

On absolutely almost convergence

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Abstract In this paper, we study (P_n, s) -absolutely almost convergence where $np_n = o(P_n)$ as $n \rightarrow \infty$, $P_n = \sum_{\nu=0}^n p_\nu$ and $s > 0$. We also give necessary and sufficient conditions for a summability matrix to be (λ_n, s) -absolutely almost conservative, and obtain some inclusion results.

Keywords sequences · series · absolutely almost convergence

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1 Introduction

Let $A = (a_{nk})$ be an infinite matrix and $\mathbf{x} = (x_k)$ a sequence of real or complex numbers. We denote $A\mathbf{x} = (A\mathbf{x})_n$, $(A\mathbf{x})_n = \sum_{k=0}^{\infty} a_{nk}x_k$, where by the existence of $A\mathbf{x}$, we mean the convergence of this last series for each $n \geq 0$. For two sequence spaces X and Y , we say that a matrix A is of type (X, Y) if $A\mathbf{x} \in Y$ whenever $\mathbf{x} \in X$. By m and c , we will denote the Banach spaces of bounded and convergent sequences $\mathbf{x} = (x_k)$, with norm $\|\mathbf{x}\| = \sup_k |x_k|$; the space of sequences of bounded variation will be denoted by bv , i.e. $bv = \{\mathbf{x} = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, (x_{-1} = 0)\}$ and bv is a Banach space with the norm $\|(x_k)\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|$.

A sequence $\mathbf{x} = (x_k)$ is said to be almost convergent if all of its Banach limits coincide. LORENTZ [7] proved that \mathbf{x} is almost convergent to a number L if and only

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if $\frac{1}{n+1} \sum_{k=p}^{p+n} x_k \rightarrow L$ as $n \rightarrow \infty$ uniformly in p . By f , we mean the set of all almost convergent sequences. L is said to be the f -limit of \mathbf{x} and we write $f - \lim \mathbf{x} = L$.

In [6], the concept of absolute almost convergence was given in the following manner: for any sequence $\mathbf{x} = (x_k) \in m$, consider $t_{n,p}(x)$ defined by $t_{n,p} = \frac{1}{n+1} \sum_{k=p}^{p+n} x_k$, ($n, p \geq 0$). Define for $p \geq 0$,

$$F_{n,p}(x) = t_{n,p}(x) - t_{n-1,p}(x) \quad (n \geq 1), \quad F_{0,p}(x) = t_{0,p}(x) = x_p. \quad (1)$$

A sequence $\mathbf{x} = (x_k)$ is absolutely almost convergent if $\sum_{n=0}^{\infty} |F_{n,p}(x)| < \infty$ uniformly in p . The set of all absolutely almost convergent sequences is denoted by $\hat{\ell}$. ÇAKALLI [2] generalized this concept to s -absolute almost convergence for $s \geq 1$ calling a sequence $\mathbf{x} = (x_k)$ s -absolutely almost convergent if $\sum_{n=0}^{\infty} (1+n)^{2(s-1)} |F_{n,p}(x)|^s < \infty$ uniformly in p . ÇAKALLI and ÇANAK [4] extended this definition for $s > 0$ calling a sequence $\mathbf{x} = (x_k)$ s -absolutely almost convergent if $\sum_{n=0}^{\infty} (1+n)^{2|s-1|} |F_{n,p}(x)|^s < \infty$ uniformly in p . ÇAKALLI and ÇANAK [3] generalized this concept to (P_n, s) -absolute almost convergence for $s \geq 1$ in the sense that a sequence $\mathbf{x} = (x_k)$ is said to be (P_n, s) -absolutely almost convergent if $\sum_{k=0}^{\infty} [(p_n)^{-1} P_n]^{2|s-1|} |F_{n,p}(x)|^s < \infty$ uniformly in p , where (p_n) is a sequence of positive real constants such that

$$np_n = 0(P_n) \text{ as } n \rightarrow \infty \text{ where } P_n = \sum_{\nu=0}^n p_\nu. \quad (2)$$

MURSALEEN [9] investigated the absolutely almost conservativity, using the concept of absolute almost convergence due to DAS [6] (see also [1], [10] and [11]).

Let $A\mathbf{x}$ be defined. Then by (1) we get for $n, p \geq 0$, $F_{n,p}(A\mathbf{x}) = \sum_{k=0}^{\infty} t(n, k, p)x_k$, where $t(n, k, p) = \frac{1}{n(n+1)} \sum_{i=1}^n i(a_{p+i,k} - a_{p+i-1,k})$ if $n \geq 1$ and $t(n, k, p) = a_{p,k}$ if $n = 0$, and $np_n = 0(P_n)$ as $n \rightarrow \infty$, $P_n = \sum_{\nu=0}^n p_\nu$, and $s > 0$.

ÇAKALLI and ÇANAK gave the following theorem, which is Theorem B in [4].

Theorem 1.1 *Let s be a real number with $0 < s \leq 1$ and $A = (a_{nk})$ be an infinite matrix. Then A is s -absolutely almost conservative if and only if:*

$$\sup_{p,k} \left| \sum_{l=k}^{\infty} a_{pl} \right| < \infty, \quad (3)$$

there exists a constant K such that for $r, p = 1, 2, \dots$,

$$\sum_{n=0}^{\infty} (1+n)^{2(1-s)} \left| \sum_{k=0}^r t(n, k, p) \right| \leq K, \quad (4)$$

$$(a_{pk}) \in \hat{\ell}(s) \text{ for each } k, \quad (5)$$

$$\left(\sum_{k=0}^{\infty} a_{pk} \right) \in \hat{\ell}(s). \quad (6)$$

For A to be s -absolutely almost conservative $f - \lim A\mathbf{x} = a \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} x_k a_k$, for every $\mathbf{x} = (x_k) \in bv$, where, for each k , (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a .

The aim of this paper is to generalize s -absolute almost convergence to (P_n, s) -absolute almost convergence, and characterize (P_n, s) -absolutely almost conservative matrices, i.e., of type $(bv, \hat{\ell}, (P_n, s))$ for $0 < s \leq 1$.

2 Results

Now we extend the definition of (P_n, s) -absolutely almost convergence to the case valid for any constant positive real number s in the sense that a sequence $\mathbf{x} = (x_n)$ is (P_n, s) -absolutely almost convergent if

$$\sum_{k=0}^{\infty} [(p_n)^{-1} P_n]^{2|s-1|} \cdot |F_{n,p}(x)|^s < \infty$$

uniformly in p , where (p_n) is a sequence of positive real constants which satisfies condition (2). By $\hat{\ell}(P, s)$ we denote the set of (P_n, s) -absolute almost convergent sequences. When $p_n = 1$ for all n , the definition of s -absolute almost convergence is obtained, that is $\hat{\ell}(n+1, s) = \hat{\ell}(s)$ given in [4], where s -absolutely almost conservative matrices were characterized for $0 < s \leq 1$.

Now we give the following theorem:

Theorem 2.1 *Let s be a real number with $0 < s \leq 1$ and $A = (a_{nk})$ be an infinite matrix. Then the matrix A is (P_n, s) -absolutely almost conservative if and only if the condition (3) holds and the following conditions are satisfied:*

There exists a constant K such that

$$\sum_{n=0}^{\infty} (p_{n-1} P_n)^{2(1-s)} \left| \sum_{k=0}^r t(n, k, p) \right| \leq K, \quad \text{for } r, p = 1, 2, \dots, \quad (7)$$

$$(a_{pk}) \in \hat{\ell}(P_n, s) \text{ for each } k, \quad (8)$$

$$\sum_{k=0}^{\infty} a_{pk} \in \hat{\ell}(P_n, s). \quad (9)$$

For A to be (P_n, s) -absolutely almost conservative,

$$f - \lim A\mathbf{x} = a \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} x_k a_k,$$

for every $\mathbf{x} = (x_k) \in bv$, where, for each k , (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a .

Proof. If $\mathbf{x} = (x_k) \in bv$, then $\sum_{n=0}^{\infty} [(p_n)^{-1} P_n]^{2(1-s)} |F_{n,p}(A\mathbf{x})|^s < \infty$ uniformly in p . Since $np_n = O(P_n)$ as $n \rightarrow \infty$, $\sum_{n=0}^{\infty} |F_{n,p}(A\mathbf{x})|^s < \infty$ uniformly in p . Hence it follows that $\sum_{n=0}^{\infty} |F_{n,p}(A\mathbf{x})| < \infty$ uniformly in p . Then $A \in (bv, \hat{\ell}(1))$, therefore A is type (bv, m) . Hence condition (6) holds. Necessities of (8) and (9) are obvious since $e_k \in bv$, $e \in bv$ where $e_k = (0, 0, \dots, 0, 1, 0, 0, \dots)$ here 1 appears in the k -th position and $e = (1, 1, \dots, 1, \dots)$. For fixed p and j , consider the following linear functional

$T_{p,j}(x) = \sum_{k=0}^j a_{pk}x_k$. For each p and j , $T_{p,j}$ is a continuous linear functional on bv . By the assumption, for all $x \in bv$, $T_{p,j}(x)$ tends to a limit as $j \rightarrow \infty$ (p fixed) and hence by the Banach Steinhaus theorem [8], this limit, $(A\mathbf{x})_p$ is also a continuous linear functional on bv . Hence, for fixed p and n ,

$$\sum_{r=0}^n (p_n^{-1}P_n)^{2(1-s)}|F_{r,p}(A\mathbf{x})|^s \text{ is a continuous } s - \text{ norm on } bv. \quad (10)$$

For any given $x \in bv$, this s -norm in (10) is bounded in n, p . Hence by another application of the Banach Steinhaus theorem, there exists a constant $K > 0$ such that

$$\sum_{r=0}^n (p_n^{-1}P_n)^{2(1-s)}|F_{r,p}(A\mathbf{x})|^s \leq K\|x\|. \quad (11)$$

Define a sequence $\mathbf{x} = (x_k)$ by $x_k = 0$, if $k > r$ and $x_k = 1$, if $k \leq r$ and applying this with (11), (7) holds.

To prove the converse, take any $\mathbf{x} = (x_k) \in bv$. By (9), it follows that $\sum_{k=0}^{\infty} t(n, k, p)$ converges for all n, p . Write $h(n, k, p) = \sum_{l=k}^{\infty} t(n, l, p)$ for each $k \geq 1$. We see that

$$\lim_{k \rightarrow \infty} h(n, k, p) = 0 \text{ for fixed } n, p. \quad (12)$$

By condition (9), $\sum_{n=0}^{\infty} (p_n^{-1}P_n)^{2(1-s)}|h(n, 0, p)|^s$ converges uniformly in p . By condition (8), for fixed k , $\sum_{n=0}^{\infty} (p_n^{-1}P_n)^{2(1-s)}|t(n, k, p)|^s$ converges uniformly in p . Hence

$$\sum_{n=0}^{\infty} (p_n^{-1}P_n)^{2(1-s)}|h(n, k, p)|^s \quad (13)$$

converges uniformly in p . By (12) and the boundedness of sequence (x_k) , one can find $F_{n,p}(A\mathbf{x}) = \sum_{k=0}^{\infty} h(n, k, p)(x_k - x_{k-1})$. We can make $\sum_{k=k_0+1}^{\infty} |x_k - x_{k-1}|$ arbitrarily small by choosing k_0 sufficiently large. Hence it follows that, given $\varepsilon > 0$, we can choose k_0 so that, for all p ,

$$\sum_{n=0}^{\infty} (p_n^{-1}P_n)^{2(1-s)} \left| \sum_{k=k_0+1}^{\infty} h(n, k, p)(x_k - x_{k-1}) \right|^s < \frac{\varepsilon}{2}.$$

By the uniform convergence of (13), it follows that once k_0 has been chosen, we can choose n_0 so that for all p ,

$$\sum_{n=n_0+1}^{\infty} (p_n^{-1}P_n)^{2(1-s)} \left| \sum_{k=0}^{k_0} h(n, k, p)(x_k - x_{k-1}) \right|^s < \frac{\varepsilon}{2}.$$

On the other hand, using the inequality $(a+b)^s \leq a^s + b^s$ for all positive real numbers a, b , and $0 \leq s \leq 1$, it follows that for all p ,

$$\sum_{n=n_0+1}^{\infty} (p_n^{-1}P_n)^{2(1-s)}|F_{n,p}(A\mathbf{x})|^s < \varepsilon.$$

Thus $A \in (bv, \hat{\ell}(P_n, s))$. \square

If $A \in (bv, \hat{\ell}(P_n, s))$, then $A \in (bv, \hat{\ell}(1))$. By the theorem in [9], we obtain that for A to be (P_n, s) -absolutely almost conservative $f - \lim A\mathbf{x} = a \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} x_k a_k$, for every $\mathbf{x} = (x_k) \in bv$, where for each k , (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a .

Now combining the theorem in [3], we get the following:

Theorem 2.2 *Let s be a real number with $s > 0$ and $A = (a_{nk})$ be an infinite matrix. Then A is (P_n, s) -absolutely almost conservative if and only if the conditions (3), (8) and (9) are satisfied and the following condition holds: there exists a constant K such that*

$$\sum_{n=0}^{\infty} (p_{n-1} P_n)^{2|1-s|} \left| \sum_{k=0}^r t(n, k, p) \right| \leq K, \text{ for } r, p = 1, 2, \dots \quad (14)$$

For A to be (P_n, s) -absolutely almost conservative $f - \lim A\mathbf{x} = a \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} x_k a_k$, for every $\mathbf{x} = (x_k) \in bv$, where, for each k , (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a .

Proof. It follows from Theorem 2.1 that the proof is obtained for all s between 0 and 1 while theorem for $s \geq 1$ is [3]. \square

Corollary 2.3 *If s_1 and s_2 are in $(0, 1]$ with $s_1 \leq s_2$ and if A is (P_n, s_1) -absolutely almost conservative, then it is (P_n, s_2) -absolutely almost conservative, that is $(bv, \hat{\ell}(P_n, s_1)) \subseteq (bv, \hat{\ell}(P_n, s_2))$.*

Corollary 2.4 *If $s \in (0, 1]$ and if A is (P_n, s) -absolutely almost conservative, then it is s -absolutely almost conservative, hence it is absolutely almost conservative, that is $(bv, \hat{\ell}(P_n, s)) \subset (bv, \hat{\ell}(s)) \subset (bv, \hat{\ell}(1))$.*

Now we generalize Theorem 2.2 defining $\hat{\ell}(\lambda_n, s)$ as the set of all sequences (x_n) such that $\sum_{n=0}^{\infty} \lambda_n^{2|s-1|} |F_{n,p}(A\mathbf{x})|^s$ converges uniformly in p , where $\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2} < \infty$ and (λ_n, s) -absolutely almost conservativity of a matrix transformation as an element of $(bv, \hat{\ell}(\lambda_n, s))$:

Corollary 2.5 *Let s be a real number with $s > 0$ and $A = (a_{nk})$ be an infinite matrix. Then A is (λ_n, s) -absolutely almost conservative if and only if the condition (3) is satisfied and the following conditions hold: there exists a constant K such that for $r, p = 1, 2, \dots$*

$$\sum_{n=0}^{\infty} \lambda_n^{2|s-1|} \left| \sum_{k=0}^r t(n, k, p) \right| \leq K, \quad (15)$$

$$(a_{pk}) \in \hat{\ell}(\lambda_n, s), \quad (16)$$

$$\left(\sum_{k=0}^{\infty} a_{pk} \right) \in \hat{\ell}(\lambda_n, s). \quad (17)$$

For A to be (λ_n, s) -absolutely almost conservative, $f - \lim A\mathbf{x} = a \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} x_k a_k$ for every $\mathbf{x} = (x_k) \in bv$, where, for each k , (a_{pk}) is almost convergent with limit a_k and $\sum_{k=0}^{\infty} a_{pk}$ is almost convergent with limit a .

Corollary 2.6 *If A is (λ_n, s) -absolutely almost conservative, then s - absolutely almost conservative, hence it is absolutely almost conservative $(bv, \hat{\ell}(\lambda_n, s)) \subset (bv, \hat{\ell}(s)) \subset (bv, \hat{\ell}(1))$.*

Corollary 2.7 *Let $\lambda_n \leq \gamma_n$ for all n . If (x_n) is (λ_n, s) -absolutely almost convergent, then it is (γ_n, s) -absolutely almost convergent, that is $\hat{\ell}(\lambda_n, s) \subseteq \hat{\ell}(\gamma_n, s)$.*

Corollary 2.8 *Let $\lambda_n \leq \gamma_n$ for all n . If A is (λ_n, s) -absolutely almost conservative, then it is (γ_n, s) -absolutely almost conservative, that is $(bv, \hat{\ell}(\lambda_n, s)) \subseteq (bv, \hat{\ell}(\gamma_n, s))$.*

We note that if A is (λ_n, s) -absolutely almost conservative for an s satisfying $0 < s < 1$, then it is (λ_n, t) -absolutely almost conservative for any $t \geq 1$, that is $(bv, \hat{\ell}(\lambda_n, s)) \subseteq (bv, \hat{\ell}(\lambda_n, t))$. We also note that for any t satisfying $t > 1$ there exist an A , a (λ_n) and an s such that $A \in (bv, \hat{\ell}(\lambda_n, t))$ but $A \notin (bv, \hat{\ell}(\lambda_n, s))$, for example for $t = 2$ there exist an $s = \frac{1}{2}$, and $(\lambda_n) = (n)$ such that $A \in (bv, \hat{\ell}(n, 2))$ but $A \notin (bv, \hat{\ell}(n, \frac{1}{2}))$.

For a further study, we suggest to extend the present work to the fuzzy setting. However due to the change in settings, the definitions and methods of proofs will not always be analogous to these of the present work (for example, see [5]). As another further study we suggest to extend the present work to cone normed spaces (see [12] for the definitions in cone normed spaces.

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