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A characterization of Finsler supermanifolds which are Riemannian supermanifolds

Esmaeil Azizpour

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Abstract Let $(\mathcal{M}, \mathcal{F})$ be a Finsler supermanifold. We introduce a linear Finsler superconnection which is h-supersymmetric but not metrical. The curvature supertensor of this connection has two components, hh-curvature component or Riemannian curvature supertensor and hv-curvature component or non-Riemannian quantity. We show that the hv-curvature tensor characterizes the Riemannian supermanifolds among Finsler supermanifolds, in other words, we show that the Finsler supermetric $\mathcal{F}$ is Riemannian if and only if the coefficients of the $hv-$curvature components of the Finsler superconnection can be expressed in terms of $A^a_{bc,d}$.

Keywords Berwald-type superconnection · Euler-Lagrange supervector field · Finsler superconnection · Finsler supermanifold · superspray

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1 Introduction

The first model of a Finsler superspace has been proposed by Vacaru [13]. He has given a general definition of locally anisotropic superspaces and formulated the theory of tangent superbundles provided with nonlinear and distinguished connections and metric structures. Such superbundles contain as particular cases the supersymmetric extensions and various prolongations of Riemann, Finsler and Lagrange spaces (see also [2], [3]).

The concept of a connection plays an important role in geometry. In a Finsler space, the curvature tensor of a linear connection gives us all non-Riemann information about this space. Examples of such connections were proposed by Berwald [4], Chern [6] and most important of all, is Elie Cartan's connection [5]. The Chern connection coincide with the Rund connection, as pointed out by Anastasiei [1]. It is torsion-free but is not completely compatible with the inner product (on π^*TM) defined by the g_{ij}'s. Shen is the person who worked on connection theory and introduced linear

Esmaeil Azizpour
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Guilan
PO Box 1914, Rasht, IRAN
E-mail: eazizpour@guilan.ac.ir

connections in Finsler geometry to characterize Riemannian manifolds among Finsler manifolds [12].

In this paper, with the help of the Berwald-type superconnection, we introduce a linear superconnection as a Finsler connection such that its curvature tensor has two components R and P. The R-term is the so-called Riemannian curvature supertensor which is a natural extension of the usual Riemann curvature supertensor of Riemannian supermetrics, while the P-term is a purely non-Riemannian quantity. The non-Riemannian quantity has a one to one correspondence with supertensor A^a_{bc}. We show that the non-Riemannian quantity (the hv-curvature tensor) characterizes the Riemannian supermanifolds among Finsler supermanifolds.

2 Preliminaries

In this section, we are going to give a brief description of the concepts of tangent supermanifolds and nonlinear superconnections.

The basic structure for building up supermanifolds is the Grassmann algebra. For each positive integer L, B will denote the Grassmann algebra over the reals with generators $1, \beta_1, ..., \beta_L$ and relations

$$1.\beta_i = \beta_i.1 = \beta_i, \quad i = 1, ..., L, \quad \beta_i.\beta_j = -\beta_j.\beta_i, \quad i, j = 1, ..., L.$$

B is a graded algebra which can be written as a direct sum $B = (B)_0 + (B)_1$ where $(B)_0$ and $(B)_1$ are the even and odd parts of (B) respectively (see [9], [8]). If the elements $A, A' \in B$ are homogeneous, then $AA' \in (B)_{|A|+|A'|}$, $AA' = (-1)^{|A||A'|}A'A$, where $|A|$ denotes the parity $(= 0, 1)$ of value A. Given the Grassmann algebra B, the corresponding $(m, n)-$dimensional superspace is defined to be the space

$$B^{m,n} = \underbrace{B_0 \times ... \times B_0}_{m\ copies} \times \underbrace{B_1 \times ... \times B_1}_{n\ copies}$$

with m is said to be the even dimension and n the odd dimension of the superspace.

Throughout this paper, $\mathcal{M}$ will denote an (m, n)-dimensional supermanifold (see [7], [11]). A vector superbundle $\mathcal{E}$ over a supermanifold $\mathcal{M}$ with the total superspace E, standard fiber V and surjective projection $\pi_E : E \mapsto M$ is defined as in the case of ordinary manifolds. The idea of construction is by using the change of coordinate functions of the supermanifold and transition functions of the superbundle. Particular example of this construction is the supermanifold $T\mathcal{M}$. The tangent superbundle $T\mathcal{M}$ over a supermanifold $\mathcal{M}$ is constructed in usual manner. If $\mathcal{M}$ has dimension (m, n) and coordinates $(x; \eta)$, then $T\mathcal{M}$ has dimension $(2m, 2n)$ and local coordinates $(x, y; \eta, \theta)$ which change according to the role

$$x'_i = x'_i(x_1, ..., x_m; \eta_1, ..., \eta_n), \quad \eta'_\alpha = \eta'_\alpha(x_1, ..., x_m; \eta_1, ..., \eta_n), \tag{2.1}$$

$$y'_j = \sum_{i=1}^{m} \frac{\partial x'_j}{\partial x_i} y_i + \sum_{\alpha=1}^{n} \frac{\partial x'_j}{\partial \eta_\alpha} \theta_\alpha, \quad \theta'_\beta = \sum_{i=1}^{m} \frac{\partial \eta'_\beta}{\partial x_i} y_i + \sum_{\alpha=1}^{n} \frac{\partial \eta'_\beta}{\partial \eta_\alpha} \theta_\alpha. \tag{2.2}$$

A vector field X on an open subset U of the supermanifold $\mathcal{M}$, is said to be even $(|X| = 0)$ if X_p is even for each point p in U, and odd $(|X| = 1)$ if each X_p is odd. For $i = 1, ..., m$ and $\alpha = 1, ..., n$, each $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial \eta_\alpha}$ is a vector field on U.

We use "a", "b", "c", ... as an index for our supertensors. Then the index "a" (and similarly for "b", "c") is $i = 1, ..., m$ and $\alpha = 1, ..., n$ where $dim\mathcal{M} = (m, n)$. For example, in index notation, we write g_{ab} instead of the coefficients of the supertensor g defined in (3.1). If H is a homogeneous geometric object, then $|H|$ denotes the parity $(= 0, 1)$ of values H. Also, we use another notation $|a|$ which is defined as bellow:

$|a| = 0$, if $a = i$, where $i = 1, ..., m$ and $|a| = 1$, if $a = \alpha$, where $\alpha = 1, ..., n$.

For each pair of vector fields X and Y on a supermanifold $\mathcal{M}$, the super commutator $[X, Y]$ is also a vector field defined by $[X, Y] = XY - (-1)^{|X||Y|}YX$.

Now consider a linear superconnection ∇ on $T\mathcal{M}$. The torsion, T, and the curvature, R, of ∇ are defined by,

$$T(X, Y) = \nabla_X Y - (-1)^{|X||Y|}\nabla_Y X - [X, Y],$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - (-1)^{|X||Y|}\nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

where $X, Y, Z \in \mathcal{X}(T\mathcal{M})$.

In the local coordinate system $(x, y; \eta, \theta)$ in $T\mathcal{M}$, the module of vertical vector fields on $T\mathcal{M}$, $\mathcal{X}^v(T\mathcal{M})$, is generated by

$$\left\{\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \theta_\alpha}, \quad i = 1, ..., m, \quad \alpha = 1, ..., n\right\}.$$

Definition 2.1 A morphism $h : \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$ is said to be a horizontal superendomorphism on $\mathcal{M}$ if it satisfies the following conditions:

(i) $h^2 = h$;

(ii) $Kerh = \mathcal{X}^v(T\mathcal{M})$.

Let h be a horizontal superendomorphism. We define $\mathcal{X}^h(T\mathcal{M}) := Imh$. Then the superendomorphism v defined by $v := (id - h) : \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$, is the vertical projection on $\mathcal{X}^v(T\mathcal{M})$ along $\mathcal{X}^h(T\mathcal{M})$ and we have $\mathcal{X}(T\mathcal{M}) = \mathcal{X}^h(T\mathcal{M}) \oplus \mathcal{X}^v(T\mathcal{M})$. They satisfy $v^2 = v$, $hv = vh = 0$. The set $\mathcal{X}^h(T\mathcal{M})$ is called the supermodule of horizontal supervector fields.

Let h be a horizontal superendomorphism, then it induces a nonlinear connection N with local coefficients $N_i^j(x, y, \eta, \theta), N_i^\beta(x, y, \eta, \theta), N_\alpha^j(x, y, \eta, \theta), N_\alpha^\beta(x, y, \eta, \theta)$, such that a local basis adapted (see [10], [13], [14]) to the given nonlinear connection N is introduced by $(\frac{\delta}{\delta x_i}, \frac{\delta}{\delta \eta_\alpha}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial \theta_\alpha})$, where

$$\begin{cases} \dfrac{\delta}{\delta x_i} := \dfrac{\partial}{\partial x_i} - N_i^j \dfrac{\partial}{\partial y_j} - N_i^\alpha \dfrac{\partial}{\partial \theta_\alpha}, \\ \dfrac{\delta}{\delta \eta_\alpha} := \dfrac{\partial}{\partial \eta_\alpha} - N_\alpha^i \dfrac{\partial}{\partial y_i} - N_\alpha^\beta \dfrac{\partial}{\partial \theta_\beta}. \end{cases} \tag{2.3}$$

Definition 2.2 Let $J : \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$ be a ninlpotent vertical superendomorphism i.e. satisfying $hJ = 0$, $J^2 = 0$. A ninlpotent morphism $\theta : \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$ is said to be the adjoint structure if $\theta o J = h, J o\theta = v$ and $\theta^2 = 0$. With respect to a alocal basis (2.3), their matrics in block form are

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & a^{-1} \\ 0 & 0 \end{pmatrix}.$$

It follows that $Jv = 0$, $\theta h = 0$, $h\theta = \theta$, $\theta v = \theta$, $v\theta = 0$.

3 Finsler supermanifolds

Definition 3.1 The function $\mathcal{F} : T\mathcal{M} \to B$ is called a *Finsler metric* (see [13]) if the following conditions are satisfied:

(1) The restriction of $\mathcal{F}$ to $T\tilde{\mathcal{M}} = T\mathcal{M} \setminus \{0\}$ is of the class G^∞ and $\mathcal{F}$ is only continuous on the image of the null cross–section in the tangent supermanifold to M.

(2) $\mathcal{F}(x, \lambda y; \eta, \lambda\theta) = \lambda\mathcal{F}(x, y; \eta, \theta)$, where λ is a real positive number.

(3) The restriction of F to the even subspace of $T\tilde{\mathcal{M}}$ is a positive function.

(4) The matrix $g = \begin{bmatrix} g_{ij} & g_{i\beta} \\ g_{\alpha j} & g_{\alpha\beta} \end{bmatrix}$ defined by

$$g_{ij} = \frac{1}{2}\frac{\partial^2\mathcal{F}^2}{\partial y_i \partial y_j},\ g_{i\beta} = \frac{1}{2}\frac{\partial^2\mathcal{F}^2}{\partial y_i \partial \theta_\beta},\ g_{\alpha j} = \frac{1}{2}\frac{\partial^2\mathcal{F}^2}{\partial \theta_\alpha \partial y_j},\ g_{\alpha\beta} = \frac{1}{2}\frac{\partial^2\mathcal{F}^2}{\partial \theta_\alpha \partial \theta_\beta} \tag{3.1}$$

is invertible. A pair $(\mathcal{M}, \mathcal{F})$ is called a Finsler supermanifold.

A Finsler supermanifold $(\mathcal{M}, \mathcal{F}(x, y; \eta, \theta))$ is called reducible to a Riemannian supermanifold if its fundamental supertensor field g does not depend on the directional variables y_i and θ_α.

The Finsler metric F induces a supermetric g, called the fundamental supertensor, with coefficients defined in (3.1).

A Finsler superconnection [13], in superbundle $\mathcal{E}$ is a linear superconnection ∇ on $\mathcal{E}$ which preserves J and θ, i.e., for each $X, Y \in \mathcal{X}(T\mathcal{M})$ and $f \in G^\infty(\mathcal{M})$,

$$\nabla_X(fY) = X(f)Y + (-1)^{|f||X|} f\nabla_X Y,\ \nabla_X JY = J\nabla_X Y,\ \nabla_X \theta Y = \theta\nabla_X Y.$$

Now we are going to introduce the Berwald-type superconnection. With the help of the Berwald-type superconnection, we can introduce a Finsler superconnection which is used to characterize the Riemannian supermanifolds among Finsler supermanifolds. As in general case, any horizontal superendomorphism gives rise to a special Finsler superconnection, called Berwald-type superconnection (see Theorem 3.2).

Theorem 3.2 *Let h be a horizontal superendomorphism. The superconnection D : $\mathcal{X}(T\mathcal{M}) \times \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$ defined by*

$$D_X Y = v[hX, vY] + h[vX, hY] + J[vX, \theta Y] + \theta[hX, JY] \tag{3.2}$$

is a linear Finsler connection.

Proof. Let f be a superfunction on $T\mathcal{M}$ and $X, Y \in \mathcal{X}(T\mathcal{M})$ two supervector fields. Then

$$\begin{aligned} D_X fY = {} & hX(f)v(vY) + (-1)^{|f||X|} fv[hX, vY] \\ & + vX(f)h(hY) + (-1)^{|f||X|} fh[vX, hY] \\ & + vX(f)J(\theta Y) + (-1)^{|f||X|} fJ[vX, \theta Y] \\ & + hX(f)\theta(JY) + (-1)^{|f||X|} f\theta[hX, JY]. \end{aligned}$$

Since $\theta o J = h$ and $J o \theta = v$, thus we have $D_X fY = X(f)Y + (-1)^{|f||X|} D_X Y$. This means that D is a linear superconnection. Also,

$$D_X(JY) = v[hX, JY] + h[vX, hY] + J[vX, hY] = JD_X Y.$$

By using the same method, we can show that $D_X(\theta Y) = \theta D_X Y$. So D is a Finsler superconnection. □

In a local supercoordinate system $(x_i, y_i; \eta_\alpha, \theta_\alpha)$, the local coefficients of the above operator D are specified by

$$\begin{cases} D_{\frac{\delta}{\delta x_i}} \dfrac{\partial}{\partial y_j} = \dfrac{\partial N_i^k}{\partial y_j}\dfrac{\partial}{\partial y_k} + \dfrac{\partial N_i^\alpha}{\partial y_j}\dfrac{\partial}{\partial \theta_\alpha},\ D_{\frac{\partial}{\partial y_i}} \dfrac{\partial}{\partial y_j} = 0, \\ D_{\frac{\delta}{\delta x_i}} \dfrac{\partial}{\partial \theta_\alpha} = \dfrac{\partial N_i^k}{\partial \theta_\alpha}\dfrac{\partial}{\partial y_k} + \dfrac{\partial N_i^\beta}{\partial \theta_\alpha}\dfrac{\partial}{\partial \theta_\beta},\ D_{\frac{\partial}{\partial y_i}} \dfrac{\partial}{\partial \theta_\alpha} = 0, \\ D_{\frac{\delta}{\delta \eta_\beta}} \dfrac{\partial}{\partial y_i} = \dfrac{\partial N_\beta^k}{\partial y_i}\dfrac{\partial}{\partial y_k} + \dfrac{\partial N_\beta^\alpha}{\partial y_i}\dfrac{\partial}{\partial \theta_\alpha},\ D_{\frac{\partial}{\partial \theta_\beta}} \dfrac{\partial}{\partial y_j} = 0, \\ D_{\frac{\delta}{\delta \eta_\beta}} \dfrac{\partial}{\partial \theta_\alpha} = \dfrac{\partial N_\beta^k}{\partial \theta_\alpha}\dfrac{\partial}{\partial y_k} + \dfrac{\partial N_\beta^\gamma}{\partial \theta_\alpha}\dfrac{\partial}{\partial \theta_\gamma},\ D_{\frac{\partial}{\partial \theta_\beta}} \dfrac{\partial}{\partial \theta_\alpha} = 0, \\ D_{\frac{\delta}{\delta x_i}} \dfrac{\delta}{\delta x_j} = \dfrac{\partial N_i^k}{\partial y_j}\dfrac{\delta}{\delta x_k} + \dfrac{\partial N_i^\alpha}{\partial y_j}\dfrac{\delta}{\delta \eta_\alpha},\ D_{\frac{\partial}{\partial y_i}} \dfrac{\delta}{\delta x_j} = 0, \\ D_{\frac{\delta}{\delta x_i}} \dfrac{\delta}{\delta \eta_\alpha} = \dfrac{\partial N_i^k}{\partial \theta_\alpha}\dfrac{\delta}{\delta x_k} + \dfrac{\partial N_i^\beta}{\partial \theta_\alpha}\dfrac{\delta}{\delta \eta_\beta},\ D_{\frac{\partial}{\partial y_i}} \dfrac{\delta}{\delta \eta_\alpha} = 0, \\ D_{\frac{\delta}{\delta \eta_\beta}} \dfrac{\delta}{\delta x_i} = \dfrac{\partial N_\beta^k}{\partial y_i}\dfrac{\delta}{\delta x_k} + \dfrac{\partial N_\beta^\alpha}{\partial y_i}\dfrac{\delta}{\delta \eta_\alpha},\ D_{\frac{\partial}{\partial \theta_\beta}} \dfrac{\delta}{\delta x_j} = 0, \\ D_{\frac{\delta}{\delta \eta_\beta}} \dfrac{\delta}{\delta \eta_\alpha} = \dfrac{\partial N_\beta^k}{\partial \theta_\alpha}\dfrac{\delta}{\delta x_k} + \dfrac{\partial N_\beta^\gamma}{\partial \theta_\alpha}\dfrac{\delta}{\delta \eta_\gamma},\ D_{\frac{\partial}{\partial \theta_\beta}} \dfrac{\delta}{\delta \eta_\alpha} = 0. \end{cases} \tag{3.3}$$

We call it the Berwald-type superconnection.

With respect to the basis (2.3), T has several components called "h" - , "v"- and "v"-torsion. The (hv)-components of the torsion tensor (as superfunctions on $T\mathcal{M}$) are given by

$$vT\left(\frac{\delta}{\delta x_k}, \frac{\partial}{\partial y_j}\right) = \bar{P}^i_{kj}\frac{\partial}{\partial y_i} + \bar{P}^\alpha_{kj}\frac{\partial}{\partial \theta_\alpha}, vT\left(\frac{\delta}{\delta \eta_\alpha}, \frac{\partial}{\partial y_j}\right) = \bar{P}^i_{\alpha j}\frac{\partial}{\partial y_i} + \bar{P}^\beta_{\alpha j}\frac{\partial}{\partial \theta_\beta} \tag{3.4}$$

$$vT\left(\frac{\delta}{\delta x_k}, \frac{\partial}{\partial \theta_\alpha}\right) = \bar{P}^i_{k\alpha}\frac{\partial}{\partial y_i} + \bar{P}^\beta_{k\alpha}\frac{\partial}{\partial \theta_\beta}, vT\left(\frac{\delta}{\delta \eta_\beta}, \frac{\partial}{\partial \theta_\alpha}\right) = \bar{P}^i_{\beta\alpha}\frac{\partial}{\partial y_i} + \bar{P}^\gamma_{\beta\alpha}\frac{\partial}{\partial \theta_\gamma}. \tag{3.5}$$

4 Characterizing Finsler supermanifolds which are Riemannian supermanifolds

We introduce a new linear superconnection and show that the hv-curvature tensor of this connection characterizes the Riemannian supermanifolds among Finsler supermanifolds.

Theorem 4.1 *Let $\mathcal{M}$ be a supermanifold with a Finsler metric $\mathcal{F}$ and a fundamental tensor g. Let D be the Berwald-type superconnection defined in (3.3) with torsion $\bar{P}^a_{bc}$ given in (3.4) and (3.5). Let in terms of $\mathcal{F}$ and g, A^a_{bc} be defined by*

$$A^a_{bc} = (-1)^{|a||b|(|c|+1)} \frac{\mathcal{F}^2}{2} \left(g^{ar} \frac{\partial}{\partial y_r}(g_{bc}) + g^{a\alpha} \frac{\partial}{\partial \theta_\alpha}(g_{bc}) \right), \tag{4.1}$$

then it defines a new connection $\nabla : \mathcal{X}(T\mathcal{M}) \times \mathcal{X}(T\mathcal{M}) \mapsto \mathcal{X}(T\mathcal{M})$, via the coefficients F^a_{bc} defined by

$$F^a_{bc} = D^a_{bc} + (-1)^{|b||a|} A^a_{bc}. \tag{4.2}$$

The new connection ∇ has the properties:
(1) $y_r F^a_{rb} + \theta_\alpha F^a_{\alpha b} = N^a_b$,
(2) *In all cases for indexes, we have* $\bar{P}^a_{bc} = -(-1)^{|a||c|} A^a_{cb}$ *except for* $\bar{P}^a_{\alpha\beta} = A^a_{\beta\alpha}$,
(3) ∇ *is h-supersymmetric, i.e.* $F^a_{bc} = (-1)^{|b||c|} F^a_{cb}$.
(4) *For all supervector fields* $X, Y \in \mathcal{X}(T\mathcal{M})$, $\nabla_{v(X)} Y = 0$.

Note that, the notation δ_a is used to refer to the supervector fields $\frac{\delta}{\delta x_i}, \frac{\delta}{\delta \eta_\alpha}$, also the notation $\dot{\partial}_a$ is used to refer to $\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \theta_\alpha}$.

Proof. First we notice that the functions (4.1) are the local components of a supertensor field since under a change of coordinates the components g_{ab} and g^{cd} transform as the components of a supertensor field on the base supermanifold.

For the given functions F^a_{bc} defined in (4.2), in all cases for indexes we have $y_r F^a_{rb} + \theta_\alpha F^a_{\alpha b} = N^a_b$, for example

$$\begin{aligned} y_r F^\gamma_{ri} + \theta_\alpha F^\gamma_{\alpha i} = y_r &\left\{ \frac{\partial N^\gamma_i}{\partial y_r} + \frac{\mathcal{F}^2}{2} \left(g^{\gamma j} \frac{\partial}{\partial y_j}(g_{ri}) + g^{\gamma\beta} \frac{\partial}{\partial \theta_\beta}(g_{ri}) \right) \right\} \\ + \theta_\alpha &\left\{ \frac{\partial N^\gamma_i}{\partial \theta_\alpha} + \frac{\mathcal{F}^2}{2} \left(g^{\gamma j} \frac{\partial}{\partial y_j}(g_{\alpha i}) + g^{\gamma\beta} \frac{\partial}{\partial \theta_\beta}(g_{\alpha i}) \right) \right\}. \end{aligned}$$

Since $y_r \frac{\partial N^\gamma_i}{\partial y_r} + \theta_\alpha \frac{\partial N^\gamma_i}{\partial \theta_\alpha} = N^i_\alpha$ so the result follows. Also the second condition of the theorem is satisfied, for example $v[T(\delta_i, \dot{\partial}_\alpha] = v[-F^j_{\alpha_i} \delta j - F^\beta_{\alpha i} \delta_\beta + \dot{\partial}_\alpha N^j_i \dot{\partial}_j + \dot{\partial}_\alpha N^\beta_i \dot{\partial}_\beta]$. So $\bar{P}^k_{i\alpha} = \dot{\partial}_\alpha N^j_i - F^j_{\alpha_i} = -A^k_{i\alpha}$ and $\bar{P}^\alpha_{ij} = \dot{\partial}_j N^\alpha_i - F^\alpha_{ji} = -A^\alpha_{ij}$. On the other hand, since $g_{ab} = (-1)^{|a||b|} g_{ba}$ and for the Berwald connection N we have $\dot{\partial}_a(N^b_c) = (-1)^{|a||c|} \dot{\partial}_c(N^a_c)$, so the third condition of the theorem is satisfied.

Using the similar way as in the proof of Theorem 3.2, the axioms (1)-(4) determine a unique linear superconnection which is also a Finsler connection. We should mention that the (hv)-components of the torsion tensor of the new connection (4.2) are given by $\bar{P}^a_{bc} = (-1)^{|b||c|} \left(\partial_c(N^a_b) - F^a_{cb} \right)$. □

The following formulas which are directly related to the definition of the superfunctions F^a_{bc} and A^a_{bc}, will be used later in our computations. Since the coefficients of the nonlinear connection N are homogeneous superfunctions of degree 1, by a straightforward computation and using the definition of A^a_{bc}, we have

$$y_i A^a_{ic} + (-1)^{|a|} \theta_\alpha A^a_{\alpha c} = 0. \tag{4.3}$$

Differentiating (4.3) with respect to y_k, we have $A^a_{kc} + y_i A^a_{ic,k} + (-1)^{|a|}\theta_\alpha A^a_{\alpha c,k} = 0$, where $A^a_{bc,i} = \frac{\partial A^a_{bc}}{\partial y_i}$. Also, differentiating (4.3) with respect to θ_γ, we have $y_i A^a_{ic,\gamma} + (-1)^{|a|}\{A^a_{\gamma c} - \theta_\alpha A^a_{\alpha c,\gamma}\} = 0$, where $A^a_{bc,\alpha} = \frac{\partial A^a_{bc}}{\partial \theta_\alpha}$.

As ∇ preserves, by parallelism, the horizontal and the vertical distributions, so the operator $R(X,Y)$ carries horizontal supervector fields into horizontal supervector fields and vertical supervector fields into verticals. So the curvature of the superconnection ∇ has six components which are obtained by horizontal or vertical supervector fields δ_a and ∂_b. We call them the h-curvature component (R), the hv-curvature (P) and the vv-curvature (Q). An important component of the curvature is hv-curvature (P) which contains non-Riemannian information.

Consider now the nonlinear connection N and the local basis $(\frac{\delta}{\delta x_i}, \frac{\partial}{\partial y_i}, \frac{\delta}{\delta \eta_\alpha}, \frac{\partial}{\partial \theta_\alpha})$, we compute the local components of the curvature R.

Theorem 4.2 *Let the curvature of the superconnection* ∇ *is given by*

$$R(\delta_a,\delta_b)\delta_c = R^d_{cba}\delta_d, \quad R(\delta_a,\delta_b)\dot\partial_c = R^d_{cba}\dot\partial_d,$$
$$R(\dot\partial_a,\delta_b)\delta_c = P^d_{cba}\delta_d, \quad R(\dot\partial_a,\delta_b)\dot\partial_c = P^d_{cba}\dot\partial_d,$$
$$R(\dot\partial_a,\dot\partial_b)\delta_c = Q^d_{cba}\delta_d, \quad R(\dot\partial_a,\dot\partial_b)\dot\partial_c = Q^d_{cba}\dot\partial_d.$$

then

$$\begin{aligned} R^i_{cba} &= \delta_a(F^i_{cb}) + (-1)^{|a|(|b|+|c|)}\left(F^j_{cb}F^i_{ja} + F^\alpha_{cb}F^i_{\alpha a}\right) \\ &\quad - \delta_b(F^i_{ca}) - (-1)^{|b|(|a|+|c|)}\left(F^j_{ca}F^i_{jb} + F^\alpha_{ca}F^i_{\alpha b}\right), \\ R^\alpha_{cba} &= \delta_a(F^\alpha_{cb}) + (-1)^{|a|(|b|+|c|)}\left(F^i_{cb}F^\alpha_{ia} + F^\beta_{cb}F^\alpha_{\beta a}\right) \\ &\quad - \delta_b(F^\alpha_{ca}) - (-1)^{|b|(|a|+|c|)}\left(F^i_{ca}F^\alpha_{ib} + F^\beta_{ca}F^\alpha_{\beta b}\right), \\ P^d_{cba} &= \dot\partial_a(F^d_{cb}), \\ Q^d_{cba} &= 0. \end{aligned} \tag{4.4}$$

Proof. In all cases for indexes, the proof is straightforward. For example $\nabla_{\delta_k}\nabla_{\delta_j}\delta_t - \nabla_{\delta_j}\nabla_{\delta_k}\delta_t - \nabla_{[\delta_k,\delta_j]}\delta_t = R^i_{tjk}\delta_i + R^\alpha_{tjk}\delta_\alpha$. The left-hand side of the above equation is

$$\begin{aligned} &\delta_k(F^r_{tj})\delta_r + F^r_{tj}\left\{F^s_{rk}\delta_s + F^\beta_{rk}\delta_\beta\right\} + \delta_k(F^\alpha_{tj})\delta_\alpha + F^\alpha_{tj}\left\{F^m_{\alpha k}\delta_m + F^\beta_{\alpha k}\delta_\beta\right\} \\ &\quad - \delta_j(F^r_{tk})\delta_r + F^r_{tk}\left\{F^s_{rj}\delta_s + F^\beta_{rj}\delta_\beta\right\} \\ &\quad + \delta_j(F^\alpha_{tk})\delta_\alpha + F^\alpha_{tk}\left\{F^m_{\alpha j}\delta_m + F^\beta_{\alpha j}\delta_\beta\right\}. \end{aligned}$$

So $R^r_{tjk} = \delta_k(F^r_{tj}) + F^s_{tj}F^r_{sk} + F^\alpha_{tj}F^r_{\alpha k} - \delta_j(F^r_{tk}) + F^s_{tk}F^r_{sj} + F^\alpha_{tk}F^r_{\alpha j}$ and $R^\alpha_{tjk} = \delta_k(F^\alpha_{tj}) + F^r_{tj}F^\alpha_{rk} + F^\beta_{tj}F^\alpha_{\beta k} - \{\delta_j(F^\alpha_{tk}) + F^r_{tk}F^\alpha_{rj} + F^\beta_{tk}F^\alpha_{\beta j}\}$. Repeat this procedure, the remaining components will be achieved. □

Lemma 4.3 *The hv−curvature components of the Finsler superconnection* (4.2) *are satisfied in the following relation.*

$$y_i P^a_{ibc} + (-1)^{|c|}\theta_\alpha P^a_{\alpha bc} = (-1)^{|b||c|}\bar P^a_{bc}. \tag{4.5}$$

Proof. If we differentiate both sides of the equation in the first property of Theorem 4.1 with respect to y_k and θ_γ, and using Theorem 4.2, we obtain (4.5). For example, if we differentiate both sides of the equation $y_r F^i_{rj} + \theta_\alpha F^i_{\alpha j} = N^i_j$ with respect to θ_β, then we have $y_r \frac{\partial F^i_{rj}}{\partial \theta_\beta} + F^i_{\beta j} - \theta_\alpha \frac{\partial F^i_{\alpha j}}{\partial \theta_\beta} = \frac{\partial N^i_j}{\partial \theta_\beta}$. Since the (hv)-components of the torsion tensor of the new connection (4.2) are given by $\bar{P}^a_{bc} = (-1)^{|b||c|}\left(\partial_c(N^a_b) - F^a_{cb}\right)$, so $y_r P^i_{rj\beta} - \theta_\alpha P^i_{\alpha j\beta} = \bar{P}^i_{j\beta}$. □

Theorem 4.4 *Let* $(\mathcal{M}, \mathcal{F})$ *be a Finsler supermanifold with the Finsler superconnection* (4.2)*. Then the Finsler supermetric* $\mathcal{F}$ *is Riemannian if and only if*

$$P^a_{rbc} = -(-1)^{|a||b||c|} A^a_{rb,c}, \tag{4.6}$$

and in all cases for indexes,

$$P^a_{\alpha bc} = -(-1)^{|a|} A^a_{\alpha b,c} \quad \textit{except } P^a_{\alpha\beta\gamma} = -A^a_{\alpha\beta,\gamma}. \tag{4.7}$$

Proof. If $\mathcal{F}$ is Riemannian supermetric, then all components A^a_{bc} are zero and the 'if'-part of the theorem is evident. Conversely, assume (4.6) and (4.7) are satisfied. Differentiating (4.3) with respect to y_k and θ_γ, we have

$$-y_i A^a_{ic,k} - (-1)^{|a|}\theta_\alpha A^a_{\alpha c,k} = A^a_{kc}, \tag{4.8}$$

$$-y_i A^a_{ic,\gamma} + (-1)^{|a|}\theta_\alpha A^a_{\alpha c,\gamma} = (-1)^{|a|} A^a_{\gamma c}. \tag{4.9}$$

We prove the theorem by dividing in the following two cases:

Case 1. We assume that for the two lower indexes "b" and "c" in P^a_{dbc}, $b \neq \beta$ or $c \neq \gamma$. Now, from relations (4.6) and (4.7), we can conclude the following relation

$$y_i P^a_{ibc} + (-1)^{|c|}\theta_\alpha P^a_{\alpha bc} = -(-1)^{|a||b||c|} y_i A^a_{ib,c} - (-1)^{|a|+|c|}\theta_\alpha A^a_{\alpha b,c}. \tag{4.10}$$

The first and second term in booth side of the above equation are obtained by contracting (4.6) and (4.7) by y_i and θ_α respectively. The left hand side of (4.10) is $(-1)^{|b||c|}\bar{P}^a_{ck}$, therefore from (4.8) and (4.9) we have:

- If c=j then $\bar{P}^a_{bj} = A^a_{jb}$. From Theorem 4.1, $\bar{P}^a_{bj} = -A^a_{jb}$, thus $A^a_{jb} = 0$.
- If $c = \beta$ then $b = j$ and we can conclude that $\bar{P}^a_{j\beta} = (-1)^{|a|} A^a_{\beta j}$. From Theorem 4.1, $\bar{P}^a_{j\beta} = -(-1)^{|a|} A^a_{\beta j}$, thus $A^a_{\beta j} = 0$.

Case 2. $b = \beta$ and $c = \gamma$.
Using the same method as above we have

$$y_i P^a_{i\beta\gamma} - \theta_\alpha P^a_{\alpha\beta\gamma} = -(-1)^{|a|} y_i A^a_{i\beta,\gamma} + \theta_\alpha A^a_{\alpha\beta,\gamma}, \tag{4.11}$$

The left hand side of (4.11) is $-\bar{P}^a_{\beta\gamma}$, therefore $-\bar{P}^a_{\beta\gamma} = A^a_{\gamma\beta}$. Thus $A^a_{\gamma\beta} = 0$. □

Here we give an example of a Finsler supermanifold which is not Riemannian supermanifold.

Example 4.1 Let $\mathcal{M} = R^{m|2}$, we use (x_i, η_α) as the coordinates on $\mathcal{M}$. Let F be a nonzero Finsler metric on R^m and let us consider $\tilde{g}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y_i \partial y_j}$ the second order fundamental tensor associated to F.

If we define $\mathcal{F} = F + \frac{1}{2F}\theta_1\theta_2$, then $(\mathcal{M}, \mathcal{F})$ is a Finsler supermanifold. According to (3.1), the coefficients of the fundamental supertensor field g are

$$\begin{bmatrix} \frac{1}{2}\frac{\partial^2 F^2}{\partial y_i \partial y_j} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

So in all cases for indexes, we have

$$A^a_{bc} = 0 \;\; except \quad A^i_{jk} = \frac{\mathcal{F}^2}{2}\tilde{g}^{ir}\frac{\partial \tilde{g}_{jk}}{\partial y_r}.$$

Similarly, for the coefficients of linear connection (4.2), we have

$$F^a_{bc} = D^a_{bc} \;\; except \quad F^i_{jk} = D^i_{jk} + A^i_{jk}.$$

From these and Theorem 4.4 it clearly follows that the Finsler supermanifold is not Riemannian supermanifold, because $A^a_{\alpha\beta,\gamma} = 0$, $P^a_{\alpha\beta\gamma} = \dot{\partial}_\alpha(F^a_{\beta\gamma}) = \dot{\partial}_\alpha(D^a_{\beta\gamma})$.

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