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# On uniform topological algebras

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**Abstract** The uniform norm on a uniform normed $Q$-algebra is the only uniform $Q$-algebra norm on it. The uniform norm on a regular uniform normed $Q$-algebra with unit is the only uniform norm on it. Let $A$ be a uniform topological algebra whose spectrum $M(A)$ is equicontinuous, then $A$ is a uniform normed algebra. Let $A$ be a regular semisimple commutative Banach algebra, then every algebra norm on $A$ is a $Q$-algebra norm on $A$.

**Keywords** uniform topological algebra · regular algebra · uniform seminorm

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## 1 Preliminaries

A topological algebra is an algebra (over the complex field) which is also a Hausdorff topological vector space such that the multiplication is separately continuous. A locally convex algebra is a topological algebra whose topology is locally convex. A uniform seminorm on an algebra $A$ is a seminorm $p$ satisfying $(i)$ : $p(x^2) = p(x)^2$, for all $x \in A$. It is shown in [3], that a seminorm satisfying $(i)$ is submultiplicative. A uniform topological algebra (uT-algebra) is a topological algebra whose topology is determined by a family of uniform seminorms. A uniform normed algebra is a normed algebra $(A, \|.\|)$ such that $\|x^2\| = \|x\|^2$, for all $x \in A$. A topological algebra is a $Q$-algebra if the set of quasi-invertible elements is open. A norm $\|.\|$ on an algebra $A$ is said to be an algebra norm (resp. $Q$-algebra norm) if $(A, \|.\|)$ is a normed algebra (resp. normed $Q$-algebra). Let $A$ be an algebra, if $x \in A$ we denote by $sp_A(x)$ the spectrum of $x$ and by $r_A(x)$ the spectral radius of $x$. An algebra $A$ is spectrally bounded if for each $x \in A, sp_A(x)$ is bounded. For a topological algebra $A$; $M(A)$ denotes the set of all nonzero continuous multiplicative linear functionals on $A, M(A)$ is endowed with the weak topology induced by the topological dual of $A$. For an arbitrary topological algebra, $M(A)$ may be empty. A topological algebra is regular if given a closed subset

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$F \subset M(A)$ and $f_1 \in M(A), f_1 \notin F$, there exists an $x \in A$ such that $\widehat{x}_{|F} = 0$ and $\widehat{x}(f_1) \neq 0; \widehat{x} : M(A) \to C, \widehat{x}(g) = g(x)$, is the Gelfand transform of $x$. Let $(B, \|.\|)$ be a uniform Banach algebra, $r_B(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|$, for all $x \in B; B$ is commutative by [4, Corollary 1], hence $\|x\| = \sup\{|f(x)|, f \in M(B)\}$, for all $x \in B$. Let $A$ be a $uT$-algebra; the completion of $A$ is a $uT$-algebra and then an inverse limit of uniform Banach algebras, thus $A$ is commutative.

## 2 Results

**Theorem 2.1** *Let $(A, \|.\|)$ be a uniform normed $Q$-algebra. If $|.|$ is a uniform norm on $A$, then $|x| \leq \|x\|$, for all $x \in A$.*

*Proof.* By [7, Proposition 7.5], $M(A, \|.\|)$ (resp. $M(A, |.|)$) is topologically isomorphic to $M(\tilde{A}, \|.\|)$ (resp. $M(\tilde{A}, |.|)), (\tilde{A}, \|.\|)$ and $(\tilde{A}, |.|)$ are respectively the completions of $(A, \|.\|)$ and $(A, |.|).(\tilde{A}, \|.\|)$ and $(\tilde{A}, |.|)$ are uniform Banach algebras, then for $x \in A$, we have $\|x\| = \sup\{|f(x)|, f \in M(\tilde{A}, \|.\|)\} = \sup\{|f(x)|, f \in M(A, \|.\|)\}$ and $|x| = \sup\{|f(x)|, f \in M(\tilde{A}, |.|)\} = \sup\{|f(x)|, f \in M(A, |.|)\}$, hence $|x| \leq \|x\|$ since $M(A, |.|) \subset M(A, \|.\|)$; the last inclusion is due to the fact that $(A, \|.\|)$ is a $Q$-algebra. □

**Corollary 2.2** *Let $(A, \|.\|)$ be a uniform normed $Q$-algebra, then $\|.\|$ is the unique uniform $Q$-algebra norm on $A$.*

**Theorem 2.3** *Let $(A, \|.\|)$ be a regular uniform normed $Q$-algebra with unit. If $|.|$ is a uniform norm on $A$, then $\|.\| = |.|$.*

*Proof.* Let $x \in A, |x| = \sup\{|f(x)|, f \in M(\tilde{A}, |.|)\} = \sup\{|f(x)|, f \in M(A, |.|)\}$. $M(A, |.|)$ is topologically isomorphic to $M(\tilde{A}, |.|)$, then $M(A, |.|)$ is compact. Put $T = M(A, |.|), T$ is a closed subset of $M(A, \|.\|)$. Suppose that $T \neq M(A, \|.\|)$, let $f_1 \in M(A, \|.\|)$ such that $f_1 \notin T$. Since $(A, \|.\|)$ is regular, there exists $x \in A$ such that $\widehat{x}_{|T} = 0$ and $\widehat{x}(f_1) \neq 0$, hence $|x| = 0$ with $x \neq 0$, a contradiction. For $x \in A, \|x\| = \sup\{|f(x)|, f \in M(\tilde{A}, \|.\|)\} = \sup\{|f(x)|, f \in M(A, \|.\|)\} = \sup\{|f(x)|, f \in M(A, |.|)\} = |x|$. □

**Theorem 2.4** *Let $A$ be a $uT$-algebra whose spectrum $M(A)$ is equicontinuous. Then $A$ is a uniform normed algebra.*

*Proof.* The topology of $A$ is determined by a family $\{p_u, u \in U\}$ of uniform seminorms. For each $u \in U$, let $N_u = \{x \in A, p_u(x) = 0\}$ and let $A_u$ be the Banach algebra obtained by completing $A/N_u$ in the norm $\|x_u\|_u = p_u(x), x_u = x + N_u$, it is clear that $A_u$ is a uniform Banach algebra. For each $u \in U$, let $M_u(A) = \{f \in M(A), |f(x)| \leq p_u(x)$, for all $x \in A\}$. Let $u \in U$ and $x \in A, p_u(x) = \|x_u\|_u = \sup\{|g(x_u)|, g \in M(A_u)\} = \sup\{|f(x)|, f \in M_u(A)\}$ by [7,Proposition 7.5]. Let $q(x) = \sup\{|f(x)|, f \in M(A)\}, x \in A$. Since $M(A)$ is equicontinuous, it follows that $q$ is a continuous seminorm on $A$. Let $x \in A$ and suppose that $q(x) = 0$, since $A$ is Hausdorff and $p_u(x) \leq q(x) = 0$, for all $u \in U$, it follows that $x = 0$. q is a continuous uniform norm on $A$, also $p_u(x) \leq q(x)$, for all $u \in U$ and $x \in A$, then the topology of $A$ can be defined by the uniform norm $q$. □

**Corollary 2.5** *Let $A$ be a $uT$-algebra that is a $Q$-algebra. Then $A$ is a uniform normed algebra.*

*Proof.* By [6, Proposition II.7.1], every topological $Q$-algebra has an equicontinuous spectrum. □

**Corollary 2.6** *Let $A$ be a spectrally bounded, barrelled, $uT$-algebra. Then $A$ is a uniform normed algebra.*

*Proof.* $\sup\{|f(x)|, f \in M(A)\} \leq r_A(x)$, for all $x \in A.M(A)$ is bounded for the weak topology, then $M(A)$ is equicontinuous. □

**Remark 2.1** These results show that completeness is not necessary for some results of S. J. Bhatt on uniform topological algebras (see [1, Theorem 1(ii)], [1, the second affirmation in the corollary], [1, Theorem 2] and [2, Corollary 2.5].

**Theorem 2.7** *Let $A$ be a topological algebra. If $p$ is a uniform seminorm on $A$, then $p(x) \leq r_A(x)$, for all $x \in A$. If $A$ is a $Q$-algebra, then every uniform seminorm on $A$ is continuous.*

*Proof.* Let $N_p = \{x \in A, p(x) = 0\}$ and $A_p$ be the Banach algebra obtained by completing $A/N_p$ in the norm $\|x_p\|_p = p(x), x_p = x + N_p.\|.\|_p$ is a uniform norm on $A_p$ since $p$ is a uniform seminorm on $A$. Consider $G : A \to A_p, G(x) = x_p$, the quotient map. Let $x \in A, p(x) = \|x_p\|_p = r_{A_p}(x_p) \leq r_A(x)$. If $A$ is a $Q$-algebra, $r_A$ is continuous at 0 by [6, Lemma II.4.2], then $p$ is continuous. □

**Remark 2.2** Theorem 2.7 shows that the hypothesis "$A$ is a unital locally convex algebra and the inversion in $A$ is continuous" is not necessary in [2, Theorem 2.3].

Now we generalize the first affirmation of [1, Corollary].

**Theorem 2.8** *Let $(A, \|.\|)$ be a regular semisimple commutative Banach algebra. Then every algebra norm on $A$ is a $Q$-algebra norm on $A$.*

*Proof.* (1) $A$ is unital: Let $|.|$ be an algebra norm on $A$. Put $K_1 = M(A, |.|)$ and $K = M(A, \|.\|), K_1 \subset K, K_1$ is homeomorphic to $M(\tilde{A}, |.|)((\tilde{A}, |.|)$ is the completion of $(A, |.|))$, then $K_1$ is compact, hence $K_1$ is closed in $K$. Suppose that $K_1 \neq K.K\backslash K_1$ is a nonempty open set, there exists an open set $G$ in $K$ such that $\bar{G} \subset K\backslash K_1$ since $(A, \|.\|)$ is regular. Also by regularity of $(A, \|.\|)$ [5, Corollary 7.3.4], there exist $x \in A, y \in A, y \neq 0$, such that $\widehat{x}_{|K_1} = 1, \widehat{x}_{|\bar{G}} = 0$ and $\widehat{y}_{|K\backslash G} = 0$. $x$ is invertible in $(\tilde{A}, |.|)$ since $\widehat{x}_{|K_1} = 1$. $\widehat{xy} = \widehat{x}\widehat{y} = 0$ on $K$, then $xy \in Rad(A) = \{0\}$, hence $y = x^{-1}xy = 0$, a contradiction. Finally, $K_1 = K$. Let $x \in A, r_A(x) = \sup\{|f(x)|, f \in K\} = \sup\{|f(x)|, f \in K_1\} \leq |x|$, then $(A, |.|)$ is a $Q$-algebra by [8, Lemma 2.1].

(2) $A$ is not unital: Let $A_1$ be the algebra obtained from $A$ by adjunction of an identity $e$. The elements of $A_1$ have the form $x + re$, where $x \in A$ and $r \in C$. $A_1$ is also regular and semisimple. Let $p$ be an algebra norm on $A$. For $x \in A$ and $r \in C$, put $p_1(x + re) = p(x) + |r|$, it is easy to show that $p_1$ is an algebra norm on $A_1$. By (1) and [8, Lemma 2.1], $r_{A_1}(x + re) \leq p_1(x + re)$, for all $x \in A$ and $r \in C$, consequently $r_A(x) = r_{A_1}(x) \leq p_1(x) = p(x)$ for all $x \in A$. Then $p$ is a $Q$-algebra norm on $A$ by [8, Lemma 2.1]. □

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