An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIII, 2017, f. 1

The mixed cubic-quartic functional equation

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Received: 3.IX.2013 / Revised: 14.X.2013 / Accepted: 23.VII.2014

Abstract In this paper, we obtain the general solution of the following generalized mixed cubic and quartic functional equation $f(x+kx)+f(x-ky)=k^2\{f(x+y)+f(x-y)\}-2(k^2-1)f(x)-2k^2(k^2-1)f(y)+\frac{1}{4}k^2(k^2-1)f(2y)$, for fixed integers k with $k \neq 0, \pm 1$. The Hyers-Ulam stability problem for the mentioned functional equation is also proved.

Keywords Cubic functional equation · Quartic functional equation · Hyers-Ulam stability

Mathematics Subject Classification (2010) 39B52 · 39B72 · 39B82

1 Introduction and preliminaries

In 1940, Ulam [15] proposed the following stability problem:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In 1941, Hyers [7] solved this stability problem for additive mappings subject to the Hyers condition $\|f(x+y)-f(x)-f(y)\| \leq \delta$ on approximately additive mappings

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$f : \mathcal{X} \longrightarrow \mathcal{Y}$ for a fixed $\delta \geq 0$ and all $x, y \in \mathcal{X}$, where $\mathcal{X}$ is a real normed space and $\mathcal{Y}$ a real Banach space. In 1950, Aoki [1] generalized the Hyers theorem for additive mappings. In 1978, Rassias [14] provided a generalized version of the Hyers theorem which permitted the Cauchy difference to become unbounded.

The cubic function $f(x) = ax^3$ satisfies the functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.1}$$

So the equation (1.1) is called a *cubic functional equation* and every solution of equation (1.1) is said to be a *cubic function*. The stability result of equation (1.1) was obtained by Jun and Kim [8] for the first time. After that, they [9] introduced the following cubic functional equation $f(x+2y)+f(x-2y)+6f(x) = 4f(x+y)+4f(x-y)$ and they established the general solution and the Hyers-Ulam stability problem for it. Recently, in [5], Bodaghi et al. introduced the following new form of cubic functional equations

$$\begin{aligned} f(x+my) + f(x-my) &= 2(2\cos(\frac{m\pi}{2}) + m^2 - 1)f(x) \\ &- \frac{1}{2}(\cos(\frac{m\pi}{2}) + m^2 - 1)f(2x) + m^2\{f(x+y) + f(x-y)\}, \end{aligned} \tag{1.2}$$

where m is an integer with $m \geq 2$. They studied the Hyers-Ulam stability of (1.2).

The quartic functional equation introduced by the third author in [13], and then was employed by other authors. Rassias [13] also investigated stability properties of the following quartic functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y) + 24f(y). \tag{1.3}$$

It is easy to show that the function $f(x) = bx^4$ is a solution of (1.3). Every solution of the quartic functional equation is said to be a *quartic mapping*. For other forms of a quartic functional equation (see [11] and [12]). The second author [10] generalized (1.3) to the following equation $f(mx+ny) + f(mx-ny) - 2m^2(m^2-n^2)f(x) = (mn)^2[f(x+y) + f(x-y)] - 2n^2(m^2-n^2)f(y)$, for fixed integers m and n such that $m \neq 0, n \neq 0, m+n \neq 0$ (for the correction of some details in [10] see [2]). The Hyers-Ulam stability and the superstability for the functional equation (1.1) and quartic functional equations via a fixed point approach under certain conditions on Banach algebras are studied in [3] and [4].

In [6], Eshaghi et al. introduced the following mixed type cubic and quartic functional equation

$$\begin{aligned} f(x+2y) + f(x-2y) &= 9\{f(x+y) + f(x-y)\} \\ &- 6f(x) - 24f(y) + 3f(2y). \end{aligned} \tag{1.4}$$

In this paper we consider the following functional equation which is a generalization of (1.4):

$$\begin{aligned} f(x+ky) + f(x-ky) &= k^2\{f(x+y) + f(x-y)\} \\ &- 2(k^2-1)f(x) - 2k^2(k^2-1)f(y) + \frac{1}{4}k^2(k^2-1)f(2y), \end{aligned} \tag{1.5}$$

where k is an integer with $k \neq 0, \pm 1$. Note that when $k = 2$, we have the equation (1.4). It is easily verified that the function $f(x) = ax^3 + bx^4$ is a solution of the functional equations (1.5).

The main purpose of the present paper is to solve and to prove the generalized Hyers-Ulam stability problem the functional equation (1.5).

2 Solution of equation (1.5)

We firsty solve the equation of (1.5) as follows:

Theorem 2.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces. Then a function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation* (1.4) *if and only if it satisfies the functional equation* (1.5).

Proof. Replacing x by $x + y$ and $x - y$ in (1.4), respectively, and adding the results we have $f(x+3y)+f(x-3y) = 9\{f(x+y)+f(x-y)\}-16f(x)-144f(y)+18f(2y)$. Similar to the above, we get $f(x+4y)+f(x-4y) = 16\{f(x+y)+f(x-y)\}-30f(x)-480f(y)+60f(2y)$. Using the above method, we can deduce that $f(x+ky)+f(x-ky) = k^2\{f(x+y)+f(x-y)\} - a_k f(x) - b_k f(y) + c_k f(2y)$ in which

$$\begin{cases} a_k = -a_{k-2} + 4(k-1)^2, \ a_2 = 6, a_3 = 16 \\ b_k = -b_{k-2} + 2b_{k-1} + 24(k-1)^2, \ b_2 = 24, b_3 = 144 \\ c_k = -c_{k-2} + 2c_{k-1} + 3(k-1)^2, \ c_2 = 3, c_3 = 18. \end{cases}$$

Solving the above recurrence equations, we obtain $a_k = 2(k^2-1)$, $b_k = 2k^2(k^2-1)$ and $c_k = \frac{1}{4}k^2(k^2-1)$, for all $x, y \in \mathcal{X}$ and each positive integer $k \geq 3$. The result for the negative integers is clear.

Conversely, suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation (1.5) for any positive integer $k \geq 3$. We wish to show its correctness for the case $k-1$. The mapping f satisfies (1.5) for each $m \geq k$, in particular for $m = k(k-1)$. Hence for each $x, y \in \mathcal{X}$, we have

$$\begin{aligned} &f(x + k(k-1)y) + f(x - k(k-1)y) \\ &= k^2\{f(x + (k-1)y) + f(x - (k-1)y)\} - 2(k^2-1)f(x) \\ &- 2k^2(k^2-1)f((k-1)y) + \frac{1}{4}k^2(k^2-1)f(2(k-1)y). \end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned} &f(x + (k^2-k)y) + f(x - (k^2-k)y) \\ &= (k^2-k)^2\{f(x+y) + f(x-y)\} - 2((k^2-k)^2-1)f(x) \\ &- 2(k^2-k)^2((k^2-k)^2-1)f(y) + \frac{1}{4}(k^2-k)^2((k^2-k)^2-1)f(2y), \end{aligned} \tag{2.2}$$

for all $x, y \in \mathcal{X}$. Since $n = k(k-1) \geq 3$, we have

$$\begin{aligned} & f(x+(k+1)(k-1)y) + f(x-(k+1)(k-1)y) \\ & = (k+1)^2\{f(x+(k-1)y) + f(x-(k-1)y)\} \\ & - 2((k+1)^2-1)f(x) - 2(k+1)^2((k+1)^2-1)f((k-1)y) \\ & + \frac{1}{4}(k+1)^2((k+1)^2-1)f(2(k-1)y), \end{aligned} \tag{2.3}$$

for all $x, y \in \mathcal{X}$. Also

$$\begin{aligned} & f(x+(k^2-1)y) + f(x-(k^2-1)y) \\ & = (k^2-1)^2\{f(x+y) + f(x-y)\} - 2((k^2-1)^2-1)f(x) \\ & - 2(k^2-1)^2((k^2-1)^2-1)f(y) + \frac{1}{4}(k^2-1)^2((k^2-1)^2-1)f(2y), \end{aligned} \tag{2.4}$$

for all $x, y \in \mathcal{X}$. It follows from (2.1) and (2.2) that

$$\begin{aligned} & f(x+(k-1)y) + f(x-(k-1)y) \\ & = (k-1)^2\{f(x+y) + f(x-y)\} - 2k(k-2)f(x) \\ & - 2(k-1)^2((k^2-k)^2-1)f(y) + \frac{1}{4}(k-1)^2((k^2-k)^2-1)f(2y) \\ & + 2(k^2-1)\{f((k-1)y) - \frac{1}{8}f(2(k-1)y)\}, \end{aligned} \tag{2.5}$$

for all $x, y \in \mathcal{X}$. Plugging (2.3) into (2.4), we get

$$\begin{aligned} & 2k(k+2)\{f((k-1)y) - \frac{1}{8}f(2(k-1)y)\} \\ & = f(x+(k-1)y) + f(x-(k-1)y) \\ & - (k-1)^2\{f(x+y) + f(x-y)\} + 2(k^2+2k)f(x) \\ & + 2k^3(k-1)^2(k-2)f(y) - \frac{1}{4}k^3(k-1)^2(k-2)f(2y), \end{aligned} \tag{2.6}$$

for all $x, y \in \mathcal{X}$. Multiplying both sides of (2.5) by $k(k+2)$ and using (2.6), we have $f(x+(k-1)y)+f(x-(k-1)y) = (k-1)^2\{f(x+y)+f(x-y)\} - 2((k-1)^2-1)f(x) - 2(k-1)^2((k-1)^2-1)f(y)$ $+ \frac{1}{4}(k-1)^2((k-1)^2-1)f(2y)$. This completes the proof. □

Lemma 2.2 *Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces.*

(i) *If an odd function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation* (1.5)*, then f is cubic.*
(ii) *If an even function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation* (1.5)*, then f is quartic.*

Proof. The result follows from Theorem 2.1 and [6, Lemma 2.1 and Lemma 2.2]. □

Theorem 2.3 *Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces. Then a function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation (1.5), for all $x, y \in \mathcal{X}$ if and only if there exists a unique function $\mathcal{C} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{Y}$ and a unique symmetric multiadditive function $\mathcal{Q} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{Y}$ such that $f(x) = \mathcal{C}(x, x, x) + \mathcal{D}(x, x, x, x)$, for all $x \in \mathcal{X}$, and that $\mathcal{C}$ is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Using Theorem 2.1 and [6, Theorem 2.3], one can obtain the desired result. □

3 Hyers-Ulam stability of (1.5) in real Banach spaces

In this section, we investigate the generalized Hyers-Ulam stability problem for the functional equation (1.5). Throughout this section, we assume that $\mathcal{X}$ is a normed real linear space with norm $\|\cdot\|_{\mathcal{X}}$ and $\mathcal{Y}$ is a real Banach space with norm $\|\cdot\|_{\mathcal{Y}}$.

Let k be an integer such that with $k \neq 0, \pm 1$. We use the abbreviation for the given mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ as follows: $\mathcal{D}_k f(x, y) := f(x + ky) + f(x - ky) - k^2\{f(x + y) + f(x - y)\} + 2(k^2 - 1)f(x) + 2k^2(k^2 - 1)f(y) - \frac{1}{4}k^2(k^2 - 1)f(2y)$, for all $x, y \in \mathcal{X}$.

Theorem 3.1 *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be an even mapping with $f(0) = 0$ for which there exists a function $\phi : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ such that*

$$\sum_{k=0}^{\infty} \frac{1}{16^k}\phi(0, 2^k x) < \infty, \lim_{k\to\infty} \frac{1}{16^k}\phi(2^k x, 2^k y) = 0 \tag{3.1}$$

and

$$\|\mathcal{D}_k f(x, y)\|_{\mathcal{Y}} \leq \phi(x, y), \tag{3.2}$$

for all $x, y \in \mathcal{X}$, where k is an integer with $k \neq 0, \pm 1$. Then, there exists a unique quartic mapping $\mathcal{Q} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(2x) - 4f(x) - \mathcal{Q}(x)\|_{\mathcal{Y}} \leq \frac{1}{16}\sum_{n=0}^{\infty} \frac{\Phi_q(2^n x)}{16^n}, \tag{3.3}$$

for all $x \in \mathcal{X}$, where the mapping $\mathcal{Q}(x)$ and $\Phi_q(2^n x)$ are defined by $\mathcal{Q}(x) = \lim\limits_{n\to\infty} \frac{1}{16^n}\{f(2^{n+1}x) - 4f(2^n x)\}$ and

$$\begin{aligned}\Phi_q(2^n x) = \frac{4}{k^2(k^2-1)} &\left[2\phi(k2^n x, 2^n x) + 2k^2\phi(2^n x, 2^n x)\right.\\ &\left.+2(k^2 - 1)\phi(0, 2^n x) + \phi(0, 2^{n+1}x)\right],\end{aligned} \tag{3.4}$$

for all $x \in \mathcal{X}$.

Proof. Replacing (x, y) by $(0, x)$ in (3.2) and using the evenness of f, we get

$$\left\|2f(kx) + 2k^2(k^2 - 2)f(x) - \frac{1}{4}k^2(k^2 - 1)f(2x)\right\|_{\mathcal{Y}} \leq \phi(0, x), \tag{3.5}$$

for all $x \in \mathcal{X}$. Interchanging (x, y) by (kx, x) in (3.2), we deduce that

$$\Big\| f(2kx) - k^2[f((k+1)x) + f((k-1)x)] + 2(k^2-1)f(kx) \\ +2k^2(k^2-1)f(x) - \frac{1}{4}k^2(k^2-1)f(2x)\Big\|_{\mathcal{Y}} \leq \phi(kx, x), \tag{3.6}$$

for all $x \in \mathcal{X}$. Putting $x = y$ in (3.2), we obtain

$$\Big\| [f((k+1)x) + f((k-1)x)] - k^2 f(2x) + 2(k^2-1)f(x) \\ +2k^2(k^2-1)f(x) - \frac{1}{4}k^2(k^2-1)f(2x)\Big\|_{\mathcal{Y}} \leq \phi(x, x),$$

for all $x \in \mathcal{X}$. Thus we have

$$\Big\| [f((k+1)x) + f((k-1)x)] + 2(k^4-1)f(x) - \frac{1}{4}k^2(k^2+3)f(2x)\Big\|_{\mathcal{Y}} \leq \phi(x, x),$$

for all $x \in \mathcal{X}$. The above inequality implies that

$$\Big\| k^2[f((k+1)x) + f((k-1)x)] + 2k^2(k^4-1)f(x) \\ -\frac{1}{4}k^4(k^2+3)f(2x)\Big\|_{\mathcal{Y}} \leq k^2\phi(x, x), \tag{3.7}$$

for all $x \in \mathcal{X}$. It follows from (3.6), (3.7) and triangular inequality that

$$\Big\| f(2kx) + 2(k^2-1)f(kx) + 2k^2(k^2-1)(k^2+2)f(x) \\ -\frac{1}{4}k^2(k^4+4k^2-1)f(2x)\Big\|_{\mathcal{Y}} \leq \phi(kx, x) + k^2\phi(x, x), \tag{3.8}$$

for all $x \in \mathcal{X}$. Multiplying both sides of (3.5) by $k^2 - 1$, we get

$$\Big\| 2(k^2-1)f(kx) + 2k^2(k^2-1)(k^2-2)f(x) \\ -\frac{1}{4}k^2(k^2-1)^2 f(2x)\Big\|_{\mathcal{Y}} \leq (k^2-1)\phi(0, x), \tag{3.9}$$

for all $x \in \mathcal{X}$. Plugging (3.8) into (3.9), we have

$$\Big\| f(2kx) + 8k^2(k^2-1)f(x) - \frac{1}{2}k^2(3k^2-1)f(2x)\Big\|_{\mathcal{Y}} \\ \leq \phi(kx, x) + k^2\phi(x, x) + (k^2-1)\phi(0, x), \tag{3.10}$$

for all $x \in \mathcal{X}$. It also follows from (3.5) that

$$\Big\| 2f(2kx) + 2k^2(k^2-2)f(2x) - \frac{1}{4}k^2(k^2-1)f(4x)\Big\|_{\mathcal{Y}} \leq \phi(0, 2x), \tag{3.11}$$

for all $x \in \mathcal{X}$. Multiplying both sides of (3.10) by 2 and then adding the result to (3.11), we obtain

$$\left\| 5k^2(k^2-1)f(2x) - 16k^2(k^2-1)f(x) - \frac{1}{4}k^2(k^2-1)f(4x) \right\|_{\mathcal{Y}} \leq 2\phi(kx,x) + 2k^2\phi(x,x) + 2(k^2-1)\phi(0,x) + \phi(0,2x),$$

for all $x \in \mathcal{X}$. Thus

$$\|20f(2x) - 64f(x) - f(4x)\|_{\mathcal{Y}} \leq \frac{4}{k^2(k^2-1)}\left[2\phi(kx,x) + 2k^2\phi(x,x) + 2(k^2-1)\phi(0,x) + \phi(0,2x)\right],$$

for all $x \in \mathcal{X}$. The above relation implies that

$$\|g(2x) - 16g(x)\|_{\mathcal{Y}} \leq \Phi_q(x), \tag{3.12}$$

for all $x \in \mathcal{X}$ in which $g(x) = f(2x) - 4f(x)$ and

$$\Phi_q(x) = \frac{4}{k^2(k^2-1)}\left[2\phi(kx,x) + 2k^2\phi(x,x) + 2(k^2-1)\phi(0,x) + \phi(0,2x)\right],$$

for all $x \in \mathcal{X}$. The equality (3.12) shows that

$$\left\| \frac{1}{16}g(2x) - g(x) \right\|_{\mathcal{Y}} \leq \frac{1}{16}\Phi_q(x), \tag{3.13}$$

for all $x \in \mathcal{X}$. Now replacing x by $2x$ and dividing by 16 in (3.13), we obtain

$$\left\| \frac{1}{16^2}g(4x) - \frac{1}{16}g(2x) \right\|_{\mathcal{Y}} \leq \frac{1}{16^2}\Phi_q(2x), \tag{3.14}$$

for all $x \in \mathcal{X}$. From (3.13) and (3.14), we arrive at

$$\left\| \frac{1}{16^2}g(4x) - g(x) \right\|_{\mathcal{Y}} \leq \frac{1}{16}\left(\Phi_q(x) + \frac{1}{16}\Phi_q(2x)\right), \tag{3.15}$$

for all $x \in \mathcal{X}$. In general for any positive integer n, we get

$$\left\| \frac{1}{16^n}g(2^n x) - g(x) \right\|_{\mathcal{Y}} \leq \frac{1}{16}\sum_{j=0}^{n-1}\frac{\Phi_q(2^j x)}{16^j}, \tag{3.16}$$

for all $x \in \mathcal{X}$. In order to prove the convergence of the sequence $\left\{\frac{g(2^n x)}{16^n}\right\}$, replace x by $2^m x$ and divide by 16^m in (3.16). For any $m, n > 0$, we have

$$\left\| \frac{g(2^{n+m}x)}{16^{n+m}} - \frac{g(2^m x)}{16^m} \right\|_{\mathcal{Y}} \leq \frac{1}{16}\sum_{j=0}^{n-1}\frac{\Phi_q(2^{j+m}x)}{16^{j+m}}, \tag{3.17}$$

for all $x \in \mathcal{X}$. Since the right hand side of the inequality (3.17) tends to 0 as m tends to infinity, the sequence $\left\{\frac{g(2^n x)}{16^n}\right\}$ is Cauchy. The completeness of $\mathcal{Y}$ allows us to assume that there exists a map $\mathcal{Q} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\mathcal{Q}(x) = \lim_{n\to\infty} \frac{g(2^n x)}{16^n} \quad (x \in \mathcal{X}). \tag{3.18}$$

Letting $n \to \infty$ in (3.16), we see that (3.3) holds for all $x \in \mathcal{X}$. To prove that $\mathcal{Q}$ satisfies (1.5), replace (x, y) by $(2^n x, 2^n y)$ and divide by 16^n in (3.2). Then, we obtain

$$\begin{aligned}
&\frac{1}{16} \cdot \frac{1}{16^n} \|\mathcal{D}_k g(2^n x, 2^n y)\|_{\mathcal{Y}} \\
&\leq \frac{1}{16^{n+1}} \|\mathcal{D}_k f(2^{n+1} x, 2^{n+1} y) - 4D_k f(2^n x, 2^n y)\|_{\mathcal{Y}} \\
&\leq \frac{1}{16^{n+1}} \|\mathcal{D}_k f(2^{n+1} x, 2^{n+1} y)\|_{\mathcal{Y}} + \frac{4}{16^n} \|D_k f(2^n x, 2^n y)\|_{\mathcal{Y}} \\
&\leq \frac{\phi(2^{n+1} x, 2^{n+1} y)}{16^{n+1}} + 4 \cdot \frac{\phi(2^n x, 2^n y)}{16^n},
\end{aligned}$$

for all $x, y \in \mathcal{X}$. Letting $n \to \infty$ in the above inequality and using (3.1), we observe that $\mathcal{D}_k \mathcal{Q}(x, y) = 0$, for all $x, y \in \mathcal{X}$. Therefore, by the part (ii) of Lemma 2.2, $\mathcal{Q}$ is a quartic mapping. Now, let $\mathcal{Q}' : \mathcal{X} \longrightarrow \mathcal{Y}$ be another quartic mapping satisfying (3.3). Then we have

$$\begin{aligned}
\|\mathcal{Q}(x) - \mathcal{Q}'(x)\|_{\mathcal{Y}} &= \frac{1}{16^n} \|\mathcal{Q}(2^n x) - \mathcal{Q}'(2^n x)\|_{\mathcal{Y}} \\
&\leq \frac{1}{16^n} \left(\|\mathcal{Q}(2^n x) - g(2^n x)\|_{\mathcal{Y}} + \|g(2^n x) - \mathcal{Q}'(2^n x)\|_{\mathcal{Y}}\right) \\
&\leq \frac{1}{16^n} \frac{1}{8} \sum_{j=0}^{\infty} \frac{\Phi_q(2^j x)}{16^j},
\end{aligned}$$

for all $x \in \mathcal{X}$. Taking $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of $\mathcal{Q}$. This finishes the proof. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.5).

Corollary 3.2 *Let α and p, q be nonnegative real numbers. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an even mapping fulfilling*

$$\|\mathcal{D}_k f(x, y)\| \leq \begin{cases} \alpha \\ \alpha \|x\|_{\mathcal{X}}^p \|y\|_{\mathcal{X}}^q, & 0 \leq p + q < 4 \\ \alpha(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p), & 0 \leq p < 4 \\ \alpha(\|x\|_{\mathcal{X}}^p \|y\|_{\mathcal{X}}^p + \|x\|_{\mathcal{X}}^{2p} + \|y\|_{\mathcal{X}}^{2p}), & 0 \leq p < 2 \end{cases},$$

for all $x, y \in \mathcal{X}$. *Then there exists a unique quartic function* $\mathcal{Q} : \mathcal{X} \longrightarrow \mathcal{Y}$ *such that*

$$\|f(2x) - 4f(x) - \mathcal{Q}(x)\|_{\mathcal{Y}} \leq \begin{cases} \lambda_1 \alpha, & \\ \dfrac{\alpha \|x\|_{\mathcal{X}}^{p+q}}{16 - 2^{p+q}} \lambda_2, & 0 \leq p + q < 4 \\ \dfrac{\alpha \|x\|_{\mathcal{X}}^{p}}{16 - 2^{p}} \lambda_3, & 0 \leq p < 4 \\ \dfrac{\alpha \|x\|_{\mathcal{X}}^{2p}}{16 - 2^{2p}} \lambda_4, & 0 \leq p < 2, \end{cases}$$

where

$$\lambda_1 = \frac{4(4k^2+1)}{15k^2(k^2-1)}, \quad \lambda_2 = \frac{8(k^p + k^2)}{k^2(k^2-1)},$$
$$\lambda_3 = \frac{8(k^p + 3k^2 + 2^{p-1})}{k^2(k^2-1)}, \quad \lambda_4 = \frac{8(k^p + 4k^2 + k^{2p} + 2^{2p-1})}{k^2(k^2-1)}.$$

In analogy with Theorem 3.1, we have the following theorem for the stability of (1.5) when f is an odd mapping.

Theorem 3.3 *Let* $f : \mathcal{X} \longrightarrow \mathcal{Y}$ *be an odd mapping with* $f(0) = 0$ *for which there exists a function* $\phi : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ *such that*

$$\sum_{j=0}^{\infty} \frac{1}{8^j} \phi(0, 2^j x) < \infty, \lim_{j \to \infty} \frac{1}{8^j} \phi(2^j x, 2^j y) = 0 \tag{3.19}$$

and

$$\|\mathcal{D}_k f(x, y)\|_{\mathcal{Y}} \leq \phi(x, y), \tag{3.20}$$

for all $x, y \in \mathcal{X}$, *where* k *is an integer with* $k \neq 0, \pm 1$. *Then, there exists a unique cubic mapping* $\mathcal{C} : \mathcal{X} \longrightarrow \mathcal{Y}$ *such that*

$$\|f(x) - \mathcal{C}(x)\|_{\mathcal{Y}} \leq \sum_{n=0}^{\infty} \frac{\psi_c(2^n x)}{8^n}, \tag{3.21}$$

for all $x \in \mathcal{X}$, *where the mapping* $\mathcal{C}(x)$ *and* $\Phi(2^n x)$ *are defined by* $\mathcal{C}(x) = \lim_{n \to \infty} \frac{1}{8^n} f(2^n x)$ *and*

$$\psi_c(2^n x) = \frac{1}{2k^2(k^2-1)} \phi(0, 2^n x), \tag{3.22}$$

for all $x \in \mathcal{X}$.

Proof. Replace (x, y) by $(0, x)$ in (3.20). By the oddness of f we have

$$\left\| 2k^2(k^2-1) f(x) - \frac{1}{4} k^2(k^2-1) f(2x) \right\|_{\mathcal{Y}} \leq \phi(0, x),$$

for all $x \in \mathcal{X}$. Hence

$$\|8f(x) - f(2x)\|_{\mathcal{Y}} \leq \frac{4}{k^2(k^2-1)}\phi(0,x), \tag{3.23}$$

for all $x \in \mathcal{X}$. In other words

$$\left\|f(x) - \frac{1}{8}f(2x)\right\|_{\mathcal{Y}} \leq \psi_c(x), \tag{3.24}$$

for all $x \in \mathcal{X}$ in which $\psi_c(x) = \frac{1}{2k^2(k^2-1)}\phi(0,x)$. Interchanging x by $2x$ and then dividing both sides by 8 in the above inequality, we deduce that

$$\left\|\frac{1}{8}f(2x) - \frac{1}{8^2}f(2^2x)\right\|_{\mathcal{Y}} \leq \frac{1}{8}\psi_c(2x), \tag{3.25}$$

for all $x \in \mathcal{X}$. The inequalities (3.24) and (3.25) imply that

$$\left\|f(x) - \frac{1}{8^2}f(2^2x)\right\|_{\mathcal{Y}} \leq \left(\psi_c(x) + \frac{1}{8}\psi_c(2x)\right), \tag{3.26}$$

for all $x \in \mathcal{X}$. This method can be repeated to obtain

$$\left\|f(x) - \frac{1}{8^n}f(2^nx)\right\|_{\mathcal{Y}} \leq \sum_{j=0}^{n-1}\frac{\psi_c(2^jx)}{8^j}, \tag{3.27}$$

for all $x \in \mathcal{X}$. Putting x by 2^mx and then dividing both sides by 8^m in (3.27), we get

$$\left\|\frac{f(2^mx)}{8^m} - \frac{f(2^{n+m}x)}{8^{n+m}}\right\|_{\mathcal{Y}} \leq \sum_{j=0}^{n-1}\frac{\psi_c(2^{j+m}x)}{8^{j+m}}, \tag{3.28}$$

for all $x \in \mathcal{X}$ and all positive integers m, n. Thus, we conclude from (3.19) and (3.28) that the sequence $\{\frac{f(2^nx)}{8^n}\}$ is Cauchy. Since the space $\mathcal{Y}$ is complete, this sequence converges in $\mathcal{Y}$ to the mapping $\mathcal{C}$. Indeed,

$$\mathcal{C}(x) = \lim_{n\to\infty}\frac{f(2^nx)}{8^n}, \quad (x \in \mathcal{X}). \tag{3.29}$$

It follows from (3.20) that $\frac{1}{8^n}\|\mathcal{D}_kf(2^nx, 2^ny)\|_{\mathcal{Y}} \leq \frac{\phi(2^nx,2^ny)}{8^n}$, for all $x, y \in \mathcal{X}$. Letting $n \to \infty$ in the above inequality and appling (3.19), (3.29), we get $\mathcal{D}_k\mathcal{C}(x,y) = 0$, for all $x, y \in \mathcal{X}$. Hence, the part (i) of Lemma 2.2 shows that $\mathcal{C}$ is a cubic mapping. Also the relations (3.27) and (3.29) imply that (3.21) holds for all $x \in \mathcal{X}$. For the uniqueness of $\mathcal{C}$, assume that $\mathcal{C}' : \mathcal{X} \longrightarrow \mathcal{Y}$ is another cubic mapping satisfying (3.21). Then, we have

$$\begin{aligned}
\|\mathcal{C}(x) - \mathcal{C}'(x)\|_{\mathcal{Y}} &= \frac{1}{8^n}\|\mathcal{C}(2^nx) - \mathcal{C}'(2^nx)\|_{\mathcal{Y}} \\
&\leq \frac{1}{8^n}\left(\|\mathcal{C}(2^nx) - f(2^nx)\|_{\mathcal{Y}} + \|f(2^nx) - \mathcal{C}'(2^nx)\|_{\mathcal{Y}}\right) \\
&\leq \frac{2}{8^n}\sum_{j=0}^{\infty}\frac{\psi_c(2^jx)}{8^j},
\end{aligned}$$

for all $x \in \mathcal{X}$. Taking $n \to \infty$ in the last inequality, we have $\mathcal{C}(x) = \mathcal{C}'(x)$, for all $x \in \mathcal{X}$. $\square$

Corollary 3.4 *Let α and p, q be nonnegative real numbers. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an odd mapping fulfilling*

$$\|\mathcal{D}_k f(x,y)\| \leq \begin{cases} \alpha \\ \alpha(\|x\|^p_{\mathcal{X}} + \|y\|^p_{\mathcal{X}}), \quad 0 \leq p < 3, \end{cases}$$

for all $x, y \in \mathcal{X}$. Then there exists a unique cubic function $\mathcal{C} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{C}(x)\|_{\mathcal{Y}} \leq \begin{cases} \lambda_5 \alpha \\ \dfrac{\alpha\|x\|^p_{\mathcal{X}}}{8-2^p}\lambda_6, \quad 0 \leq p < 3, \end{cases}$$

where $\lambda_5 = \frac{4}{7k^2(k^2-1)}$, $\lambda_6 = \frac{4}{k^2(k^2-1)}$.

In particular, if $\|\mathcal{D}_k f(x,y)\| \leq \alpha\|x\|^p_{\mathcal{X}}\|y\|^q_{\mathcal{X}}$ where $p + q \neq 3$, then the mapping f is cubic.

Theorem 3.5 *Let $\phi : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{1}{8^n}\phi(0, 2^n x) < \infty, \sum_{n=0}^{\infty} \frac{1}{16^n}\phi(0, 2^n x) < \infty \tag{3.30}$$

and

$$\lim_{n\to\infty} \frac{1}{8^n}\phi(2^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{16^n}\phi(2^n x, 2^n y) = 0. \tag{3.31}$$

Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an mapping with $f(0) = 0$ satisfies

$$\|\mathcal{D}_k f(x,y)\|_{\mathcal{Y}} \leq \phi(x,y), \tag{3.32}$$

for all $x, y \in \mathcal{X}$, where k is an integer with $k \neq 0, \pm 1$. Then there exists a unique cubic mapping $\mathcal{C} : \mathcal{X} \longrightarrow \mathcal{Y}$ and a unique quartic mapping $\mathcal{Q} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\begin{aligned} &\|f(2x) - 4f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\|_{\mathcal{Y}} \\ &\leq \frac{1}{32}\sum_{n=0}^{\infty} \frac{\Phi_q(2^n x) + \Phi_q(-2^n x)}{16^n} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{\Phi_c(2^n x) + \Phi_c(-2^n x)}{8^n}, \end{aligned} \tag{3.33}$$

for all $x \in \mathcal{X}$, where $\Phi_c(2^n x) = \frac{1}{2k^2(k^2-1)}[\phi(0, 2^{n+1}x) + 4\phi(0, 2^n x)]$ and $\Phi_q(2^n x)$ is given in (3.4).

Proof. We decompose f into the even part and odd part by setting $f_e(x) = \frac{f(x)+f(-x)}{2}$, $f_o(x) = \frac{f(x)-f(-x)}{2}$, for all $x \in \mathcal{X}$. Obviously, $f(x) = f_e(x) + f_o(x)$, for all $x \in \mathcal{X}$. Then

$\|\mathcal{D}_k f_{\rm e}(x,y)\|_{\mathcal{Y}} = \frac{1}{2}\|\mathcal{D}_k f(x,y) + \mathcal{D}_k f(-x,-y)\|_{\mathcal{Y}} \le \frac{1}{2}(\|\mathcal{D}_k f(x,y)\|_{\mathcal{Y}} + \|\mathcal{D}_k f(-x,-y)\|_{\mathcal{Y}}) \le \frac{1}{2}(\phi(x,y) + \phi(-x,-y))$ and

$$\begin{aligned}\|\mathcal{D}_k f_{\rm o}(x,y)\|_{\mathcal{Y}} &= \frac{1}{2}\|\mathcal{D}_k f(x,y) - \mathcal{D}_k f(-x,-y)\|_{\mathcal{Y}}\\ &\le \frac{1}{2}(\|\mathcal{D}_k f(x,y)\|_{\mathcal{Y}} + \|\mathcal{D}_k f(-x,-y)\|_{\mathcal{Y}})\\ &\le \frac{1}{2}(\phi(x,y) + \phi(-x,-y)),\end{aligned}$$

for all $x \in \mathcal{X}$. By Theorems 3.1 and 3.3, there exists a unique quratic function $\mathcal{Q}_0 : \mathcal{X} \longrightarrow \mathcal{Y}$ and a unique cubic function $\mathcal{C}_0 : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f_{\rm e}(2x) - 4f_{\rm e}(x) - \mathcal{Q}_0(x)\|_{\mathcal{Y}} \le \frac{1}{32}\sum_{n=0}^{\infty} \frac{\Phi_q(2^n x) + \Phi_q(-2^n x)}{16^n} \tag{3.34}$$

and

$$\|f_{\rm o}(x) - \mathcal{C}_0(x)\|_{\mathcal{Y}} \le \frac{1}{2}\sum_{n=0}^{\infty} \frac{\Phi_c(2^n x) + \Phi_c(-2^n x)}{8^n}, \tag{3.35}$$

for all $x \in \mathcal{X}$. Put $\mathcal{Q}(x) = \mathcal{Q}_0(x)$ and $\mathcal{C}(x) = 4\mathcal{C}_0(x)$. Since $\mathcal{C}_0(x)$ is odd and satisfies the equation (1.5), it is easy to check that $C_0(2x) = 8C_0(x)$. Thus we have

$$\begin{aligned}&\|f(2x) - 4f(x) - \mathcal{Q}(x) - \mathcal{C}(x)\|_{\mathcal{Y}} = \|f(2x) - 4f(x) - \mathcal{Q}_0(x) - 4\mathcal{C}_0(x)\|_{\mathcal{Y}}\\ &= \|(f_{\rm e}(2x) - 4f_{\rm e}(x) - \mathcal{Q}_0(x)) + (f_{\rm o}(2x) - 4f_{\rm o}(x) - 4\mathcal{C}_0(x))\|_{\mathcal{Y}}\\ &\le \|f_{\rm e}(2x) - 4f_{\rm e}(x) - \mathcal{Q}_0(x)\|_{\mathcal{Y}}\\ &+ \|f_{\rm o}(2x) - 8\mathcal{C}_0(x)\| + 4\|f_{\rm o}(x) - \mathcal{C}_0(x)\|_{\mathcal{Y}}\\ &= \|f_{\rm e}(2x) - 4f_{\rm e}(x) - \mathcal{Q}_0(x)\|_{\mathcal{Y}}\\ &+ \|f_{\rm o}(2x) - \mathcal{C}_0(2x)\|_{\mathcal{Y}} + 4\|f_{\rm o}(x) - \mathcal{C}_0(x)\|_{\mathcal{Y}}\\ &\le \frac{1}{32}\sum_{n=0}^{\infty} \frac{\Phi_q(2^n x) + \Phi_q(-2^n x)}{16^n} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{\Phi_c(2^n x) + \Phi_c(-2^n x)}{8^n}\end{aligned}$$

in which $\Phi_c(2^n x) = \frac{1}{2k^2(k^2-1)}[\phi(0, 2^{n+1}x) + 4\phi(0, 2^n x)]$ and $\Phi_q(2^n x)$ is given in (3.4). □

The following corollary is a direct consequence of Theorem 3.5 concerning the stability of (1.5).

Corollary 3.6 *Let α and p, q be nonnegative real numbers. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an even mapping fulfilling*

$$\|\mathcal{D}_k f(x,y)\| \le \begin{cases} \alpha, & \\ \alpha\|x\|_{\mathcal{X}}^p \|y\|_{\mathcal{X}}^q, & 0 \le p+q < 3\\ \alpha(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p), & 0 \le p < 3\\ \alpha(\|x\|_{\mathcal{X}}^p \|y\|_{\mathcal{X}}^p + \|x\|_{\mathcal{X}}^{2p} + \|y\|_{\mathcal{X}}^{2p}), & 0 \le p < \frac{3}{2}, \end{cases}$$

for all $x, y \in \mathcal{X}$. *Then there exists a unique cubic mapping* $\mathcal{C} : \mathcal{X} \longrightarrow \mathcal{Y}$ *and a unique quartic mapping* $\mathcal{Q} : \mathcal{X} \longrightarrow \mathcal{Y}$ *such that*

$$\|f(2x) - 4f(x) - \mathcal{C}(x) - \mathcal{Q}(x)\|_{\mathcal{Y}} \leq \begin{cases} (\lambda_1 + \lambda_5)\alpha, & \\ \dfrac{\alpha\|x\|_{\mathcal{X}}^{p+q}}{16 - 2^{p+q}}\lambda_2, & 0 \leq p + q < 3 \\ \left(\dfrac{1}{16 - 2^p}\lambda_3 + \dfrac{4 + 2^p}{8 - 2^p}\lambda_6\right)\alpha\|x\|_{\mathcal{X}}^p, & 0 \leq p < 3 \\ \left(\dfrac{1}{16 - 2^{2p}}\lambda_4 + \dfrac{4 + 2^{2p}}{8 - 2^{2p}}\lambda_6\right)\alpha\|x\|_{\mathcal{X}}^{2p}, & 0 \leq p < \frac{3}{2}, \end{cases}$$

where $\lambda_j\,(j = 1, 2, 3, 4, 5, 6)$ *are given in Corollaries* 3.2 *and* 3.4.

Acknowledgements The authors express their sincere thanks to the referee for the careful and detailed reading of the manuscript and very helpful comments. The first author would like to thanks the Young Researchers and Elite Club of Islamic Azad University of Islamshahr for its financial support.

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