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Some Tauberian theorems for weighted means of bounded double sequences

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Abstract In this paper we define double weighted generator sequence and prove some Tauberian theorems under which the convergence of bounded double sequences follows from its summability by weighted means.

Keywords Tauberian theorem · summability · weighted mean method

Mathematics Subject Classification (2010) 40C05 · 40E05

1 Introduction

Recently, the concept of weighted generator sequence of a single sequence, which is the difference between the sequence and its weighted mean, has been introduced by Çanak and Totur [6]. They proved that certain conditions in terms of weighted generator sequences are Tauberian conditions for the weighted mean method (see also [14]). Tauberian theorems for double weighted mean summability method have been examined by several authors such as Baron and Stadtmüller [2], Chen and Hsu [3], Chen and Chang [4,5], Móricz and Stadtmüller [11], Stadtmüller [13]. The purpose of this paper is to introduce the double analogue of weighted generator sequence and show that some conditions based on the double weighted generator sequence are Tauberian conditions that Pringsheim's convergence of a bounded complex (real) double sequence follows from its weighted summability.

Let $p = (p_j)$, $q = (q_k)$ be two sequences of nonnegative numbers such that $p_0 > 0$, $q_0 > 0$ and $P_m := \sum_{j=0}^{m} p_j \to \infty$ as $m \to \infty$, $Q_n := \sum_{k=0}^{n} q_k \to \infty$ as $n \to \infty$. The weighted means $\sigma_{mn}^{\alpha\beta}$ of a given double sequence (u_{jk}) are defined by

$$\sigma_{mn}^{11}:=\frac{1}{P_mQ_n}\sum_{j=0}^{m}\sum_{k=0}^{n}p_jq_ku_{jk},\sigma_{mn}^{10}:=\frac{1}{P_m}\sum_{j=0}^{m}p_ju_{jn},\quad \sigma_{mn}^{01}:=\frac{1}{Q_n}\sum_{k=0}^{n}q_ku_{mk},$$

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where $m,n\geq 0$. We say that (u_{jk}) is $(\overline{N},p,q;\alpha,\beta)$ summable to L if $\lim_{m,n\to\infty}\sigma_{mn}^{\alpha\beta}=L$, where $(\alpha,\beta)=(1,1)\,,\,(1,0)$ or $(0,1)$. Here we mean the convergence in the sense of Pringsheim, i.e., m and n tend to infinity independently of each other. Under the conditions $P_m\to\infty$ $(m\to\infty)$ and $Q_n\to\infty$ $(n\to\infty)\,,$ we have by a theorem of Kojima-Robison (see, e.g. [7]) that for bounded double sequences the corresponding weighted mean method is bounded regular (or RH-regular).

Let SVA_+ be the set of all nonnegative sequences such that $p_0>0$ and $\liminf_{m\to\infty}|\frac{P_{\lambda_m}}{P_m}-1|>0$, for all $\lambda>0$ with $\lambda\neq 1$, where $\lambda_m:=[\lambda m]$ denotes the integral part of λm.

Lemma 1.1 ([3]) *$p\in SVA_+$ is equivalent to any of the following conditions:*

$$\liminf_{m\to\infty}\frac{P_{\lambda_m}}{P_m}>1\quad(\lambda>1)\,,\qquad\qquad \liminf_{m\to\infty}\frac{P_m}{P_{\lambda_m}}>1\quad(0<\lambda<1)\,,$$

$$\limsup_{m\to\infty}\frac{P_{\lambda_m}}{P_m}<1\quad(0<\lambda<1)\,,\qquad\qquad \limsup_{m\to\infty}\frac{P_m}{P_{\lambda_m}}<1\quad(\lambda>1)\,.$$

We shall write throughout for simplicity in notation for all j,k that $\Delta_{10}u_{jk}=u_{jk}-u_{j-1,k}$, $\Delta_{01}u_{jk}=u_{jk}-u_{j,k-1}$, and $\Delta_{11}u_{jk}=u_{jk}-u_{j,k-1}-u_{j-1,k}+u_{j-1,k-1}$.
Now consider the difference $u_{mn}-\sigma_{mn}^{10}$. From the equation

$$\begin{aligned}\sum_{j=0}^{m}p_j u_{jn}&=\sum_{j=0}^{m}\left(P_j-P_{j-1}\right)u_{jn}=\sum_{j=1}^{m+1}P_{j-1}u_{j-1,n}-\sum_{j=0}^{m}P_{j-1}u_{j,n}\\&=\sum_{j=1}^{m}P_{j-1}\left(u_{j-1,n}-u_{j,n}\right)+P_m u_{mn},\end{aligned}$$

where $P_{-1}=Q_{-1}=0$, we have

$$u_{mn}-\sigma_{mn}^{10}=\frac{1}{P_m}\sum_{j=1}^{m}P_{j-1}\Delta_{10}u_{jn}=:V_{mn}^{10^{(0)}}\left(\Delta_{10}u\right).$$

Similarly we have

$$u_{mn}-\sigma_{mn}^{01}=\frac{1}{Q_n}\sum_{k=1}^{n}Q_{k-1}\Delta_{01}u_{mk}=:V_{mn}^{01^{(0)}}\left(\Delta_{01}u\right).$$

The sequences $(V_{mn}^{10^{(0)}}(\Delta_{10}u))$ and $(V_{mn}^{01^{(0)}}(\Delta_{01}u))$ are called weighted generator sequence of (u_{mn}) in the sense (1,0) and (0,1), respectively. By applying the Abel generalized transformation for double sequences to $\sum_{j=0}^{m}\sum_{k=0}^{n}p_j q_k u_{jk}$ (see [1]), we obtain

$$\begin{aligned}\sum_{j=0}^{m}\sum_{k=0}^{n}p_j q_k u_{jk}&=-P_m\sum_{k=1}^{n}Q_{k-1}\Delta_{01}u_{mk}-Q_n\sum_{j=1}^{m}P_{j-1}\Delta_{10}u_{jn}\\&\quad+\sum_{j=1}^{m}\sum_{k=1}^{n}P_{j-1}Q_{k-1}\Delta_{11}u_{jk}+P_mQ_nu_{mn}.\end{aligned}$$

Then multiplying each side of the last equality by $1/P_m Q_n$, we have the double Kronecker identity given by $u_{mn} - \sigma_{mn}^{11} =: V_{mn}^{11^{(0)}}(\Delta_{11}u)$ where

$$V_{mn}^{11^{(0)}}(\Delta_{11}u) = V_{mn}^{10^{(0)}}(\Delta_{10}u) + V_{mn}^{01^{(0)}}(\Delta_{01}u) - \frac{1}{P_m Q_n}\sum_{j=1}^{m}\sum_{k=1}^{n} P_{j-1}Q_{k-1}\Delta_{11}u_{jk}.$$

The sequence $(V_{mn}^{11^{(0)}}(\Delta_{11}u))$ is called weighted generator sequence of (u_{mn}) in the sense $(1,1)$. For each $v \geq 0$, we define $\sigma_{mn}^{11^{(v)}}$ and $V_{mn}^{11^{(v)}}$ by

$$\sigma_{mn}^{11^{(v)}} := \sigma_{mn}^{11^{(v)}}(u) = \begin{cases} \dfrac{1}{P_m Q_n}\displaystyle\sum_{j=0}^{m}\sum_{k=0}^{n} p_j q_k \sigma_{jk}^{11^{(v-1)}}(u), & v \geq 1 \\ u_{mn}, & v = 0 \end{cases}$$

and

$$V_{mn}^{11^{(v)}} := V_{mn}^{11^{(v)}}(\Delta_{11}u) = \begin{cases} \dfrac{1}{P_m Q_n}\displaystyle\sum_{j=0}^{m}\sum_{k=0}^{n} p_j q_k V_{jk}^{11^{(v-1)}}(\Delta_{11}u), & v \geq 1 \\ V_{mn}^{11^{(0)}}(\Delta_{11}u), & v = 0 \end{cases}$$

respectively. Throughout the paper, we will use the notation σ_{mn}^{11} instead of $\sigma_{mn}^{11^{(1)}}$.

The following lemma gives us two different representations of the difference $u_{mn} - \sigma_{mn}^{11}$ and it can be easily proved by Lemma 2 of [10] with suitable modifications.

Lemma 1.2 *(i) For $\lambda > 1$, $\lambda_m > m$ and $\lambda_n > n$*

$$\begin{aligned} u_{mn} - \sigma_{mn}^{11} = {} & \frac{P_{\lambda_m}}{P_{\lambda_m} - P_m}\left(\sigma_{\lambda_m,n}^{11} - \sigma_{mn}^{11}\right) + \frac{Q_{\lambda_n}}{Q_{\lambda_n} - Q_n}\left(\sigma_{m,\lambda_n}^{11} - \sigma_{m,n}^{11}\right) \\ & + \frac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)}\left(\sigma_{\lambda_m,\lambda_n}^{11} - \sigma_{\lambda_m,n}^{11} - \sigma_{m,\lambda_n}^{11} + \sigma_{mn}^{11}\right) \\ & - \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n} p_j q_k (u_{jk} - u_{mn}). \end{aligned}$$

(ii) For $0 < \lambda < 1$, $\lambda_m < m$ and $\lambda_n < n$,

$$\begin{aligned} u_{mn} - \sigma_{mn}^{11} = {} & \frac{P_{\lambda_m}}{P_m - P_{\lambda_m}}\left(\sigma_{mn}^{11} - \sigma_{\lambda_m,n}^{11}\right) + \frac{Q_{\lambda_n}}{Q_m - Q_{\lambda_n}}\left(\sigma_{mn}^{11} - \sigma_{m,\lambda_n}^{11}\right) \\ & + \frac{P_{\lambda_m}Q_{\lambda_n}}{(P_m - P_{\lambda_m})(Q_m - Q_{\lambda_n})}\left(\sigma_{mn}^{11} - \sigma_{\lambda_m,n}^{11} - \sigma_{m,\lambda_n}^{11} + \sigma_{\lambda_m,\lambda_n}^{11}\right) \\ & + \frac{1}{(P_m - P_{\lambda_m})(Q_m - Q_{\lambda_n})}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n} p_j q_k (u_{mn} - u_{jk}). \end{aligned}$$

2 Main results

In this section we state and prove Tauberian theorems for $(\overline{N}, p, q; 1, 1)$ method.

Theorem 2.1 *Let $p, q \in SVA_+$ and*

$$\frac{p_m}{P_{m-1}} = O(\frac{1}{m}) \text{ and } \frac{q_n}{Q_{n-1}} = O(\frac{1}{n}). \tag{2.1}$$

If a bounded double sequence of real numbers (u_{mn}) is $(\overline{N}, p, q; 1,1)$ summable to L and

$$\frac{P_{m-1}}{p_m} \Delta_{10} V_{mn}^{11^{(0)}} (\Delta_{11} u) \geq -H \text{ and } \frac{Q_{n-1}}{q_n} \Delta_{01} V_{mn}^{11^{(0)}} (\Delta_{11} u) \geq -H, \tag{2.2}$$

for some $H \geq 0$, then (u_{mn}) is convergent to L.

Proof. Assume that (u_{mn}) is $(\overline{N}, p, q; 1, 1)$ summable to L and conditions (2.1)-(2.2) hold. Since (σ_{mn}^{11}) is convergent to L and the weighted mean method is bounded regular, we obtain that $(\sigma_{mn}^{11^{(2)}})$ is convergent to L. It follows from the double weighted Kronecker identity that $(V_{mn}^{11^{(1)}} (\Delta_{11} u))$ is convergent to zero. If we replace u_{mn} by $V_{mn}^{11^{(0)}} (\Delta_{11} u)$ in Lemma 1.2 (i), we have

$$\begin{aligned} V_{mn}^{11^{(0)}} - V_{mn}^{11^{(1)}} &= \frac{P_{\lambda_m}}{P_{\lambda_m} - P_m} \left(V_{\lambda_m,n}^{11^{(1)}} - V_{mn}^{11^{(1)}} \right) + \frac{Q_{\lambda_n}}{Q_{\lambda_n} - Q_n} \left(V_{m,\lambda_n}^{11^{(1)}} - V_{mn}^{11^{(1)}} \right) \\ &+ \frac{P_{\lambda_m} Q_{\lambda_n}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \left(V_{\lambda_m,\lambda_n}^{11^{(1)}} - V_{\lambda_m,n}^{11^{(1)}} - V_{m,\lambda_n}^{11^{(1)}} + V_{mn}^{11^{(1)}} \right) \\ &- \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k \left(V_{jk}^{11^{(0)}} - V_{mn}^{11^{(0)}} \right). \end{aligned} \tag{2.3}$$

Since $p \in SVA_+$ we have by Lemma 1.1 that

$$\limsup_{m \to \infty} \frac{P_{\lambda_m}}{P_{\lambda_m} - P_m} = \frac{1}{\liminf_{m \to \infty} \left(1 - \frac{P_m}{P_{\lambda_m}} \right)} < \infty.$$

Hence

$$\limsup_{m,n \to \infty} \frac{P_{\lambda_m}}{P_{\lambda_m} - P_m} \left(V_{\lambda_m,n}^{11^{(1)}} - V_{mn}^{11^{(1)}} \right) = 0.$$

Similar conclusions can be obtained for the second and third terms in the right-hand side of (2.3). Since

$$\begin{aligned} V_{jk}^{11^{(0)}} - V_{mn}^{11^{(0)}} &= \sum_{\mu=m+1}^{j} \Delta_{10} V_{\mu k}^{11^{(0)}} + \sum_{l=n+1}^{k} \Delta_{01} V_{ml}^{11^{(0)}} \\ &\geq -H \left(\sum_{\mu=m+1}^{j} \frac{p_\mu}{P_{\mu-1}} + \sum_{l=n+1}^{k} \frac{q_l}{Q_{l-1}} \right), \end{aligned}$$

we have

$$-\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_k\left(V_{jk}^{11^{(0)}}-V_{mn}^{11^{(0)}}\right)$$
$$\leq H_1\left(\log\frac{\lambda_m}{m}+\log\frac{\lambda_n}{n}\right),$$

for some $H_1>0$. Hence taking $\limsup_{m,n\to\infty}$ and then limit as $\lambda\to 1^+$ of both sides of (2.3), we obtain that

$$\limsup_{m,n\to\infty}\left(V_{mn}^{11^{(0)}}-V_{mn}^{11^{(1)}}\right)\leq 0. \tag{2.4}$$

From Lemma 1.2 (ii), we have

$$\begin{aligned}V_{mn}^{11^{(0)}}-V_{mn}^{11^{(1)}}=&\frac{P_{\lambda_m}}{P_m-P_{\lambda_m}}\left(V_{mn}^{11^{(1)}}-V_{\lambda_m,n}^{11^{(1)}}\right)+\frac{Q_{\lambda_n}}{Q_n-Q_{\lambda_n}}\left(V_{mn}^{11^{(1)}}-V_{m,\lambda_n}^{11^{(1)}}\right)\\&+\frac{P_{\lambda_m}Q_{\lambda_n}}{(P_{\lambda_m}-P_m)(Q_n-Q_{\lambda_n})}\left(V_{mn}^{11^{(1)}}-V_{\lambda_m,n}^{11^{(1)}}-V_{m,\lambda_n}^{11^{(1)}}+V_{\lambda_m,\lambda_n}^{11^{(1)}}\right)\\&+\frac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}p_jq_k(V_{mn}^{11^{(0)}}-V_{jk}^{11^{(0)}}).\end{aligned} \tag{2.5}$$

Again by using Lemma 1.1, it is obvious that the first, second and third terms in the right-hand side of (2.5) vanishes under the operator $\liminf\limits_{m,n\to\infty}$. In addition, since

$$\begin{aligned}V_{mn}^{11^{(0)}}-V_{jk}^{11^{(0)}}&=\sum_{\mu=j+1}^{m}\Delta_{10}V_{\mu k}^{11^{(0)}}+\sum_{l=k+1}^{n}\Delta_{01}V_{ml}^{11^{(0)}}\\&\geq -H\left(\sum_{\mu=m+1}^{j}\frac{p_\mu}{P_{\mu-1}}+\sum_{l=k+1}^{n}\frac{q_l}{Q_{l-1}}\right),\end{aligned}$$

we have

$$\frac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}p_jq_k\left(V_{mn}^{11^{(0)}}-V_{jk}^{11^{(0)}}\right)$$
$$\geq -H_1\left(\log\frac{\lambda_m}{m}+\log\frac{\lambda_n}{n}\right),$$

for some $H_1>0$. Hence taking $\liminf_{m,n\to\infty}$ and then limit as $\lambda\to 1^-$ of both sides of (2.5), we obtain that

$$\liminf_{m,n\to\infty}\left(V_{mn}^{11^{(0)}}-V_{mn}^{11^{(1)}}\right)\geq 0. \tag{2.6}$$

From (2.4) and (2.6) we have $\lim_{m,n\to\infty}V_{mn}^{11^{(0)}}=\lim_{m,n\to\infty}V_{mn}^{11^{(1)}}=0$. Hence (u_{mn}) is convergent to L by the double weighted Kronecker identity. □

Corollary 2.2 *Let $p, q \in SVA_+$ and $\frac{p_m}{P_{m-1}} = O(\frac{1}{m})$ and $\frac{q_n}{Q_{n-1}} = O(\frac{1}{n})$. If a bounded double sequence of complex numbers (u_{mn}) is $(\overline{N}, p, q; 1, 1)$ summable to L and $\frac{P_{m-1}}{p_m} \Delta_{10} V_{mn}^{11^{(0)}}(\Delta_{11} u) = O(1)$ and $\frac{Q_{n-1}}{q_n} \Delta_{01} V_{mn}^{11^{(0)}}(\Delta_{11} u) = O(1)$ then (u_{mn}) is convergent to L.*

Following Schmidt [12] and also Móricz [9], we say that a double sequence (u_{jk}) is slowly decreasing with respect to the first index if

$$\inf_{\lambda>1} \limsup_{m,n\to\infty} \min_{m<j\le\lambda_m} (u_{jn} - u_{mn}) \ge 0 \quad \text{or} \tag{2.7}$$

$$\inf_{0<\lambda<1} \limsup_{m,n\to\infty} \min_{\lambda_m<j\le m} (u_{mn} - u_{jn}) \ge 0. \tag{2.8}$$

Note that conditions (2.7) and (2.8) are equivalent (see [5, 10]).
Analogously we say that (x_{jk}) is slowly decreasing with respect to the second index if $\inf_{\lambda>1} \limsup_{m,n\to\infty} \min_{n<k\le\lambda_n} (u_{mk} - u_{mn}) \ge 0$. We also say that (u_{jk}) is slowly decreasing in the strong sense with respect to the first index if $\inf_{\lambda>1} \limsup_{m,n\to\infty} \min_{\substack{m<j\le\lambda_m \\ n<k\le\lambda_n}} (u_{jk} - u_{mk}) \ge 0$ and (u_{jk}) is slowly decreasing in the strong sense with respect to the second index if $\inf_{\lambda>1} \limsup_{m,n\to\infty} \min_{\substack{m<j\le\lambda_m \\ n<k\le\lambda_n}} (u_{jk} - u_{jn}) \ge 0$. In these definitions we can replace $\inf_{\lambda>1}$ and $\inf_{0<\lambda<1}$ by $\lim_{\lambda\to1^+}$ and $\lim_{\lambda\to1^-}$, respectively (see [5, 10]). For $\lambda > 1$ we have

$$\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k \left(V_{jk}^{11^{(0)}} - V_{mn}^{11^{(0)}}\right)$$
$$\ge \min_{\substack{m<j\le\lambda_m \\ n<k\le\lambda_n}} \left(V_{jk}^{11^{(0)}} - V_{mk}^{11^{(0)}}\right) + \min_{n<k\le\lambda_n} \left(V_{mk}^{11^{(0)}} - V_{mn}^{11^{(0)}}\right).$$

Hence, if $(V_{mn}^{11^{(0)}}(\Delta_{11} u))$ is slowly decreasing with respect to the second index and slowly decreasing in the strong sense with respect to the first index, then (2.4) holds. Analogously (2.6) holds. Hence we have the following result which can be proved by the same idea in proof of Theorem 2.1.

Theorem 2.3 *Let $p, q \in SVA_+$. If a bounded sequence of real numbers (u_{mn}) is $(\overline{N}, p, q; 1, 1)$ summable to L and $(V_{jk}^{11^{(0)}}(\Delta_{11} u))$ is slowly decreasing with respect to both indices and, in addition, slowly decreasing in the strong sense with respect to one of indices, then (u_{mn}) is convergent to L.*

Following Hardy [8] and also Móricz [9], we say that a double sequence (u_{jk}) of complex numbers is slowly oscillating with respect to the first index if $\lim_{\lambda\to1^+} \limsup_{m,n\to\infty} \max_{m<j\le\lambda_m} |u_{jn} - u_{mn}| = 0$ and (u_{jk}) is slowly oscillating in the strong sense with respect to the first index if

$$\lim_{\lambda\to1^+} \limsup_{m,n\to\infty} \max_{\substack{m<j\le\lambda_m \\ n<k\le\lambda_n}} |u_{jk} - u_{mk}| = 0.$$

Slowly oscillating property with respect to the second index can be defined analogous to that of slowly decreasing property. Since

$$\left|\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n} p_j q_k \left(V_{jk}^{11^{(0)}}-V_{mn}^{11^{(0)}}\right)\right|$$
$$\leq \max_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}\left|V_{jk}^{11^{(0)}}-V_{mk}^{11^{(0)}}\right| + \max_{n<k\leq\lambda_n}\left|V_{mk}^{11^{(0)}}-V_{mn}^{11^{(0)}}\right|,$$

according to the proof of Theorem 2.1 we have $\limsup_{m,n\to\infty}|V_{mn}^{11^{(0)}} - V_{mn}^{11^{(1)}}| \leq \lim_{\lambda\to 1^+}\limsup_{m,n\to\infty}\max_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}|V_{jk}^{11^{(0)}} - V_{mk}^{11^{(0)}}| + \lim_{\lambda\to 1^+}\limsup_{m,n\to\infty}\max_{n<k\leq\lambda_n}|V_{mk}^{11^{(0)}} - V_{mn}^{11^{(0)}}|$ if $p,q\in SVA_+$ and if the bounded sequence of complex numbers (u_{mn}) is $(\overline{N},p,q;1,1)$ summable to some L. Also, if we assume that $(V_{jk}^{11^{(0)}}(\Delta_{11}u))$ is slowly oscillating with respect to both indices and, in addition, slowly oscillating in the strong sense with respect to one of indices, we obtain from the last inequality that $V_{mn}^{11^{(0)}}(\Delta_{11}u) = o(1)$. Hence we proved the following result:

Theorem 2.4 *Let $p,q\in SVA_+$. If a bounded sequence of complex numbers (u_{mn}) is $(\overline{N},p,q;1,1)$ summable to L and $(V_{jk}^{11^{(0)}}(\Delta_{11}u))$ is slowly oscillating with respect to both indices and, in addition, slowly oscillating in the strong sense with respect to one of indices, then (u_{mn}) is convergent to L.*

Consider the following two-sided Landau's conditions for the complex case:

$$j\left|V_{jn}^{11^{(0)}}-V_{j-1,n}^{11^{(0)}}\right|\leq H \quad (j,n>N) \tag{2.9}$$
$$k\left|V_{mk}^{11^{(0)}}-V_{m,k-1}^{11^{(0)}}\right|\leq H \quad (j,n>N) \tag{2.10}$$

where $N>0$ and H are suitable constants. For $\lambda>1$ and $m,n>N$, we have

$$\max_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}\left|V_{jk}^{11^{(0)}}-V_{mk}^{11^{(0)}}\right| \leq \max_{\substack{m<j\leq\lambda_m\\ n<k\leq\lambda_n}}\left\{\sum_{\mu=m+1}^{j}\frac{1}{\mu}\left(\sup_{m<\mu\leq j}\mu\left|V_{\mu k}^{11^{(0)}}-V_{\mu-1,k}^{11^{(0)}}\right|\right)\right\}$$
$$\leq \sum_{\mu=m+1}^{j}\frac{H}{\mu}\leq H\log\frac{\lambda_m}{m}.$$

This indicates that if (2.9) holds then $(V_{jk}^{11^{(0)}}(\Delta_{11}u))$ is slowly oscillating in the strong sense with respect to the first index. Similarly (2.10) implies slow oscillation property with respect to the second index. Hence (2.9) and (2.10) are Tauberian conditions for convergence followed by $(\overline{N},p,q;1,1)$ summability.

Remark 2.1 Analogously to $(\overline{N}, p, q; 1, 1)$ summability one can obtain Tauberian theorems for $(\overline{N}, p, q; 1, 0)$ and $(\overline{N}, p, q; 0, 1)$ summability of double sequences. In these cases, our Tauberian conditions are replaced either by a Schmidt type slow decrease/ oscillation condition or a Landau type one/two sided boundedness condition for the weighted generator sequences in the senses $(1, 0)$ and $(0, 1)$, respectively.

Remark 2.2 In case of $p_j = 1$ and $q_k = 1$, for all j, k, our results are also valid for Cesàro summability of double sequences.

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