

Fractional order Pettis integral equations with multiple time delay in Banach spaces

Mouffak Benchohra ·
Fatima-Zohra Mostefai

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Abstract This paper is devoted to study the existence of solutions under the Pettis integrability assumption for an integral equation of fractional order with multiple time delay in Banach space by using the technique of measure of weak noncompactness.

Keywords Integral equation · Measure of weak noncompactness · Left sided mixed Pettis integral · Weak solution

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1 Introduction

In the last 20 years, the theory of differential and integral equations of fractional orders has become a new important branch and significant development has been done; see for instance the books by ABBAS ET AL [1], KILBAS ET AL. [9], and LAKSHMIKANTHAM ET AL. [11]. Let us mention that this theory has many applications in describing numerous events and problems of the real world. For example, fractional differential equations are often applicable in engineering, physics, chemistry, and biology. See BALEANU ET AL. [2], HILFER [8], PODLUBNY [16] and TARASOV [19].

Mouffak Benchohra
Laboratoire de Mathématiques,
Université de Sidi Bel-Abbès,
B.P. 89, 22000, Sidi Bel-Abbès, Algérie and
Department of Mathematics,
Faculty of Science,
King Abdulaziz University, Saudi Arabia
E-mail: benchohra@univ-sba.dz

Fatima-Zohra Mostefai
Laboratoire de Mathématiques,
Université de Saïda,
B.P. 138, 20000, Saïda, Algérie
E-mail: f.z.mostefai@gmail.com

In this paper, we will investigate the existence of solutions for the following fractional integral equation

$$u(x, y) = \sum_{i=1}^m g_i(x, y)u(x - \xi_i, y - \mu_i) + I_{\theta}^{\alpha} f(x, y, u(x, y)); (x, y) \in J_a \times J_b, \quad (1.1)$$

$$u(x, y) = \Psi(x, y); (x, y) \in \tilde{J} = [-\xi, a] \times [-\mu, b] \setminus (0, a] \times (0, b], \quad (1.2)$$

where $J_a = [0, a]$, $J_b = [0, b]$ for $a, b > 0$, $\theta = (0, 0)$, $\xi = \max_{i=1 \dots m} \{\xi_i\}$, $\mu = \max_{i=1 \dots m} \{\mu_i\}$, I_{θ}^{α} is the left sided mixed Pettis integral of order α , $\alpha = (\alpha_1, \alpha_2) \in (0, \infty) \times (0, \infty)$, $f : J_a \times J_b \times E \rightarrow E$ is a given function satisfying some assumptions that will be specified later, $g_i : J_a \times J_b \rightarrow \mathbb{R}$, $i = 1, \dots, m$ are given functions, and $\Psi : \tilde{J} \rightarrow E$ is a given continuous function such that

$$\begin{aligned} \Psi(0, y) &= \sum_{i=1}^m g_i(0, y)\Psi(-\xi_i, y - \mu_i); y \in [0, b], \\ \Psi(x, 0) &= \sum_{i=1}^m g_i(x, 0)\Psi(x - \xi_i, -\mu_i); x \in [0, a], \end{aligned}$$

E is a real Banach space with norm $\|\cdot\|$. In our investigation we apply the method associated with the technique of measures of weak noncompactness and a fixed point theorem of Mönch type. This technique was mainly initiated in the monograph of BANAŠ and GOEBEL [3] and subsequently developed and used in many papers; see, for example, BANAŠ ET AL. [4], GUO ET AL. [7], KRZYSKA and KUBIACZYK [10], MÖNCH [12], O'REGAN [13,14], SZUFLA [17], SZUFLA and SZUKALA [18], and the references therein.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this survey paper.

Let \mathbb{R} denote the real line and let $J_a = [0, a]$ and $J_b = [0, b]$ be two closed and bounded intervals in \mathbb{R} for some real numbers $a > 0$ and $b > 0$. Throughout the paper, E is a Banach space with norm $\|\cdot\|$ and dual E^* . Also $(E, w) = (E, \sigma(E, E^*))$ denotes the space E with its weak topology.

We take $C(J_a \times J_b, E)$ to be the Banach space of continuous functions $u : J_a \times J_b \rightarrow E$, with the usual supremum norm $\|u\|_{\infty} = \sup\{\|u(x, y)\|, (x, y) \in J_a \times J_b\}$.

Definition 2.1 ([15]) The function $x : J_a \times J_b \rightarrow E$ is said to be Pettis integrable on $J_a \times J_b$ if and only if there is an element $x_{I \times J} \in E$ corresponding to each $I \times J \subset J_a \times J_b$ (I and J are measurable), such that $\varphi(x_{I \times J}) = \int_I \int_J \varphi(x(s, t)) ds dt$ for all $\varphi \in E^*$ where the integral on the right is assumed to exist in the sense of Lebesgue (by definition, $x_{I \times J} = \int_I \int_J x(s, t) ds dt$).

We let $L^1(J_a \times J_b, E)$ denote the Banach space of measurable functions $u : J_a \times J_b \rightarrow E$ that are Pettis integrable, equipped with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y)\| dx dy.$$

Definition 2.2 A function $h : E \rightarrow E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to weakly convergent sequence in E (i.e., for any $(x_n)_n$ in E with $x_n \rightarrow x$ in E_w , $h(x_n) \rightarrow h(x)$ in E_w).

Definition 2.3 ([6]) Let E be a Banach space, Ω_E be the family of all bounded subsets of E and B_E the unit ball of E . The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \rightarrow [0, \infty)$ defined by $\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E : X \subset \epsilon B_E + \Omega\}$.

Properties: The De Blasi measure of noncompactness satisfies the following properties.

- (a) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,
- (b) $\beta(A) = 0 \Leftrightarrow A$ is weakly relatively compact,
- (c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,
- (d) $\beta(\overline{A}^w) = \beta(A)$, (\overline{A}^w denotes the weak closure of A),
- (e) $\beta(A + B) \leq \beta(A) + \beta(B)$,
- (f) $\beta(\lambda A) = |\lambda|\beta(A)$, $\lambda \in \mathbb{R}$,
- (g) $\beta(\text{conv}(A)) = \beta(A)$,
- (h) $\beta(\cup_{|\lambda| \leq h} \lambda A) = h\beta(A)$.

The following result follows directly from the Hahn-Banach theorem.

Proposition 2.4 Let E be a normed space with $x_0 \neq 0$ then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.

For completeness, we recall the definition of the fractional Pettis-integral of order $\alpha > 0$. Let $\alpha_1, \alpha_2 > 0$ and $\alpha = (\alpha_1, \alpha_2)$. For $h \in L^1(J_a \times J_b, E)$, the expression

$$(I_{\theta}^{\alpha} h)(x, y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} h(s, t) ds dt,$$

where the sign “ \int ” denotes the Pettis integral and $\Gamma(\cdot)$ is the Euler gamma function, is called the left sided mixed Pettis integral of order α .

For our purpose we will need the following fixed point theorem.

Theorem 2.5 ([13]) Let E be a Banach space with Q a nonempty, bounded, closed, convex and equicontinuous subset of metrizable locally convex vector space $C(J_a \times J_b, E)$ such that $0 \in Q$. Assume that $T : Q \rightarrow Q$ is weakly-sequentially continuous. If the implication

$$\overline{V} = \overline{\text{conv}}(\{0\} \cup T(V)) \Rightarrow V \text{ is relatively weakly compact,} \quad (2.1)$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of solutions

First of all, we define what we mean by a solution of problem (1.1)-(1.2). Set $J = [-\xi, a] \times [-\mu, b]$.

Definition 3.1 A function $u \in C(J, E)$ is said to be a solution of (1.1)-(1.2) if u satisfies equation (1.1) on $J_a \times J_b$ and condition (1.2) on \tilde{J}

Set $G = \max_{i=1\dots m} \{\sup_{(x,y) \in J_a \times J_b} |g_i(x,y)|\}$. We are now in the position to state and prove our existence result for the problem (1.1)-(1.2). We first list the following hypotheses.

- (H1) For each $(x, y) \in J_a \times J_b$, $f(x, y, \cdot)$ is weakly sequentially continuous.
- (H2) For each $u \in C(J_a \times J_b, E)$, $f(\cdot, \cdot, u(\cdot, \cdot))$ is Pettis integrable on $J_a \times J_b$.
- (H3) There exists $p \in L^\infty(J_a \times J_b, \mathbb{R}_+)$ such that $\|f(x, y, u)\| \leq p(x, y)$, for $(x, y) \in J_a \times J_b$ and each $u \in E$.
- (H4) Let $r_0 > 0$ be arbitrary (but fixed). For any $\epsilon > 0$ and for any subset $X \subset B_{r_0}$, there exists a closed subset $I_\epsilon \subset J_a \times J_b$ such that $\mu(J_a \times J_b \setminus I_\epsilon) < \epsilon$ and $\beta(f(T \times X)) \leq \sup_{(x,y) \in T} p(x,y)\beta(X)$, for each closed subset T of I_ϵ , where μ denotes the Lebesgue measure in \mathbb{R}^2 .
- (H5) The functions $g_i : J_a \times J_b \rightarrow \mathbb{R}$, $i = 1, \dots, m$ are continuous.

Remark 3.1 (H4) is satisfied if the set $f(x, y, B)$ is weakly relatively compact in E for each $(x, y) \in J_a \times J_b$ and B a bounded set of E .

Remark 3.2 If E is reflexive then (H4) is automatically satisfied since a subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.

The main result in this paper reads as follows.

Theorem 3.2 *Assume that assumptions (H1) – (H5) hold. If*

$$mG + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} < 1, \quad (3.1)$$

where $p^* = \|p\|_\infty$, then problem (1.1) – (1.2) has at least one solution on J .

Proof. To transform the problem (1.1)-(1.2) into a fixed point problem, we define the operator $T : C(J, E) \rightarrow C(J, E)$ as

$$\begin{aligned} T(u)(x, y) &= \begin{cases} \Psi(x, y); (x, y) \in \tilde{J}, \\ \sum_{i=1}^m g_i(x, y)u(x-\xi_i, y-\mu_i) + I_\theta^\alpha f(x, y, u(x, y)); (x, y) \in J_a \times J_b. \end{cases} \end{aligned} \quad (3.2)$$

Now we prove that T satisfies all the assumptions of Theorem 2.5 and thus T has a fixed point which is a solution of problem (1.1)-(1.2).

First notice that, for all $u \in C(J_a \times J_b, E)$, $f(x, y, u(x, y))$ is Pettis integrable for a.e. $(x, y) \in J_a \times J_b$ (Assumption (H2)) then $\varphi(f(x, y, u(x, y))) \in L^1(J_a \times J_b)$ for any $\varphi \in E^*$. From the definition of the integral of fractional order we have

$$\begin{aligned} I^\alpha \varphi(f(x, y, u(x, y))) &= \int_0^x \int_0^y \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \varphi(f(s, t, u(s, t))) ds dt \\ &= \int_0^x \int_0^y \varphi \left(\frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} f(s, t, u(s, t)) \right) ds dt \end{aligned}$$

exists for almost every $(x, y) \in J_a \times J_b$ and is an element of $L^1(J_a \times J_b)$, that is, for almost every $(x, y) \in J_a \times J_b$, $s \in (0, x)$, $t \in (0, y)$ the measurable function

$$\begin{aligned} & \varphi \left(\frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} f(s, t, u(s, t)) \right) \\ &= \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \varphi(f(s, t, u(s, t))) \end{aligned}$$

is Lebesgue integrable, hence the function

$$(s, t) \rightarrow \frac{(x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} f(s, t, u(s, t))$$

is Pettis integrable on $J_a \times J_b$, and thus the operator T is well defined.

Let $R > 0$, be such that $R \geq \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)(1 - mG)}$, and consider the set $Q = \{u \in C(J, E) : \|u\|_\infty \leq R \text{ and } \|u(x_2, y_2) - u(x_1, y_1)\| \leq R \sum_{i=1}^m |g_i(x_2, y_2) - g_i(x_1, y_1)| + \frac{p^*}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} [x_2^{\alpha_1} y_2^{\alpha_2} - x_1^{\alpha_1} y_1^{\alpha_2}]; \text{ for } (x_1, y_1), (x_2, y_2) \in J_a \times J_b\}$. Clearly, the subset Q is closed, convex and equicontinuous. The remainder of the proof will be given in three steps.

Step 1: T maps Q into itself.

To see this, take $u \in Q$, $(x, y) \in J_a \times J_b$ and assume that $Tu(x, y) \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ such that $\|Tu(x, y)\| = \varphi(Tu(x, y))$. Thus, we obtain:

$$\begin{aligned} \|Tu(x, y)\| &= \varphi(Tu(x, y)) = \varphi \left(\sum_{i=1}^m g_i(x, y) u(x - \xi_i, y - \mu_i) \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt \right) \\ &= \varphi \left(\sum_{i=1}^m g_i(x, y) u(x - \xi_i, y - \mu_i) \right) \\ &\quad + \varphi \left(\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt \right) \\ &\leq \sum_{i=1}^m |g_i(x, y)| \|u(x - \xi_i, y - \mu_i)\| \\ &\quad + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} \|f(s, t, u(s, t))\| ds dt \\ &\leq mGR + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} p^* ds dt \\ &\leq mGR + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \leq R. \end{aligned}$$

On the other hand, for $(x, y) \in \tilde{J}$, we have $\|Tu(x, y)\| = \varphi(Tu(x, y)) = \varphi(\Psi(x, y)) \leq R$. Next, suppose that $(x_1, y_1), (x_2, y_2) \in J_a \times J_b$ with $x_1 < x_2$ and $y_1 < y_2$, and let $u \in Q$, so $Tu(x_1, y_1) - Tu(x_2, y_2) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|Tu(x_1, y_1) - Tu(x_2, y_2)\| = \varphi(Tu(x_1, y_1) - Tu(x_2, y_2))$ and $\|\varphi\| = 1$. Thus

$$\begin{aligned}
& \|Tu(x_2, y_2) - Tu(x_1, y_1)\| = \varphi \left(\sum_{i=1}^m g_i(x_2, y_2)u(x_2 - \xi_i, y_2 - \mu_i) \right. \\
& + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_2} \int_0^{y_2} (x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt \\
& - \sum_{i=1}^m g_i(x_1, y_1)u(x_1 - \xi_i, y_1 - \mu_i) \\
& \left. + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} \int_0^{y_1} (x_1 - s)^{\alpha_1-1} (y_1 - t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt \right) \\
& = \varphi \left(\sum_{i=1}^m g_i(x_2, y_2)u(x_2 - \xi_i, y_2 - \mu_i) - \sum_{i=1}^m g_i(x_1, y_1)u(x_1 - \xi_i, y_1 - \mu_i) \right) \\
& + \varphi \left(\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt \right. \\
& + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} \int_0^{y_1} [(x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} - (x_1 - s)^{\alpha_1-1} (y_1 - t)^{\alpha_2-1}] \\
& \times f(s, t, u(s, t)) ds dt + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot \\
& \cdot \int_0^{x_1} \int_{y_1}^{y_2} (x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt \\
& \left. + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt \right) \\
& \leq \sum_{i=1}^m \|g_i(x_2, y_2)u(x_2 - \xi_i, y_2 - \mu_i) - g_i(x_1, y_1)u(x_1 - \xi_i, y_1 - \mu_i)\| \\
& + \frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} \int_0^{y_1} [(x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} - (x_1 - s)^{\alpha_1-1} (y_1 - t)^{\alpha_2-1}] ds dt \\
& + \frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} ds dt \\
& + \frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} ds dt \\
& + \frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2 - s)^{\alpha_1-1} (y_2 - t)^{\alpha_2-1} ds dt \\
& \leq R \sum_{i=1}^m |g_i(x_2, y_2) - g_i(x_1, y_1)| + \frac{p^*}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} [x_2^{\alpha_1} y_2^{\alpha_2} - x_1^{\alpha_1} y_1^{\alpha_2}].
\end{aligned}$$

Thus $T(Q) \subset Q$.

Step 2: T is weakly sequentially continuous.

Let (u_n) be a sequence in Q with $u_n(x, y) \rightarrow u(x, y)$ in (E, ω) for each $(x, y) \in J$. Obviously, $Tu_n \rightarrow Tu$ for any $(x, y) \in [-\xi, 0] \times [-\mu, 0]$. Fix $(x, y) \in J_a \times J_b$, we have $\sum_{i=1}^m g_i(x, y)u_n(x - \xi_i, y - \mu_i) \rightarrow \sum_{i=1}^m g_i(x, y)u(x - \xi_i, y - \mu_i)$ and since $f(x, y, \cdot)$ is weakly sequentially continuous (Assumption (H1)) we have immediately $f(x, y, u_n(x, y))$ converging weakly uniformly to $f(x, y, u(x, y))$. Then, Lebesgue Dominated Convergence Theorem for the Pettis integral implies that $Tu_n(x, y)$ converging weakly uniformly to $Tu(x, y)$ in (E, ω) . Since this holds, for each $(x, y) \in J_a \times J_b$ we have $Tu_n \rightarrow Tu$, i.e $T : Q \rightarrow Q$ is weakly sequentially continuous.

Step 3: The implication (2.1) holds.

Let V be a subset of Q such that $\bar{V} = \overline{\text{conv}}(T(V) \cup \{0\})$. Obviously $V(x, y) \subset \overline{\text{conv}}(T(V(x, y)) \cup \{0\})$, $\forall (x, y) \in J$. Further, as V is bounded and equicontinuous, by Ambrosetti Lemma (cf. [5], Lemma 3) the function $(x, y) \rightarrow v(x, y) = \beta(V(x, y))$ is continuous on J .

Since Ψ is continuous on $[-\xi, 0] \times [-\mu, 0]$, the set $\overline{\{\Psi(x, y), (x, y) \in [-\xi, 0] \times [-\mu, 0]\}} \subset E$ is compact. By (H3) and the properties of the measure β , for any $(x, y) \in J_a \times J_b$, we have

$$\begin{aligned}
 v(x, y) &\leq \beta(T(V)(x, y) \cup \{0\}) \leq \beta(T(V)(x, y)) \\
 &\leq \beta \left(\left\{ \sum_{i=1}^m g_i(x, y)u(x - \xi_i, y - \mu_i) \right. \right. \\
 &\quad \left. \left. + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt; u \in V \right\} \right) \\
 &\leq \beta \left(\left\{ \sum_{i=1}^m g_i(x, y)u(x - \xi_i, y - \mu_i); u \in V \right\} \right) \\
 &+ \beta \left(\left\{ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt; u \in V \right\} \right) \\
 &\leq \sum_{i=1}^m \beta(\{g_i(x, y)u(x - \xi_i, y - \mu_i); u \in V\}) \\
 &+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \beta \left(\left\{ \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} f(s, t, u(s, t)) ds dt; u \in V \right\} \right) \\
 &\leq \sum_{i=1}^m g_i(x, y) \beta(V(x, y)) \\
 &+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} p(s, t) \beta(V(s, t)) ds dt \\
 &\leq mG \|v\|_\infty + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \|v\|_\infty.
 \end{aligned}$$

In particular, $\|v\|_\infty \leq \|v\|_\infty (mG + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)})$. By (3.1) it follows that $\|v\|_\infty = 0$, that is $v(x, y) = \beta(V(x, y)) = 0$, for each $(x, y) \in J$ and then V is weakly relatively

compact in $C(J, E)$. Applying now Theorem 2.5 we conclude that T has a fixed point which is a solution of problem (1.1)-(1.2). \square

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