

## Tauberian theorems for the product of Borel and Hölder summability methods

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**Abstract** In this paper we prove some Tauberian theorems for the product of Borel and Hölder summability methods which improve the classical Tauberian theorems for the Borel summability method.

**Keywords** Borel method · Hölder method · Tauberian theorems

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### 1 Introduction

A number of authors including LORD [13], HARDY and LITTLEWOOD [9,8], JAKIMOVSKI [1], RAJAGOPAL [15], ZELLER and BEEKMANN [18], PARAMESWARAN [14], KWEE [12], and BORWEIN and MARKOVICH [2] obtained some results related to the Borel summability method. In [3] ÇANAK and TOTUR have investigated conditions under which Borel summability implies convergence (see Section 2 below for the definition and notations).

In this paper our object is to prove some new Tauberian theorems for the product of Borel and Hölder summability methods which improve classical Tauberian theorems of HARDY and LITTLEWOOD [9].

Our results are based on the following theorems.

**Theorem 1.1** ([7]) *If  $s_n \rightarrow s(B)$  and  $n\Delta s_n = o(\sqrt{n})$  ( $n \rightarrow \infty$ ), then  $(s_n)$  converges to  $s$ .*

**Theorem 1.2** ([7]) *If  $s_n \rightarrow s(B)$  and  $n\Delta s_n = O(\sqrt{n})$  ( $n \rightarrow \infty$ ), then  $(s_n)$  converges to  $s$ .*

The generalization of Theorem 1.1 above is given by HARDY and LITTLEWOOD [10]. For another proof, we refer the reader to HARDY and LITTLEWOOD [8].

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## 2 Definition and notations

A sequence  $(s_n)$  is said to be summable by Borel's method to  $s$  and we write  $s_n \rightarrow s(B)$  if the corresponding series  $\sum_{n=0}^{\infty} \frac{s_n}{n!} x^n$  converges for all  $x$  and  $e^{-x} \sum_{n=0}^{\infty} \frac{s_n}{n!} x^n \rightarrow s$ ,  $x \rightarrow \infty$ . Borel summability method is regular; that is, if  $s_n \rightarrow s$  as  $n \rightarrow \infty$ , then  $s_n \rightarrow s(B)$ .

The Hölder summability method was introduced by HÖLDER [11] as a generalization of the limitation method of the arithmetical averages. For a sequence  $(s_n)$ , we define  $\sigma_n^{(1)} = \frac{1}{n+1} \sum_{j=0}^n s_j$  for all  $n \geq 0$ . Repeating this averaging process, we define  $\sigma_n^{(k)}$  for all  $k \geq 0$  by  $\sigma_n^{(0)} = s_n$  and  $\sigma_n^{(k+1)} = \frac{1}{n+1} \sum_{j=0}^n \sigma_j^{(k)}$ . A sequence  $(s_n)$  is said to be summable by the Hölder's method  $(H, k)$  to  $s$  and we write  $s_n \rightarrow s(H, k)$  if  $\sigma_n^{(k)} \rightarrow s$  as  $n \rightarrow \infty$ .

Note that  $(H, 0)$  summability of a sequence means that it converges in the ordinary sense and  $(H, 1)$  method is the limitation method of the arithmetical averages. The Hölder method  $(H, k)$  are regular for any  $k$ . It is well known that if  $s_n \rightarrow s(H, k)$ , where  $k \geq 0$ , and  $k' > k$ , then  $s_n \rightarrow s(H, k')$  (see [7]).

If  $(\sigma_n^{(k)})$  is Borel summable to  $s$ , we say that  $(s_n)$  is  $(B)(H, k)$  summable to  $s$  and we write  $s_n \rightarrow s(B)(H, k)$ .

We use the symbols  $s_n = o(1)$  ( $n \rightarrow \infty$ ) and  $s_n = O(1)$  ( $n \rightarrow \infty$ ) to mean that  $s_n \rightarrow 0$  as  $n \rightarrow \infty$  and that  $(s_n)$  is bounded for large enough  $n$ , respectively. The backward difference of a sequence  $(s_n)$  is defined for all  $n \geq 0$  by  $\Delta s_0 = s_0$  and  $\Delta s_{n+1} = s_{n+1} - s_n$ . For any  $k \geq 0$ , the identity

$$\sigma_n^{(k)} - \sigma_n^{(k+1)} = V_n^{(k)}, \quad (2.1)$$

where

$$V_n^{(k)} = \begin{cases} \frac{1}{n+1} \sum_{j=0}^n V_j^{(k-1)}, & k \geq 1 \\ \frac{1}{n+1} \sum_{j=0}^n j \Delta s_j, & k = 0, \end{cases}$$

is frequently used in the proofs of our results.

Since  $\sigma_n^{(k+1)}$  can be expressed in the form

$$\sigma_n^{(k+1)} = s_0 + \sum_{j=1}^n \frac{V_j^{(k)}}{j}, \quad (2.2)$$

we may rewrite (2.1) as  $\sigma_n^{(k)} = V_n^{(k)} + \sum_{j=1}^n \frac{V_j^{(k)}}{j} + s_0$ . A sequence  $(s_n)$  is said to be slowly oscillating (see [5, 6]) if  $\lim_{\lambda \rightarrow 1^+} \limsup_{n \rightarrow \infty} \max_{n+1 \leq k \leq [\lambda n]} |s_k - s_n| = 0$ , where  $[\lambda n]$  denotes the integer part of  $\lambda n$ .

A sequence  $(s_n)$  is said to be moderately oscillating (see [5, 6]) if for  $\lambda > 1$ ,  $\limsup_{n \rightarrow \infty} \max_{n+1 \leq k \leq [\lambda n]} |s_k - s_n| < \infty$ .

It can be easily seen that if  $(s_n)$  is moderately oscillating, then  $V_n^{(0)} = O(1)$  ( $n \rightarrow \infty$ ) and  $(\sigma_n^{(1)})$  is slowly oscillating (see Dik [5]).

We now recall what we mean by Tauberian conditions and Tauberian theorems.

If a sequence is convergent, then it is summable by any regular summability method. However, the converse is not true in general. The converse may be valid under some conditions which are called Tauberian conditions. Any theorem which states that convergence of a sequence follows from a summability method and some Tauberian condition(s) is called a Tauberian theorem.

### 3 Lemmas

We need the following lemmas required in the proof of our theorems in the next section.

**Lemma 3.1** For any  $k \geq 0$ ,  $V_n^{(k)} = n\Delta\sigma_n^{(k+1)}$ .

*Proof.* Take the backward difference of (2.2) and then multiply by  $n$ .  $\square$

**Lemma 3.2** For any  $k \geq 0$ ,

$$n\Delta V_n^{(k+1)} = V_n^{(k)} - V_n^{(k+1)}. \quad (3.1)$$

*Proof.* We have

$$V_n^{(k)} - V_n^{(k+1)} = n\Delta(\sigma_n^{(k+1)} - \sigma_n^{(k+2)}) = n\Delta V_n^{(k+1)} \quad (3.2)$$

for all  $k$ .  $\square$

**Lemma 3.3** ([4]) If  $s_n \rightarrow s(H, 1)$  and  $(s_n)$  is slowly oscillating, then  $(s_n)$  converges to  $s$ .

**Lemma 3.4** ([16]) If  $s_n \rightarrow s(B)$ , then  $s_n \rightarrow s(B)(H, 1)$ .

**Lemma 3.5** ([5, 17]) If  $(s_n)$  is slowly oscillating, then  $(V_n^{(0)})$  is bounded and slowly oscillating.

*Proof.* Let  $(s_n)$  be slowly oscillating. For  $\lambda > 1$ , we define

$$w_n(s, \lambda) = \max_{n+1 \leq k \leq [\lambda n]} \left| \sum_{j=n+1}^k \Delta s_j \right| \quad (3.3)$$

and rewrite the finite sum  $\sum_{k=1}^n k\Delta s_k$  as the series

$$\sum_{j=0}^{\infty} \sum_{\frac{n}{2^{j+1}} \leq k < \frac{n}{2^j}} k\Delta s_k. \quad (3.4)$$

Hence

$$\begin{aligned} \left| \sum_{k=1}^n k\Delta s_k \right| &\leq \sum_{j=0}^{\infty} \left| \sum_{\frac{n}{2^{j+1}} \leq k < \frac{n}{2^j}} k\Delta s_k \right| \leq \left( \sum_{j=0}^{\infty} \frac{n}{2^j} \right) w_{[\frac{n}{2^{j+1}}]}(s, \lambda) \\ &\leq nC_1 \sum_{j=0}^{\infty} \frac{1}{2^j} = 2nC_1 = nC, \end{aligned}$$

where  $C > 0$ . Consequently, we have  $V_n^{(0)} = O(1) (n \rightarrow \infty)$ . Therefore  $(\sigma_n^{(1)}) = (s_0 + \sum_{k=1}^n \frac{V_k^{(1)}}{k})$  is slowly oscillating. Thus  $(V_n^{(0)})$  is slowly oscillating.  $\square$

As a consequence of Lemma 3.5, we note that if a sequence  $(s_n)$  is slowly oscillating, then  $(V_n^{(k)})$  is bounded and therefore  $(\sigma_n^{(k)})$  is slowly oscillating for all  $k$ .

#### 4 Main results

We state and prove the following Tauberian theorems for  $(B)(H, k)$  summability method.

**Theorem 4.1** *If  $s_n \rightarrow s(B)(H, k + 1)$  for some  $k \geq 0$  and*

$$V_n^{(k)} = O(\sqrt{n}) (n \rightarrow \infty), \quad (4.1)$$

*then  $s_n \rightarrow s(H, k + 1)$ .*

*Proof.* By hypothesis we have

$$\sigma_n^{(k+1)} \rightarrow s(B). \quad (4.2)$$

By (4.1) and Lemma 3.1 we have

$$V_n^{(k)} = n\Delta\sigma_n^{(k+1)} = O(\sqrt{n}) (n \rightarrow \infty). \quad (4.3)$$

We obtain  $\sigma_n^{(k+1)} \rightarrow s$  from Theorem 1.2, which means that  $s_n \rightarrow s(H, k + 1)$ .  $\square$

From Theorem 4.1 we deduce the following corollary.

**Corollary 4.2** *If  $(s_n) \rightarrow s(B)(H, k + 1)$  for some  $k \geq 0$  and  $(s_n)$  is slowly oscillating, then  $(s_n)$  converges to  $s$ .*

*Proof.* Slow oscillation of  $(s_n)$  implies both  $V_n^{(k)} = O(1) (n \rightarrow \infty)$  and slow oscillation of  $(\sigma_n^{(k)})$  for all  $k \geq 1$  as a consequence of Lemma 3.5. Hence, we have  $V_n^{(k)} = n\Delta\sigma_n^{(k+1)} = O(\sqrt{n}) (n \rightarrow \infty)$ . Since  $(s_n)$  is  $(H, k + 1)$  summable to  $s$ ,  $\sigma_n^{(k+1)} \rightarrow s$  by Theorem 1.2. It follows by the slow oscillation of  $(\sigma_n^{(k)})$  that  $\sigma_n^{(k)} \rightarrow s$ . Continuing in this vein, we obtain  $\sigma_n^{(1)} \rightarrow s$ . By Lemma 3.3,  $(s_n)$  converges to  $s$ .  $\square$

**Remark 4.1** If we replace slow oscillation of  $(s_n)$  by moderate oscillation of  $(s_n)$  in Corollary 4.2, we recover convergence of  $(\sigma_n^{(1)})$  from  $(B)(H, k + 1)$  summability of  $(s_n)$ .

Parallel with the Hardy and Littlewood Tauberian condition  $n\Delta s_n = O(\sqrt{n}) (n \rightarrow \infty)$  for Borel summability method, the condition  $n\Delta V_n^{(k)} = O(\sqrt{n}) (n \rightarrow \infty)$  suffices to recover  $(H, k)$  summability of  $(s_n)$  from its  $(B)(H, k)$  summability for some  $k \geq 0$ .

**Theorem 4.3** *If  $s_n \rightarrow s(B)(H, k)$  for some  $k \geq 0$  and*

$$n\Delta V_n^{(k)} = O(\sqrt{n}) (n \rightarrow \infty), \quad (4.4)$$

*then  $(s_n)$  is  $(H, k)$  summable to  $s$ .*

*Proof.* By hypothesis,  $\sigma_n^{(k)} \rightarrow s (B)$ . By Lemma 3.4, we have

$$\sigma_n^{(k+1)} \rightarrow s (B). \quad (4.5)$$

It follows by (2.1) that

$$\sigma_n^{(k)} - \sigma_n^{(k+1)} = V_n^{(k)} \rightarrow 0 (B). \quad (4.6)$$

If we apply Theorem 1.2 to  $(V_n^{(k)})$ , we conclude from (4.4) that

$$V_n^{(k)} = o(1) (n \rightarrow \infty). \quad (4.7)$$

By Lemma 3.1, we have

$$V_n^{(k)} = n\Delta\sigma_n^{(k+1)} = o(1) (n \rightarrow \infty). \quad (4.8)$$

It follows from (4.8) that

$$n\Delta\sigma_n^{(k+1)} = o(\sqrt{n}) (n \rightarrow \infty). \quad (4.9)$$

If we apply Theorem 1.1 to  $(\sigma_n^{(k+1)})$ , we conclude that

$$\sigma_n^{(k+1)} \rightarrow s. \quad (4.10)$$

Substituting (4.7) and (4.10) into (2.1), we obtain  $\sigma_n^{(k)} \rightarrow s$  which means that  $s_n \rightarrow s(H, k)$ .  $\square$

If condition (4.4) is replaced by the condition  $n\Delta V_n^{(k+1)} = o(\sqrt{n}) (n \rightarrow \infty)$ , we recover  $(H, k)$  summability of  $(s_n)$  from its  $(B)(H, k+1)$  summability for some  $k \geq 0$ .

**Theorem 4.4** *If  $s_n \rightarrow s(B)(H, k+1)$  for some  $k \geq 0$  and*

$$n\Delta V_n^{(k+1)} = o(1) (n \rightarrow \infty), \quad (4.11)$$

*then  $s_n \rightarrow s(H, k)$ .*

*Proof.* By hypothesis,  $\sigma_n^{(k+1)} \rightarrow s (B)$ . By Lemma 3.4, we have

$$\sigma_n^{(k+2)} \rightarrow s (B). \quad (4.12)$$

Replacing  $k$  by  $k+1$  in (2.1), we obtain

$$\sigma_n^{(k+1)} - \sigma_n^{(k+2)} = V_n^{(k+1)} \rightarrow 0 (B). \quad (4.13)$$

It follows from (4.11) that

$$n\Delta V_n^{(k+1)} = o(\sqrt{n}) (n \rightarrow \infty). \quad (4.14)$$

If we apply Theorem 1.1 to  $(V_n^{(k+1)})$ , we obtain

$$V_n^{(k+1)} = o(1) (n \rightarrow \infty). \quad (4.15)$$

Replacing  $k$  by  $k + 1$  in Lemma 3.1, we have

$$V_n^{(k+1)} = n\Delta\sigma_n^{(k+2)} = o(1) \ (n \rightarrow \infty). \quad (4.16)$$

It follows from (4.16) that

$$n\Delta\sigma_n^{(k+2)} = o(\sqrt{n}) \ (n \rightarrow \infty). \quad (4.17)$$

By (4.12) and (4.17), we get

$$\sigma_n^{(k+2)} \rightarrow s \quad (4.18)$$

from Theorem 1.1. Replacing  $k$  by  $k + 1$  in (2.1), we have

$$\sigma_n^{(k+1)} - \sigma_n^{(k+2)} = V_n^{(k+1)}. \quad (4.19)$$

By (4.15), (4.18) and (4.19), we obtain

$$\sigma_n^{(k+1)} \rightarrow s. \quad (4.20)$$

By Lemma 3.2 we have

$$V_n^{(k)} - V_n^{(k+1)} = n\Delta V_n^{(k+1)}. \quad (4.21)$$

By (4.11), (4.15) and from identity (4.21), we have

$$V_n^{(k)} = o(1) \ (n \rightarrow \infty). \quad (4.22)$$

From (4.20) and (4.22), we obtain  $\sigma_n^{(k)} \rightarrow s$  by (2.1), which means that  $s_n \rightarrow s(H, k)$ .  
□

Also we have Theorem 4.4 when condition (4.11) is replaced by the condition  $V_n^{(k)} = o(1) \ (n \rightarrow \infty)$  for some  $k \geq 0$ .

**Corollary 4.5** *If  $s_n \rightarrow s(B)(H, k + 1)$  for some  $k \geq 0$  and*

$$V_n^{(k)} = o(1) \ (n \rightarrow \infty), \quad (4.23)$$

*then  $s_n \rightarrow s(H, k)$ .*

*Proof.* The proof easily follows from (3.1) and the regularity of the Hölder's method of summability. □

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