose that $\phi := \phi_{(f_\lambda)} : F_{\mathbf{X}} \rightarrow F_{\mathbf{Y}}$ is a uniformly movable natural transformation, and let us consider an index $\lambda \in \Lambda$. Then there is an index $\lambda' \in \Lambda, \lambda' = M_\lambda^\phi$, with $\lambda' \geq \lambda$ (in order to define $m_\lambda^\phi = (\lambda, \lambda')$), for which we consider $q_{\lambda\lambda'} : Y_{\lambda'} \rightarrow Y_\lambda$. Now we consider an arbitrary index $\nu \in \Lambda$, and we define a morphism $r_\nu : X_{\lambda'} \rightarrow Y_\nu$ by $r_\nu = q_{\nu\mu} \circ u^\phi(\lambda, \mu) : X_{\lambda'} \xrightarrow{u^\phi(\lambda, \mu)} Y_\mu \xrightarrow{q_{\nu\mu}} Y_\nu$, for an index $\mu \in \Lambda$, with $\mu \geq \lambda', \mu \geq \nu$. Then we can see that the set of these morphisms $\{r_\nu | \nu \in \Lambda\}$, in the category \mathcal{C} , defines a morphism of inverse systems $\mathbf{r} : X_{\lambda'} \rightarrow \mathbf{Y}$. For this, at first

we observe that if $\mu \geq \lambda', \mu \geq \nu$ and $\mu' \geq \mu$, then by the second part of condition (iii) of uniform movability of natural transformation ϕ we have $q_{\nu\mu} \circ u^\phi(\lambda, \mu) = q_{\nu\mu} \circ q_{\mu\mu'} \circ u^\phi(\lambda, \mu') = q_{\nu\mu'} \circ u^\phi(\lambda, \mu')$. Then, if $\nu, \nu' \in \Lambda$ are in the relation $\nu' \geq \nu$, we can consider an index $\bar{\mu} \in \Lambda$ satisfying the conditions $\bar{\mu} \geq \lambda', \bar{\mu} \geq \nu, \bar{\mu} \geq \nu'$, and we can take $r_\nu = q_{\nu\bar{\mu}} \circ u^\phi(\lambda, \bar{\mu})$ and $r_{\nu'} = q_{\nu'\bar{\mu}} \circ u^\phi(\lambda, \bar{\mu})$. This implies the equality $q_{\lambda\nu'} \circ r_{\nu'} = q_{\lambda\nu'} \circ q_{\nu'\bar{\mu}} \circ u^\phi(\lambda, \bar{\mu}) = q_{\lambda\bar{\mu}} \circ u^\phi(\lambda, \bar{\mu}) = r_\nu$. Therefore $\mathbf{r} = (r_\nu) : X_{\lambda'} \rightarrow \mathbf{Y}$ is a morphism of inverse systems. Afterwards, since for $\nu = \lambda$ we can take $\mu = \lambda'$ and because $u^\phi(\lambda, \lambda') = f_{\lambda'}$, we deduce that $r_\lambda = q_{\lambda\lambda'} \circ f_{\lambda'} = f_\lambda \circ p_{\lambda\lambda'}$. Then this implies that, if $\mathbf{q}_\lambda : \mathbf{Y} \rightarrow Y_\lambda$ is the morphism of $pro - \mathcal{C}$ given by the identity morphism 1_{Y_λ} , we have $q_{\lambda\lambda'} \circ f_{\lambda'} = \mathbf{q}_\lambda \circ \mathbf{r}$. Thus condition (c) from Definition 3.3 of the uniform movability of the level morphism (f_λ) is verified.

Conversely, if (f_λ) is uniformly movable, in order to verify that the natural transformation $\phi = \phi_{(f_\lambda)} : F_{\mathbf{X}} \rightarrow F_{\mathbf{Y}}$ is uniformly movable, for an index $\lambda \in \Lambda$ assign $\lambda' \geq \lambda$ as for (f_λ) , and if $\nu \geq \lambda$, we take $u^\phi((\lambda, \nu)) = r_\nu$. Then, since $r_\lambda = q_{\lambda\lambda'} \circ f_{\lambda'}$ and $q_{\lambda\nu} \circ r_\nu = r_\lambda$, we deduce that $q_{\lambda\nu} \circ r_\nu = q_{\lambda\lambda'} \circ f_{\lambda'}$. that is $q_{\lambda\lambda'} \circ f_{\lambda'} = q_{\lambda\nu} \circ u^\phi((\lambda, \nu))$. Moreover, with this notation, if $\nu' \geq \nu \geq \lambda'$, then we have $u^\phi((\lambda, \nu)) = q_{\nu\nu'} \circ u^\phi((\lambda, \nu'))$ or $u^\phi((\lambda, \nu)) = F_{\mathbf{Y}}((\nu, \nu')) \circ u^\phi((\lambda, \nu'))$. These facts show that $\phi_{(f_\lambda)}$ is uniformly movable in the sense of Definition 3.1, (iii). \square

Remark 3.5. In [12] it is proved that if $(f_\lambda), (g_\lambda) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow \mathbf{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ are two equivalent level morphisms, and (f_λ) is (uniformly, strongly) movable in the sense of Definition 3.3, then so is (g_λ) . This result allows to consider some corresponding properties of movability in a pro-category for the morphisms which can be represented by level morphisms of inverse systems.

Remark 3.6. A functor F is (uniformly, strongly) movable if and only if the identity natural morphism 1_F is movable. Particularly, an inverse system \mathbf{X} in a category \mathcal{C} is (uniformly, strongly) movable if and only if the identity morphism $1_{\mathbf{X}}$ is movable in the sense of our definition.

Example 3.7. Let $F, G, H : \mathcal{K} \rightarrow \mathcal{K}'$ be three covariant functors and $\phi : F \rightarrow G, \psi : G \rightarrow H$ natural transformations. If ψ is movable (uniformly movable, strongly movable), then the composition natural transformation $\psi \circ \phi : F \rightarrow H$ is movable (strongly movable, uniformly movable). Particularly, if $(f_\lambda) : \mathbf{X} \rightarrow \mathbf{Y}$ and $(g_\lambda) : \mathbf{Y} \rightarrow \mathbf{Z}$ are two level morphisms of inverse

systems, with (g_λ) movable (uniformly movable, strongly movable) then the composition $(g_\lambda f_\lambda) : \mathbf{X} \rightarrow \mathbf{Z}$ has the corresponding movability property.

Example 3.8. Let $F, G : \mathcal{K} \rightarrow \mathcal{K}'$ be covariant functors and let $\phi : F \rightarrow G$ be a movable (strongly movable, uniformly movable) natural transformation. Then, for an arbitrary covariant functor $H : \mathcal{K}' \rightarrow \mathcal{K}''$, the natural transformation $H(\phi) : H \circ F \rightarrow H \circ G$ has the corresponding movability property. Particularly, if $(f_\lambda) : \mathbf{X} \rightarrow \mathbf{Y}$ is a level morphism of inverse systems in a category \mathcal{C} , and $H : \mathcal{C} \rightarrow \mathcal{D}$ is a covariant functor, then the level morphism of inverse systems $(H(f_\lambda)) : H(\mathbf{X}) \rightarrow H(\mathbf{Y})$ preserves the movability properties of (f_λ) .

4. Movable generalized natural transformations

Definition 4.1. Let $F : \mathcal{F} \rightarrow \mathcal{C}$, $G : \mathcal{G} \rightarrow \mathcal{C}$ be covariant functors with the same codomain and let $(\Phi, \varphi) : F \rightarrow G$ be a generalized natural transformation. Then we say that (Φ, φ) is movable (uniformly movable, strongly movable) if the following conditions are satisfied:

(I) The natural transformation $\varphi : F\Phi \rightarrow G$ is movable (uniformly movable, strongly movable) in the sense of Definition 3.1,

(II) The functor Φ satisfies the following property: every morphism $f : X \rightarrow \Phi(Y)$ in \mathcal{F} , with $X \in \mathcal{F}$, $Y \in \mathcal{G}$, admits a morphism $g : Y' \rightarrow Y$ in \mathcal{G} and a morphism $h : \Phi(Y') \rightarrow X$ in \mathcal{F} , such that $f \circ h = \Phi(g)$, i.e., the following diagram commutes

$$\begin{array}{ccc}
 X & \xrightarrow{f} & \Phi(Y) \\
 & \swarrow h & \nearrow \Phi(g) \\
 & \Phi(Y') &
 \end{array}$$

Remark 4.2. If $\mathcal{F} = \mathcal{G}$ and $\Phi = 1_{\mathcal{F}}$, then condition (II) is trivially verified and we recover the definition of movability for natural transformations. More generally, condition (II) is verified if the functor Φ is full and surjective on the objects of \mathcal{F} .

Theorem 4.3. Let $(f_\mu, \varphi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be a "natural" morphism of inverse systems in a category \mathcal{C} (see definition in Example 2.11). Then the generalized natural transformation defined by (f_μ, φ) , from $F_{\mathbf{X}}$ to $F_{\mathbf{Y}}$, is movable in the sense of Definition 4.1 if and only the following conditions are satisfied.

(i) Every $\mu \in M$ admits a $\mu' \geq \mu$ (called a movability index of μ relative to (f_μ, φ)) such that each $\mu'' \geq \mu$ admits a morphism $r_{\mu''} : X_{\varphi(\mu')} \rightarrow Y_{\mu''}$ in \mathcal{C} which satisfies

$$(4.1) \quad q_{\mu\mu'} \circ f_{\mu'} = q_{\mu\mu''} \circ r_{\mu''},$$

i.e., makes the following diagram commutative

$$\begin{array}{ccc} Y_{\mu'} & \xrightarrow{q_{\mu\mu'}} & Y_{\mu} \\ f_{\mu'} \uparrow & & \uparrow q_{\mu\mu''} \\ X_{\varphi(\mu')} & \xrightarrow{r_{\mu''}} & Y_{\mu''} \end{array}$$

(ii) Every pair $(\lambda, \mu) \in \Lambda \times M$, with $\lambda \geq \varphi(\mu)$, admits an index $\mu' \in M$, $\mu' \geq \mu$, satisfying $\varphi(\mu') \geq \lambda$.

Proof. Denote $F_{\mathbf{X}} : (\Lambda, \leq)^0 \rightarrow \mathcal{C}$, $F_{\mathbf{Y}} : (M, \leq)^0 \rightarrow \mathcal{C}$, the functor $\Phi : (M, \leq)^0 \rightarrow (\Lambda, \leq)^0$, with $\Phi(\mu) = \varphi(\mu)$, $\Phi((\mu, \mu')) = (\varphi(\mu), \varphi(\mu'))$, for $\mu, \mu' \in M, \mu \leq \mu'$, and the natural transformation $\phi : F_{\mathbf{X}} \circ \Phi \rightarrow F_{\mathbf{Y}}$, with $\phi_\mu = f_\mu : F_{\mathbf{X}}(\Phi(\mu)) = X_{\varphi(\mu)} \rightarrow F_{\mathbf{Y}}(\mu) = Y_\mu, \forall \mu \in M$. Now the condition of movability for the natural transformation ϕ is equivalent with condition (i) (see the proof of Theorem 3.4). The second condition from Definition 4.1 refers to the functor Φ , and in our case this means the following: a morphism from $\lambda \in \Lambda$ to $\varphi(\mu)$, $\mu \in M$, in the category $(\Lambda, \leq)^0$ implies $\lambda \geq \varphi(\mu)$. And this must involve the existence of a morphism in $(M, \leq)^0$ with the target μ , i.e., a $\mu' \in M, \mu' \geq \mu$, and a morphism from $\varphi(\mu')$ to λ , i.e., $\varphi(\mu') \geq \lambda$. Therefore, condition (ii) is equivalent with condition (II) from Definition 4.1. \square

Using this model of proof and the results of Theorem 3.4, one can prove the following theorem.

Theorem 4.4. Suppose that $(f_\mu, \varphi): \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is a "natural" morphism of inverse systems in a category \mathcal{C} satisfying condition (ii) from Theorem 4.3.

(a) The generalized natural transformation defined by (f_μ, φ) , from $F_{\mathbf{X}}$ to $F_{\mathbf{Y}}$, is uniformly movable in the sense of Definition 4.1 if and only if every $\mu \in M$ admits a $\mu' \in M, \mu' \geq \mu$ and a morphism of inverse systems $\mathbf{r} : X_{\varphi(\mu')} \rightarrow \mathbf{Y}$ satisfying

$$(4.2) \quad q_{\mu\mu'} \circ f_{\mu'} = \mathbf{q}_\mu \circ \mathbf{r},$$

$$\begin{array}{ccc}
 Y_{\mu'} & \xrightarrow{q_{\mu\mu'}} & Y_{\mu} \\
 f_{\mu'} \uparrow & & \uparrow \mathbf{q}_{\lambda} \\
 X_{\varphi(\mu')} & \xrightarrow{r} & \mathbf{Y}
 \end{array}$$

where $\mathbf{q}_{\lambda} : \mathbf{Y} \rightarrow Y_{\lambda}$ is the morphism of $\text{inv} - \mathcal{C}$ given by $1_{Y_{\lambda}} : Y_{\lambda} \rightarrow Y_{\lambda}$, i.e., \mathbf{q}_{λ} is the restriction of \mathbf{Y} to Y_{λ} .

(b) The generalized natural transformation defined by (f_{μ}, φ) , from $F_{\mathbf{X}}$ to $F_{\mathbf{Y}}$, is strongly movable in the sense of Definition 4.1 if and only if every $\mu \in M$ admits a $\mu' \geq \mu$ such that for every $\mu'' \geq \mu$ there exist a $\mu^* \geq \mu', \mu''$ and a morphism $r : X_{\varphi(\mu')} \rightarrow Y_{\mu''}$ satisfying

$$(4.3) \quad q_{\mu\mu'} \circ f_{\mu'} = q_{\mu\mu''} \circ r$$

$$(4.4) \quad r \circ p_{\varphi(\mu')\varphi(\mu^*)} = q_{\mu''\mu^*} \circ f_{\mu^*}$$

$$\begin{array}{ccccc}
 X_{\varphi(\mu^*)} & \xrightarrow{p_{\varphi(\mu')\varphi(\mu^*)}} & X_{\varphi(\mu')} & \xrightarrow{f_{\mu'}} & Y_{\mu'} \\
 f_{\mu^*} \downarrow & & \downarrow r & & \downarrow q_{\mu\mu'} \\
 Y_{\mu^*} & \xrightarrow{q_{\mu''\mu^*}} & Y_{\mu''} & \xrightarrow{q_{\mu\mu''}} & Y_{\mu}
 \end{array}$$

Remark 4.5. If \mathbf{Y} is a movable system, then condition (i) is satisfied for any (natural) morphism $(f_{\mu}, \varphi) : \mathbf{X} \rightarrow \mathbf{Y}$.

Remark 4.6. Condition (II) from Definition 4.1 translated as condition (ii) from Theorem 4.3 in the case of morphisms of inverse was used in [12] to extend the movability properties from the 'natural' and general morphisms of inverse systems to pro-morphisms. We can call such a condition a *pre-movability condition*. And to give a topological example, let us consider Morita's shape theory [7]. In this context the condition of pre-movability for the morphism of inverse systems induced by a continuous map $f : X \rightarrow Y$ is the following: if \mathcal{U} is a locally finite normal open covering of X and \mathcal{V} is a locally finite normal open covering of Y such that \mathcal{U} is a refinement of $f^{-1}(\mathcal{V})$, $\mathcal{U} \preceq f^{-1}(\mathcal{V})$, then there exists a locally finite normal open covering \mathcal{V}' of Y satisfying the conditions $\mathcal{V}' \preceq \mathcal{V}$ and $f^{-1}(\mathcal{V}') \preceq \mathcal{U}$. We also make the remark that such a morphism is a 'natural' morphism of inverse systems.

Example 4.7. Let $(f_\mu) : X \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be a morphism of inverse systems from a rudimentary inverse system X . Condition (ii) from Theorem 4.3 is trivially satisfied. Condition (i) from Theorem 4.3 is also satisfied since for every index $\mu \in M$ a movable index relative to (f_μ) is itself. Therefore we have that (f_μ) is always a movable morphism. It is also strongly movable and uniformly movable.

Example 4.8. Let $(f_\mu, \varphi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be a morphism of inverse systems with $\varphi : M \rightarrow \Lambda$ a constant function, $\varphi(\mu) = \lambda_0$, for any index $\mu \in M$. For this morphism condition (ii) from Theorem 4.3 is satisfied only if Λ contains a single element, i.e., \mathbf{X} is a rudimentary system. But as we have seen in Example 4.7 we can have a movable morphism of inverse systems $(f_\mu, \varphi) : \mathbf{X} \rightarrow \mathbf{Y}$ without \mathbf{Y} being movable. Consider for example the morphism of inverse sequences of groups $(f_n) : 0 \rightarrow \mathbf{G}$, with $\mathbf{G} = (G_n, p_{nn+1}, \mathbf{N})$, where p_{nn+1} is multiplication by 2. This is a movable morphism, but \mathbf{G} is a non-movable inverse system of groups (see [6], Ch. I, §6.1, Example 1).

Example 4.9. Let $(f, \varphi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow Y$, be a morphism of inverse systems in a rudimentary system Y , and with $f : X_{\lambda_0} \rightarrow Y$. Then condition (ii) from Theorem 4.3 is satisfied if and only if λ_0 is a maximal element in (Λ, \leq) . Condition (i) is trivially verified, and thus such a morphism is movable.

Example 4.10. Let $F : \mathcal{F} \rightarrow \mathcal{C}$, $G : \mathcal{G} \rightarrow \mathcal{C}$ be covariant functors, and let $(\Phi, \varphi) : F \rightarrow G$ be a generalized natural transformation. Then for an arbitrary functor $K : \mathcal{C} \rightarrow \mathcal{D}$ the generalized natural transformation $(\Phi, K(\varphi)) : K \circ F \rightarrow K \circ G$ preserves the movability properties of (Φ, φ) . This fact results from Example 3.8 and because the functor Φ is preserved, condition (II) from Definition 4.1 does not change. Particularly, if $(f_\mu, \varphi) : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \rightarrow \mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is a 'natural' morphism of inverse systems in a category \mathcal{C} , and if $K : \mathcal{C} \rightarrow \mathcal{D}$ is an arbitrary covariant functor, then the 'natural' morphism of inverse systems $(K(f_\mu), \varphi) : K(\mathbf{X}) \rightarrow K(\mathbf{Y})$, in the category \mathcal{D} , preserves the movability properties of (f_μ, φ) . For example, a 'natural' morphism of inverse systems of pointed topological spaces is movable only if the unpointed morphism is movable.

Remark 4.11. Example 3.7 can also be stated for generalized natural transformations. If $F : \mathcal{F} \rightarrow \mathcal{C}$, $G : \mathcal{G} \rightarrow \mathcal{C}$, $H : \mathcal{H} \rightarrow \mathcal{C}$ are covariant

functors and $(\Phi, \varphi) : F \rightarrow G$, $(\Psi, \psi) : G \rightarrow H$ are generalized natural transformation, then the composition $(\Phi \circ \Psi, \chi) : F \rightarrow H$, with $\chi_Z = \psi_Z \circ \varphi_{\Psi(Z)}$, $Z \in \mathcal{H}$, preserves the movability properties of (Ψ, ψ) if the functor $\Phi \circ \Psi$ satisfies condition (II) from Definition 4.1. Particularly if $(f_\mu, \varphi) : \mathbf{X} \rightarrow \mathbf{Y}$ and $(g_\nu, \psi) : \mathbf{Y} \rightarrow \mathbf{Z}$ are 'natural' morphisms of inverse systems such that $\varphi \circ \psi$ verifies condition (ii) from Theorem 4.3, then the composition morphism $(g_\nu \circ f_{\psi(\nu)}, \varphi \circ \psi) : \mathbf{X} \rightarrow \mathbf{Z}$ preserves the movability properties of (g_ν, ψ) .

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